Lecture 17 11/2

Central simple algebras

k $\rightsquigarrow$ division algebras$/k$.
field.

k-algebra: R means $k \subset Z(R)$.

central k-alg R means $k = Z(R)$.
e.g. $M_n(k)$.

simple : no nontrivial 2-sided ideals.

R is simple $\iff$ R as an R-bimod is a simple bimod.

($R' \subset R \Rightarrow R' = 0$ or R).
sub bimod
same as a 2-sided ideal.

R/k c.s.a. (central simple algebra).
($Z(R) = k$, and R is a simple ring)

Examples
- $M_n(k)$.
- Assume $\dim_k R < \infty$. $\Rightarrow$ R = semisimple. and simple.

$$R \simeq M_n(D).$$

$$Z(R) = k \iff Z(D) = k.$$

Need to understand central div. alg. / k.

- $k = \mathbb{R}$. $\mathbb{H}$.

  f.d. c.s.a. / $\mathbb{R}$: $M_n(\mathbb{R})$.

  $M_n(\mathbb{H})$.

- $k = \mathbb{F}_q$. finite field.

  Wedderburn's Thm: f.d. division alg / $\mathbb{F}_q$ are fields.

  $\Rightarrow$ f.d.c.s.a. / $\mathbb{F}_q$: $M_n(\mathbb{F}_q)$.

- $k = \overline{k}$. No nontrivial f.d. division alg / k.

  $D \supset k$.

  - $x \in D \setminus k$.

  $\Rightarrow$ x is algebraic over k.

  $k[x] \subset D$.

  finite-dim'l

  $x^n + a_1 x^{n-1} + \cdots + a_n = 0$.

  factorizes.

  $(x - \lambda_1) \cdots (x - \lambda_n) = 0 \in D$.

  D has no zero div $\Rightarrow$ $x = \lambda_i \in k$ $\times$

  f.d. c.s.a / $k = \overline{k}$: $M_n(k)$.

<u>Ex</u>. (Cyclic algebras).

k contains a primitive n^{th} root of 1, ζ.

($\operatorname{char}(k) = 0$ or prime to n).

$$a, \ b \in k.$$

$$R_{a,b} = k\langle x, y\rangle \Big/ \begin{pmatrix} \underline{x^n - a} \\ y\underline{^n -} \ b \\ yx - \zeta xy \end{pmatrix}$$

e.g. $\mathbb{H} = \mathbb{R}\langle x, y\rangle \Big/ \begin{pmatrix} x^2+1 \\ y^2+1 \\ yx+xy \end{pmatrix}$ $\quad x = i$, $y = j$

as k-v.s.

$$yx = \zeta xy.$$

$$\left\{ \sum c_{m,l}\, x^m y^l \right\} = k\langle x, y\rangle \big/ (\underline{yx - \zeta xy})$$

$$\left\{ \sum_{m,l=0}^{n-1} c_{m,l}\, x^m y^l \right\} = R_{a,b}.$$

$$\dim_k R_{a,b} = n^2.$$

$$Z(R_{a,b}) = k.$$

<u>Fact</u>: if $ab \neq 0$. $\Rightarrow$ $R_{a,b}$ is c.s.a$/k$.

Idea: we will see: R/k c.s.a

$\Longleftrightarrow (R \otimes_k \bar{k})/\bar{k}$ is c.s.a.

$k = \bar{k}$. $a \neq 0$. $x^n - a = \prod_{i=0}^{n-1} (x - \zeta^i \alpha)$

$$R_{a,b} \supset k[x]/(x^n - a) \xrightarrow{\sim} \underbrace{k \times k \times \cdots \times k}_{n}.$$

want : $R_{a,b} \simeq M_n(k)$.

Try to construct an $\underline{n\text{-dim'l}}$ module of $R_{a,b}$.
(over k).

$$\underset{\underset{x}{\circlearrowright}}{V} = \underset{\underset{\alpha}{\circlearrowright}}{k(0)} \oplus \underset{\underset{\zeta\alpha}{\circlearrowright}}{k(1)} \oplus \cdots \oplus \underset{\underset{\zeta^{n-1}\alpha}{\circlearrowright}}{k(n-1)} \qquad \dim_k k(i) = 1$$

$$x = \begin{pmatrix} \alpha & & & \\ & \zeta\alpha & & \\ & & \ddots & \\ & & & \zeta^{n-1}\alpha \end{pmatrix}.$$

$yx = \zeta xy$. $\Longrightarrow$ $xv = \lambda v$.

$$\begin{array}{ccc} y(xv) & = & \zeta x(yv) \\ \| & & * \\ \lambda yv & = & \end{array}$$

$\Rightarrow$ yv is in the eigensp of x with eigenval $= \lambda \cdot \zeta^{-1}$.

$$\boxed{y: \ k(i) \longrightarrow k(i-1).}$$

$$y = \begin{pmatrix} & \beta_1 & & & \\ & & \beta_2 & & \\ & & & \ddots & \\ & & & & \beta_{n-1} \\ \beta_n & & & & \end{pmatrix} \qquad \beta_i \in k.$$

any y of this form satisfies $\boxed{yx = \zeta x y}$

To construct the action of $y \circlearrowright V$.

just need $y^n = b.$

$$\Updownarrow$$

$$\prod \beta_i = b.$$

Take $y = \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ b & & & & \end{pmatrix}$ $\qquad b \neq 0.$

$$\Rightarrow \quad \underset{n^2}{R_{a,b}} \longrightarrow \underset{n^2}{\mathrm{End}_k(V)}$$

Check this is $\cong$.

$$\underset{\underset{k\times\cdots\times k.}{\shortparallel}}{k[x]} \xrightarrow{\sim} \begin{pmatrix} * & & & \\ & * & & \\ & & \ddots & \\ & & & * \end{pmatrix}. \qquad E_{ii} \ ✓$$

$E_{ii}\, y^j$

get E_{ij}.

surj ✓

More general cyclic alg construction:

$L \leftarrow$ a product of field extensions of k, e.g. $\underbrace{k\times k\times\cdots\times k}_{n} \circlearrowleft \sigma =$ cyclic perm.

$|$ n cyclic extension.

k $\quad \left(\iff \sigma : L \longrightarrow L \text{ of order } n \text{ s.t. } k = L^{\sigma}. \right)$

Skew-polynomial ring

$$L\langle x;\sigma\rangle = \left\{ \sum_{\substack{i\geq 0\\ \text{finite}}} c_i x^i \,\middle|\, c_i \in L \right\}$$

$$x\cdot c = \sigma(c)\cdot x \quad \forall\, c\in L.$$

$\forall\, a\in k$, $L\langle x;\sigma\rangle \big/ (x^n - a)$.

Fact: when $a\neq 0$, this is a c.s.a. of dim n^2/k.

Previous ex. $\quad L = \underbrace{k[y]/(y^n-b)}_{}$. cyclic extn of deg n.

$|$

$k \qquad \sigma : L \longrightarrow L,$

$\qquad\qquad y \longmapsto \boxed{\zeta^{-1}y}.$

$$L\langle x;\sigma\rangle \big/ (x^n - a)$$

$$= \underbrace{k[y]\langle x;\sigma\rangle}_{} \big/ (x^n-a,\, y^n-b).$$

$$\underset{\text{"const"}}{k[y]}\langle x;\sigma\rangle = k\langle x,y\rangle \Big/ \left(\underline{xy - \overset{\zeta^{-1}yx}{\overset{\shortparallel}{\sigma(y)}}x}\right)$$

$$\Longleftrightarrow \zeta xy - yx.$$

Tensor product of algebras / k.

$$\boxed{R \underset{k}{\otimes} S}$$

$(r\otimes s)(r'\otimes s') = rr'\otimes ss'$ defines a ring structure.

$$\begin{tikzcd} R \arrow[dr, "r \mapsto r\otimes 1"] & \\ & R\otimes S \\ S \arrow[ur, "s \mapsto 1\otimes s"'] & \end{tikzcd}$$

ring homo.

Universal property. $\underline{k\text{-Alg}}$.

T : k-alg.

$$\begin{tikzcd} \operatorname{Hom}_{k\text{-Alg}}(\underbrace{R \underset{k}{\otimes} S}_{\substack{R,\,S \text{ commute} \\ \text{inside } R\otimes_k S}}, T) \arrow[r] \arrow[dr] & \operatorname{Hom}_{k\text{-Alg}}(R,T) \times \operatorname{Hom}_{k\text{-Alg}}(S,T) \\ & \left\{(\varphi_1,\varphi_2) \,\middle|\, \substack{\operatorname{Im}(\varphi_1),\,\operatorname{Im}(\varphi_2) \\ \text{commute in } T}\right\} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

not a bijection.

$$\varphi : (R \underset{k}{\otimes} S) \longrightarrow T$$

$\Rightarrow \varphi(R), \varphi(S)$ commute within T.

Conversely. $\varphi_1 : R \longrightarrow T$

$\varphi_2 : S \longrightarrow T$

s.t. $\operatorname{Im}(\varphi_1), \operatorname{Im}(\varphi_2)$ commute.

$$R \underset{k}{\otimes} S \longrightarrow T$$

$$r\otimes s \longmapsto \varphi_1(r)\varphi_2(s).$$

R, S $\underline{k\text{-alg}}$

(R,S)-bimod. M. $\Longleftrightarrow$ $\underline{R \underset{k}{\otimes} S}^{\overset{\swarrow}{op}}$-module M.

Lemma R, S: k-algebras.
$(Z(R), Z(S)$: comm k-algs$)$.

Then $Z(R \underset{k}{\otimes} S) = Z(R) \underset{k}{\otimes} Z(S)$.

Pf. $\supset$ clear.

$\subset$: $\zeta = \sum_{\alpha \in I} r_\alpha \otimes \boxed{s_\alpha}$ $\quad$ $s_\alpha = 0$ for almost all α.

assume $\zeta \in Z(R \otimes S)$ $\quad$ $\{r_\alpha\}_{\alpha \in I}$ k-basis of R.

ζ comm. with $S \ni s$.

$\zeta s = s \zeta$.

$$\Rightarrow \quad \sum r_\alpha \otimes (s_\alpha s) = \sum r_\alpha \otimes (s s_\alpha).$$

$$\Rightarrow \quad s_\alpha s = s s_\alpha \quad \forall s.$$

i.e. $s_\alpha \in Z(S)$.

$\Rightarrow$ Choose a k-basis $\{\sigma_\beta\}$ for $Z(S)$.
write

$$\zeta = \sum \rho_\beta \otimes \sigma_\beta$$

$\rho_\beta \in R$.

use $r\zeta = \zeta r \quad \forall\, r \in R.$

$$\Rightarrow \quad r\rho_\beta = \rho_\beta r \quad \forall\, r \in R$$

i.e., $\rho_\beta \in Z(R).$

$$\Rightarrow \quad \zeta \in Z(R) \otimes Z(S).$$

Goal: $R/k,\ S/k$ c.s.a.

$\Rightarrow \quad R \otimes_k S$ is c.s.a.

(centrality ✓)

Thm: R/k is c.s.a.
S any k-alg.
There's a bijection

$$\begin{tikzcd}
\left\{ \substack{\text{ideals of } S \\ \text{2-sided}} \right\} \arrow[r, leftrightarrow] \arrow[d, phantom, "\cup"] & \left\{ \substack{\text{2-sided} \\ \text{ideals of } R \otimes_k S} \right\} \arrow[d, phantom, "\cup"] \\
I \arrow[r, mapsto, shift left] & R \otimes_k I \arrow[l, shift left] \\
J \cap (1 \otimes S) & J \subset R \otimes S \arrow[l, mapsto]
\end{tikzcd}$$

Cor. R/k is c.s.a. then

1) $R \otimes_k S$ is simple $\Longleftrightarrow$ S is simple.

2) $R \otimes_k S$ is c.s.a $\Longleftrightarrow$ S/k c.s.a.

Cor. L/k field extension.
then R/k is c.s.a. $\Longleftrightarrow$ $(R \otimes_k L)/L$ is c.s.a.

Pf: Apply previous Cor to $S = L$.

R/k is c.s.a $\Rightarrow$ $R \otimes_k L$ is simple.

with center

$\parallel$

$Z(R) \otimes_k L = L$.

$\Leftarrow$ $R \otimes_k L / L$ is c.s.a.

center: $\utilde{Z(R)} \otimes_k L = Z(R \otimes_k L) = L$.

$\Rightarrow Z(R) = k$.

simple: If R is not simple. $0 \subsetneq I \subsetneq R$

$\Rightarrow 0 \subsetneq I \otimes_k L \subsetneq R \otimes_k L$

$\Rightarrow R \otimes_k L$ is not simple. $\times$.