Lecture 16 — 10/28

- Morita context

- Time shift from the next week.
- Next Homework is due Nov. 9. (Monday)

Morita equivalence

$$\text{R-mod} \underset{P \otimes_S (-) = G}{\overset{S \circlearrowleft Q \otimes_R (-) = F}{\rightleftarrows}} \text{S-mod}$$

$$\begin{array}{c} R \\ P \end{array} \rightleftarrows \begin{array}{c} Q \\ S \end{array}$$

$$G \circ F = \underbrace{\left(P \otimes_S Q\right)}_{\text{R-bimod}} \otimes_R (-)$$

$G \circ F \simeq id_{R\text{-mod}} \iff \lambda : P \otimes_S Q \xrightarrow{\sim} R$ as (R,R)-bimod

$F \circ G \simeq id_{S\text{-mod}} \iff \mu : Q \otimes_R P \xrightarrow{\sim} S$ as (S,S)-bimod.

+ conditions.
$$P\,Q\,P \overset{\sim}{\rightrightarrows} P$$
$$Q\,P\,Q \overset{\sim}{\rightrightarrows} Q$$

existence of λ and μ $\Longrightarrow$ ${}_RP$ is progenerator.

${}_SQ$ —— " ——

P_S —— " ——

Q_R —— " ——

Morita equiv can be summarized as

$$(R, S, P, Q, \underset{=}{\lambda}, \underset{=}{\mu}).$$

Morita context: not requiring λ, μ to be $\cong$.

+ PQP and QPQ conditions.

$$\Longleftrightarrow \begin{pmatrix} R & P \\ Q & S \end{pmatrix} \text{ is a ring.}$$

Derived Morita context:

Start from $\boxed{(R, P)}$ P: left R-mod.

$S := \mathrm{End}_R(P)^{op}$ $(R \circlearrowright P \circlearrowleft S)$

$Q := \mathrm{Hom}_R(P, \underset{=}{R})$ $(S \circlearrowright Q \circlearrowleft R)$

$$\lambda = ev: \quad P \underset{S}{\otimes} Q = P \underset{S}{\otimes} \mathrm{Hom}_R(P, R) \longrightarrow R$$

$$p \otimes q \longmapsto q(p).$$

$q : P \to R.$

$$\mu : \quad Q \otimes_R P = \mathrm{Hom}_R(P, R) \otimes_R P \longrightarrow \mathrm{End}_R(P)$$

$$q \otimes p \longmapsto (P \xrightarrow{q} R \xrightarrow{1 \mapsto p} P)$$

Check: PQP, QPQ ✓

<u>Ex</u>. R-mod.

$\boxed{P = \text{f.g. proj. } R\text{-mod}}$. (not nec. generator)

$$\rightsquigarrow (R, P, S = \mathrm{End}_R(P)^{op}, Q, \ldots)$$

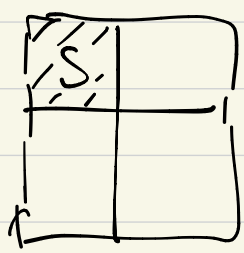
$$R\text{-mod} \xrightarrow[\mathrm{Hom}_R(P, -)]{} S\text{-mod}$$

e : idempotent in R. $(e^2 = e)$

$P = R \cdot e$, $\quad S = eRe$

(unit of S is e).

$$\mathrm{Hom}_R(P, M) = eM.$$

R = 

$$Re \to M.$$

$$e \mapsto x = ex.$$

if $eM = M$, $\quad M \overset{S}{\longmapsto} M$.

if $eM = 0$. $\quad M \longmapsto 0$.

This is a quotient functor.

$$R\text{-mod} / \langle \cdots \rangle \xrightarrow{\sim} S\text{-mod}.$$

Serre quotient.

M s.t. $eM = 0$.

closed under taking extensions and subquotients.

$$\begin{tikzcd}
R\text{-mod} \arrow[r, bend left, "\operatorname{Hom}_R(P, -)"] & S\text{-mod} \arrow[l, bend right, "P \otimes_S (-)"']
\end{tikzcd}$$

$$\begin{tikzcd}
P \otimes_S (M) \arrow[dr, bend right] & M \arrow[l] \\
& \operatorname{Hom}_R(P, P \otimes_S M) \arrow[u, "S"']
\end{tikzcd}$$

Proof: $Q = \operatorname{Hom}_R(P, R)$

$$(*) \quad \operatorname{Hom}_R(P, -) \simeq Q \otimes_R (-)$$

(write P as summand of $R^{\oplus n}$)

$$\begin{tikzcd}
\operatorname{Hom}_R(P, P \otimes_S (-)) \simeq Q \otimes_R P \otimes_S (-) \arrow[d, "\mu"] \\
S \otimes_S (-).
\end{tikzcd}$$

want $\mu : Q \otimes_R P \longrightarrow S$ is $\cong$.

$$\begin{array}{c} S \\ /\!/ \end{array}$$

$(*)$: $\mathrm{Hom}_R(P, P) \simeq Q \otimes_R P$.

Conversely, if

$(R, P) \rightsquigarrow (R, S, P, Q, \lambda, \mu)$ any Morita context.

Then TFAE:

1) $Q \otimes_R P \xrightarrow[\mu]{\sim} S$

2) $Q \otimes_R P \overset{\mu}{\twoheadrightarrow} S$.

3) P is a f.g. projective R-module

Pf: 2) $\Rightarrow$ 3). λ gives $Q \longrightarrow \mathrm{Hom}_R(P, R)$
μ can be written as composition.

$Q \otimes_S P \longrightarrow \mathrm{Hom}_R(P, R) \otimes_S P \xrightarrow{\bar{\mu}} S$. (using identities on λ & μ)

$\Rightarrow \mathrm{Hom}_R(P, R) \otimes_S P \overset{\bar{\mu}}{\twoheadrightarrow} S$

$\Downarrow$

$$\boxed{\sum_{i=1}^{n} q_i \otimes p_i} \boxed{\longmapsto 1}$$

$\bar{\mu}(\sum q_i \otimes p_i) = \sum_i p_i q_i \in \mathrm{End}_R(P)$

$$\begin{tikzcd} P \arrow[r, "(q_1,\dots,q_n)"] \arrow[rr, bend right, "id_P"'] & R^{\oplus n} \arrow[r, "\begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}"] & P \end{tikzcd}$$

$\Rightarrow$ P is a direct summand of $R^{\oplus n}$.

Any Morita context $(R, S, P, Q, \lambda, \mu)$.

$Q \otimes_R P \xrightarrow{\mu} S$ is an $\cong$ $\Longleftrightarrow$ surj.

$P \otimes_S Q \xrightarrow{\lambda} R$ is $\cong$ $\Longleftrightarrow$ surj.

<u>Categorical properties / invariants of a ring:</u>

injective module ✓
projective module. ✓
free module. ✗ 　　 k-v.s. $\xrightarrow{\sim}$ $M_n(k)$-mod.
　　 $k \longmapsto k^n$ not free.

f.g. ✓

finite presentation. 　 $R^{\oplus a} \to R^{\oplus b} \to M \to 0$.
$\Longleftrightarrow$ $\mathrm{Hom}_R(M, -)$ comm with <u>filtered colim</u>.

M is f.g. $\Longleftrightarrow$ any $\{M_\alpha\}_{\alpha \in I}$ <u>subobj of M</u>
s.t. $\bigoplus M_\alpha \twoheadrightarrow M$.
then $\exists$ finite $I' \subset I$.
$\bigoplus_{\alpha \in I'} M_\alpha \twoheadrightarrow M$.

$Z(R)$ categorical invariant.

$$Z(R) \cong \mathrm{End}(\mathrm{id}_{R\text{-mod}}).$$

Dually. (cocenter).

$$[R, R] \subset R$$

$[R,R]$: $\mathbb{Z}$-span of $ab - ba$. $a, b \in R$.

⚠ Not an ideal in general.

$$\cancel{r(ab-ba)}$$

$$Tr(R) := R/[R,R]$$ abelian gp. (not a ring in general)

$$M_n(k) \xrightarrow{Tr} k = M_n(k)/[M_n(k), M_n(k)].$$

$$Tr(ab) = Tr(ba).$$

$$\begin{tikzcd} R \arrow[r, two heads] \arrow[dr, "t"'] & R/[R,R] \arrow[d, dashed, "\exists!"] \\ & A \end{tikzcd}$$

A = ab. gp.

$$t(ab) = t(ba) \quad \forall\, a, b \in R$$

Ex. $R = k\langle x, y\rangle$, $\longrightarrow\!\!\!\!\to k[x, y]$.

$R/[R,R]$: $\quad xy \equiv yx$

$$xyxy \not\equiv x^2y^2$$

$$xyxy \equiv yxyx$$

$$[xyx, y]$$

$$x^2y^2 \equiv xy^2x \equiv y^2x^2 \equiv yx^2y$$

$$R/[R,R] = \mathrm{Span}\left\{\begin{matrix}\text{words in } x, y \\ \text{up to cyclic permutation}\end{matrix}\right\}.$$

<u>Prop.</u> Cocenter is a categorical invt.

<u>Pf.</u> $R \underset{\text{Morita}}{\sim} S$

$(R, S, P, Q, \lambda, \mu)$. Morita context realizing equiv.

$$\lambda: P \underset{S}{\otimes} Q \xrightarrow{\sim} R$$

$$\mu: Q \underset{R}{\otimes} P \xrightarrow{\sim} S$$

Define $R \longrightarrow S/[S,S]$.

$$\lambda\left(\sum p_i \otimes q_i\right) \longmapsto \mu\left(\sum q_i \otimes p_i\right)$$

only well-defined mod $[S,S]$.

$$\lambda(p_i \otimes sq_i) \longmapsto \mu(sq_i \otimes p_i) = s\mu(q_i \otimes p_i)$$

$$\shortparallel$$

$$\lambda(p_i s \otimes q_i) \longmapsto \mu(q_i \otimes p_i s) = \mu(q_i \otimes p_i)s$$

Factors through $R/[R,R] \longrightarrow S/[S,S]$

$$M \otimes_R N.$$

M $\underline{R\text{-bimod}}$.

$$M \otimes_R M$$

$$\boxed{\left(M \otimes_R \right)}$$

$$\|$$

$$HH_0(R,M) = \boxed{M / \left(\begin{array}{l} rm - mr \\ \forall\, m \in M,\ r \in R \end{array} \right)}$$

$M = R$.

$$\left(R \otimes_R \right) = R/[R,R]$$

$$P \otimes_S Q \xrightarrow{\sim} R.$$

$$P \otimes_S Q \otimes_R$$

$$P - \otimes_S - Q$$

$$\left(R \otimes_R \right) = \left(\boxed{P \otimes_S Q} \otimes_R \right) = P \otimes_S Q \otimes_R P.$$

$$\|$$

$$Q \otimes_R P \otimes_S Q$$

$$P \otimes_S Q / \left(r(p \otimes q) - (p \otimes q) r \right)$$

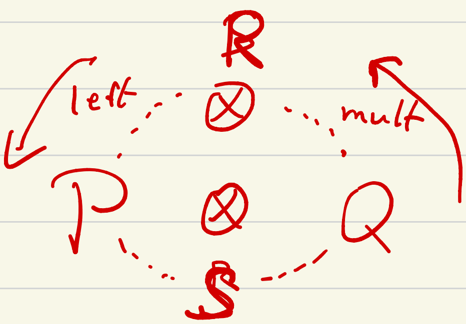
$$\boxed{rp\otimes q - p\otimes qr}$$

$$\left(S \underset{S}{\otimes} \right) = \left(Q \underset{R}{\otimes} P \underset{S}{\otimes} \right)$$

$$\mathrm{Fun}^{\mathrm{cont}}_{\mathrm{co}}(R\text{-mod},\ \underset{S\text{-mod}}{R\text{-mod}}) \xrightarrow{\ \sim\ } \underset{(S,R)\text{-bimod}}{R\text{-bimod}}.$$

(right exact, comm $\oplus$)

$$\text{composition} \longleftrightarrow \underset{R}{\otimes}$$



$$= \boxed{\dfrac{P \underset{\mathbb{Z}}{\otimes} Q}{\left(\begin{array}{c} ps\otimes q - p\otimes sq \\ rp\otimes q - p\otimes qr \end{array} \right)}}$$