Lecture 15

10/26

- Morita equivalence
- Morita context

Def. An R-mod P is a *progenerator* if

- P is f.g. projective ;
- $\exists$ surjection $P^{\oplus n} \twoheadrightarrow R$.
  $\sum x_i \longmapsto 1$

Def. An R-mod M is a *generator* (for the category of R-mod) if satisfies one of the following equiv. conditions

1) The functor

$$\mathrm{Hom}_R(M, -) : R\text{-mod} \longrightarrow \underline{Ab}.$$

$\mathcal{C}$

is faithful, i.e., injective on Hom sets.

$$X \ , Y \in R\text{-mod} \quad \mathcal{C}$$

$$\mathrm{Hom}_R(X, Y) \xrightarrow[\text{injective}]{} \mathrm{Hom}(\mathrm{Hom}_R(M, X), \mathrm{Hom}_R(M, Y))$$

$$\Updownarrow$$

if $f : X \xrightarrow[R\text{-linear}]{} Y$, $f \neq 0$

then $\mathrm{Hom}_R(M, X) \longrightarrow \mathrm{Hom}_R(M, Y)$
$\mathrm{Hom}_R(M, f) \neq 0$

2). Any R-mod can be written as a quotient

(poss. ∞).
of a direct sum of M.

3) $M^{\oplus n} \twoheadrightarrow R$

Ex. R itself is a progenerator.

Ex. D = div. ~~alg~~ ring.
D-mod
progen = f.d. D-v.s.

Ex. R = ss ring. $\{S_i\}$ simple obj.
$\bigoplus S_i$ is a progenerator.
smaller than R.

Ex. R artinian. $P_{S_i} \twoheadrightarrow S_i$.
$\bigoplus P_{S_i}$ is a progen

Thm P : progenerator for R-mod
$S := \mathrm{End}_R(P)^{op}$.
$(R \circlearrowright P \circlearrowleft S)$

Then
$G = \boxed{\mathrm{Hom}_R(P, -)}$: R-mod $\longrightarrow$ S-mod.
right adj
is an equivalence with inverse

R-mod $\longleftarrow$ S-mod : $\boxed{P \otimes_S (-)}$ $= F$
left adj

$(\otimes, \mathrm{Hom})$ - adjunction

$R \circlearrowright M \circlearrowleft S$

$$M \otimes_S (-) : \underset{\text{(left)}}{S\text{-mod}} \rightleftarrows \underset{\text{(left)}}{R\text{-mod}} : \mathrm{Hom}_R(M, -)$$

$X \in S\text{-mod}$, $Y \in R\text{-mod}$

canonical isom of ab. gps

$$\mathrm{Hom}_R(M \otimes_S X, Y) \simeq \mathrm{Hom}_S(X, \mathrm{Hom}_R(M, Y))$$

Pf of Thm.

By adjunction (F, G).

$$\mathrm{id}_{S\text{-mod}} \longrightarrow G \circ F$$

$$F \circ G \longrightarrow \mathrm{id}_{R\text{-mod}}.$$

Need to show both are $\cong$.

Observe: $F = P \otimes_S (-)$.

$\left.\begin{array}{l}\text{is right exact.} \\ \text{and commutes with } \bigoplus_{\uparrow \text{ poss. } \infty}.\end{array}\right\}$ F commutes with (small) colimits.

$$G = \mathrm{Hom}_R(P, -)$$

since P is proj. $\Rightarrow$ G is exact.

since P is f.g. $\Rightarrow$ G commutes with arbitrary $\oplus$.

M = f.g. R-mod $\Rightarrow$

$$\bigoplus_{i \in I} \mathrm{Hom}_R(M, X_i) \xrightarrow{\sim} \mathrm{Hom}\Big(M, \bigoplus_{i \in I} X_i\Big) \hookrightarrow \prod_{i \in I} \mathrm{Hom}(M, X_i)$$

[$\mathrm{Hom}_R(M, -)$ commutes with filtered direct limits with monomorphisms $\Longleftrightarrow$ M is a f.g. R-mod.

$\Rightarrow$ FG, GF are also right exact, comm w/ $\oplus$

<u>Claim:</u> two functors $\Phi_1, \Phi_2 : \underset{\mathcal{C}}{R\text{-mod}} \longrightarrow \underset{\mathcal{D}}{S\text{-mod}}$.

both right ex, comm w/ $\oplus$.

$$\Phi_1 \overset{T}{\Longrightarrow} \Phi_2$$

Then T is an isom iff for some progen $P \in \mathcal{C}$.

$$T(P) : \Phi_1(P) \xrightarrow{\sim} \Phi_2(P)$$

(in $\mathcal{D}$).

Apply claim to $\mathrm{id}_{S\text{-mod}} \overset{T}{\Longrightarrow} G\circ F$

$$T(S): \quad S \longrightarrow G\circ F(S) = G(P \otimes_S S)$$
$$= G(P)$$
$$= \mathrm{Hom}_R(P, P) = S.$$

Check: this is the id map on S. as left S-mod.

Apply claim to $F\circ G \overset{T'}{\Longrightarrow} \mathrm{id}_{R\text{-mod}}$,

$$G(R) = \mathrm{Hom}_R(P, R)$$

Check on P. instead of R.

$$T'(P): \quad F\circ G(P) = F(S) = P \longrightarrow P$$

Check this is id.

$$\begin{tikzcd} FG(P) \arrow[r] \arrow[d, Leftrightarrow] & P \arrow[d, equal] \\ G(P) \arrow[r, equal, "id"] & G(P) \end{tikzcd}$$

Pf of claim: Check $T(X): \Phi_1(X) \longrightarrow \Phi_2(X)$ is an $\cong$, $\forall X \in \mathcal{C}$.

$$0 \to \ker(\pi) \to P^{\oplus I_0} \overset{\pi}{\twoheadrightarrow} X \to 0$$

$$P^{\oplus I_1} \twoheadrightarrow \ker(\pi)$$

$$\Longrightarrow \quad \begin{array}{c} P^{\oplus I_1} \\ \uparrow \end{array} \longrightarrow P^{\oplus I_0} \longrightarrow X \longrightarrow 0$$

apply Φ_i

$$\begin{tikzcd}
\Phi_1(P)^{\oplus I_1} \arrow[r] \arrow[d, "T(P)^{\oplus I_1} = T(P^{\oplus I_1})"'] & \Phi_1(P)^{\oplus I_0} \arrow[r] \arrow[d, "T(P)^{\oplus I_0} = T(P^{\oplus I_0})"'] & \Phi_1(X) \arrow[r] \arrow[d, "T(X)"] & 0 \\
\Phi_2(P)^{\oplus I_1} \arrow[r] & \Phi_2(P)^{\oplus I_0} \arrow[r] & \Phi_2(X) \arrow[r] & 0
\end{tikzcd}$$

red arrows are $\cong$. $\Longrightarrow$ $T(X)$ is $\cong$.

$\blacksquare$

$$P \otimes_S (-) : S\text{-mod} \underset{G}{\overset{F}{\rightleftarrows}} R\text{-mod} : \mathrm{Hom}_R(P, -)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \;\; Q \otimes_R (-)$$

Let $Q = \mathrm{Hom}_R({}^{S}\circlearrowleft P, R \circlearrowleft^{R}) = G(R)$

(dual of P, it is a right R-mod)

$R \circlearrowright P \circlearrowleft S$

$S \circlearrowright Q \circlearrowleft R$,

<u>Claim</u> natural isom of functors : $R\text{-mod} \longrightarrow S\text{-mod}$

$$\mathrm{Hom}_R(P, -) \xleftarrow{\cong} Q \otimes_R (-)$$

$\underline{\text{Pf}}$ $\quad X \in R\text{-mod}$.

$$Q \underset{R}{\otimes} X = \mathrm{Hom}_R(P, R) \underset{R}{\otimes} X \longrightarrow \mathrm{Hom}_R(P, X)$$

$$\underline{\alpha \otimes x} \longmapsto \left(p \mapsto \underset{\substack{\cap \\ R}}{\underline{\alpha(p)}}\, x\right)$$

with $\alpha \in \mathrm{Hom}_R(P,R)$, $x \in X$, $p \in P$.

To show this is $\cong$

use $\quad R^{\oplus n} = P \oplus P'$

(P is f.g. projective).

reduce to prove same for $R^{\oplus n}$ instead of P. ∎

Properties of Q: $\quad S \circlearrowright Q \circlearrowleft R$

$$\mathcal{D} = S\text{-mod} \overset{\sim}{\longleftrightarrow} R\text{-mod} = \mathcal{C}$$

$$S \longleftrightarrow P$$

$$Q \longleftrightarrow R$$

(*) progen. for $\mathcal{D} \Longleftarrow$ progen. for $\mathcal{C}$.

$\Rightarrow$ Q is a f.g. proj S-mod.

In fact, Q is a progenerator as a right R-mod.

P —— " —— right S-mod.

(*) f.g. is invariant under Morita equiv.

M is f.g. $\iff \mathrm{Hom}_{\mathcal{C}}(M,-)$ commutes with filtered direct limit.

finitely presented

$\mathfrak{m} = k[x_1, \dots, x_n, \dots]/\mathfrak{m}$

$$\longrightarrow \varinjlim k[x_1, \dots]/(x_1, \dots, x_n)$$

$M \in \mathcal{C}$ is f.g. if for any collection of subobjects $\{M_\alpha\}_{\alpha \in I}$

s.t. $\sum M_\alpha = M$.

$(\bigoplus M_\alpha \twoheadrightarrow M)$.

then $\exists$ finite subset $I' \subset I$

s.t. $\sum_{\alpha \in I'} M_\alpha = M$.

Claim: Q is a progen as right R-mod.

Pf. $P \oplus P' = R^{\oplus n}$

$\mathrm{Hom}_R(-, R)$

$Q \oplus Q' = R^{\oplus n}$ as right R-mod

$\Rightarrow$ Q is f.g. proj right R-mod.

$$R \oplus M = P^{\oplus n}$$

$\mathrm{Hom}_R(-, R)$

get $\quad R \oplus \mathrm{Hom}(M, R) = Q^{\oplus n}$ as right R-mod.

$\Rightarrow$ Q is a generator.

$$\begin{array}{ccc} P \otimes_S (-) & & \mathrm{Hom}_R(P, -) \\ \| & S\text{-mod} \longleftrightarrow R\text{-mod} & \wr\| \\ \mathrm{Hom}_S(Q, -) & & Q \otimes_R (-) \end{array}$$

$$Q = \mathrm{Hom}_R(P, R)$$

$$P = \mathrm{Hom}_S(Q, S)$$

<u>Morita context</u>:

$R \circlearrowright P \circlearrowleft S$

$S \circlearrowright Q \circlearrowleft R$

<u>Data</u>:

$$\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$$

$\lambda : P \otimes_S Q \longrightarrow R \quad$ (as R-bimod)

$\mu : Q \otimes_R P \longrightarrow S \quad$ (as S-bimod)

Conditions: (associativity)

$$\begin{tikzcd}
 & R \otimes_R P \arrow[dr, equal] \arrow[d, phantom, "\circlearrowright"] & \\
P \otimes_S Q \otimes_R P \arrow[ur, "\lambda \otimes id_P"] \arrow[dr, "id_P \otimes \mu"'] & {} & P \\
 & P \otimes_S S \arrow[ur, equal] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & Q \otimes_R R \arrow[dr, equal] \arrow[d, phantom, "\circlearrowright"] & \\
Q \otimes_R P \otimes_S Q \arrow[ur, "id_Q \otimes \lambda"] \arrow[dr, "\mu \otimes id_Q"'] & {} & Q \\
 & S \otimes_S Q \arrow[ur, equal] &
\end{tikzcd}$$

$$\mathcal{C} = R\text{-mod}, \quad \mathcal{D} = S\text{-mod}.$$

Morita context $(R, S, P, Q, \lambda, \mu)$.

$\Longleftrightarrow$ $\qquad \mathcal{C} \xrightarrow{F} \mathcal{D}$

$\qquad \mathcal{D} \xrightarrow{G} \mathcal{C}$ $\qquad$ both are right ex comm w/ $\oplus$.

and $\qquad G \circ F \Longrightarrow id_{\mathcal{C}} \qquad (\lambda)$

$\qquad F \circ G \Longrightarrow id_{\mathcal{D}} \qquad (\mu).$

and

$$\begin{tikzcd} G \circ F \circ G \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "="] & G \end{tikzcd}$$

$$\begin{tikzcd} F \circ G \circ F \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "="] & F \end{tikzcd}.$$