

# Lecture 15

10/26

- Morita equivalence
- Morita context

Def. An  $R$ -mod  $P$  is a progenerator if

- $P$  is f.g. projective ;
- $\exists$  surjection  $P^{\oplus n} \twoheadrightarrow R$   
 $\sum x_i \mapsto 1$

Def. An  $R$ -mod  $M$  is a generator  
(for the category of  $R$ -mod)  
if satisfies one of the following equiv. conditions

1) The functor

$\text{Hom}_R(M, -) : \mathcal{C} \rightarrow \text{Ab}$   
is faithful, i.e., injective on Hom sets.

$X, Y \in \mathcal{C}$

$\text{Hom}_R(X, Y) \xrightarrow{\text{injective}} \text{Hom}_R(\text{Hom}_R(M, X), \text{Hom}_R(M, Y))$

$\Updownarrow$

if  $f: X \rightarrow Y$ ,  $f \neq 0$   
 $R$ -linear

then  $\text{Hom}_R(M, X) \rightarrow \text{Hom}_R(M, Y)$   
 $\text{Hom}_R(M, f) \neq 0$

2). Any  $R$ -mod can be written as a quotient

(poss.  $\infty$ ),  
of a direct sum of  $M$ .

3)  $M^{\oplus n} \twoheadrightarrow R$

Ex.  $R$  itself is a progenerator.

Ex.  $D = \text{div. ring}$ .

$D\text{-mod}$

progen = f.d.  $D\text{-v.s.}$

Ex.  $R = \text{ss ring}$ .  $\{S_i\}$  simple obj.

$\bigoplus S_i$  is a progenerator.

smaller than  $R$ .

Ex.  $R$  artinian.  $P_{S_i} \twoheadrightarrow S_i$ .

$\bigoplus P_{S_i}$  is a progen

Thm  $P$ : progenerator for  $R\text{-mod}$

$S := \text{End}_R(P)^{\text{op}}$ .

$(R \subset P \subset S)$

Then

$G = \boxed{\text{Hom}_R(P, -)}$ :  $R\text{-mod} \longrightarrow S\text{-mod}$ ,  
right adj

is an equivalence with inverse

$R\text{-mod} \longleftarrow S\text{-mod} : \boxed{\begin{matrix} P \otimes (-) \\ S \end{matrix}}$   
left adj

$(\otimes, \text{Hom})$  - adjunction

$$R \supseteq M \supseteq S$$

$$M \otimes_S (-) : S\text{-mod} \xrightleftharpoons{\text{(left)}} R\text{-mod} : \text{Hom}_R(M, -)$$

$X \in S\text{-mod}, Y \in R\text{-mod}$   
canonical isom of ab. gps

$$\text{Hom}_R(M \otimes_S X, Y) \cong \text{Hom}_S(X, \text{Hom}_R(M, Y))$$

Pf of Thm

By adjunction  $(F, G)$ .

$$\text{id}_{S\text{-mod}} \longrightarrow G \circ F$$

$$F \circ G \longrightarrow \text{id}_{R\text{-mod}}$$

Need to show both are  $\cong$ .

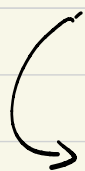
Observe:  $F = P \otimes_S (-)$ .

is right exact. }  $F$  commutes  
and commutes with  $\bigoplus$  } with (small)  
 $\uparrow$  } colimits.  
poss.  $\infty$ .

$$G = \text{Hom}_R(P, -)$$

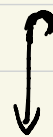
Since  $P$  is proj.  $\Rightarrow G$  is exact.

since  $P$  is f.g.  $\Rightarrow G$  commutes with arbitrary  $\bigoplus$ .



$M = \text{f.g. } R\text{-mod} \Rightarrow \bigoplus$

$$\bigoplus_{i \in I} \text{Hom}_R(M, X_i) \xrightarrow{\sim} \text{Hom}(M, \bigoplus_{i \in I} X_i)$$



$$\prod_{i \in I} \text{Hom}(M, X_i)$$

$\left[ \begin{array}{l} \text{Hom}_R(M, -) \text{ commutes with filtered direct limits} \\ \text{with monomorphisms} \\ \iff M \text{ is a f.g. } R\text{-mod.} \end{array} \right.$

$\Rightarrow FG, GF$  are also right exact, comm w/  $\bigoplus$

Claim: two functors  $\mathcal{C} \quad \mathcal{D}$   
 $\Phi_1, \Phi_2 : R\text{-mod} \longrightarrow S\text{-mod.}$

both right ex, comm w/  $\bigoplus$ .

$$\Phi_1 \xrightarrow{T} \Phi_2$$

Then  $T$  is an isom iff for some projen  $P \in \mathcal{C}$ .

$$T(P) : \Phi_1(P) \xrightarrow{\sim} \Phi_2(P)$$

(in  $\mathcal{D}$ ).

Apply claim to  $\text{id}_{S\text{-mod}} \xrightarrow{T} G \circ F$

$$\begin{aligned} T(S) : S &\longrightarrow G \circ F(S) = G(P \otimes_S S) \\ &= G(P) \\ &= \text{Hom}(P, P) = S. \end{aligned}$$

Check: this is the id map on  $S$ .  
as left  $S$ -mod.

Apply claim to  $F \circ G \xrightarrow{T'} \text{id}_{R\text{-mod}}$ ,

$$G(R) = \text{Hom}_R(P, R)$$

Check on  $P$ . instead of  $R$ .

$$T'(P) : F \circ G(P) = F(S) = P \longrightarrow P$$

check this is id.

$$\begin{array}{ccc} \parallel & FG(P) & \longrightarrow P \\ & \updownarrow & \\ & G(P) & \xlongequal{\text{id}} G(P) \end{array}$$

Pf of claim: Check  $T(X) : \mathbb{F}_1(X) \longrightarrow \mathbb{F}_2(X)$   
 is an  $\cong$ ,  $\forall X \in \mathcal{C}$ .

$$\begin{array}{ccccccc} 0 \rightarrow & \ker(\pi) & \rightarrow & P^{\oplus I_0} & \xrightarrow{\pi} & X & \rightarrow 0 \\ & & & P^{\oplus I_1} & \rightarrow & \ker(\pi) & \end{array}$$

$$\Rightarrow P^{\oplus I_1} \longrightarrow P^{\oplus I_0} \longrightarrow X \longrightarrow 0$$

↑

apply  $\Phi_i$

$$\begin{array}{ccccc} \Phi_1(P)^{\oplus I_1} & \longrightarrow & \Phi_1(P)^{\oplus I_0} & \longrightarrow & \Phi_1(X) \longrightarrow 0 \\ \downarrow T(P)^{\oplus I_1} & \cong & \downarrow T(P)^{\oplus I_0} & & \downarrow T(X) \\ \Phi_2(P)^{\oplus I_1} & \longrightarrow & \Phi_2(P)^{\oplus I_0} & \longrightarrow & \Phi_2(X) \longrightarrow 0 \end{array}$$

red arrows are  $\cong$ .  $\Rightarrow T(X)$  is  $\cong$ .

$$P \otimes_S (-) : S\text{-mod} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} R\text{-mod} : \text{Hom}_R(P, -)$$

$Q \otimes_R (-)$   
 $\parallel$

Let  $Q = \text{Hom}_R(P, R) = G(R)$

(dual of  $P$ , it is a right  $R$ -mod)

$$R \supseteq P \supseteq S$$

$$S \supseteq Q \supseteq R$$

Claim natural isom of functors:  $R\text{-mod} \rightarrow S\text{-mod}$   
 $\text{Hom}_R(P, -) \xleftarrow{\cong} Q \otimes_R (-)$

Pf  $X \in R\text{-mod}$ .

$$Q \otimes_R X = \text{Hom}_R(P, R) \otimes_R X \longrightarrow \text{Hom}_R(P, X)$$

$$\downarrow \quad \downarrow$$

$$\alpha \otimes x \longmapsto \left( p \mapsto \alpha(p)x \right)$$

$\underbrace{\hspace{10em}}_{\substack{\cap \\ R}}$

To show this is  $\cong$

use  $R^{\oplus n} = P \oplus P'$

( $P$  is f.g. projective).

reduce to prove same for  $R^{\oplus n}$  instead of  $P$ .

Properties of  $Q$ :  $S \subseteq Q \subseteq R$

$$\mathcal{D} = S\text{-mod} \xrightarrow{\sim} R\text{-mod} = \mathcal{C}$$

$$S \longleftrightarrow P$$

$$Q \longleftrightarrow R$$

$$(*) \quad \begin{array}{ccc} \uparrow & & \uparrow \\ \text{progen. for } \mathcal{D} & \Leftarrow & \text{progen. for } \mathcal{C} \end{array}$$

$\Rightarrow Q$  is a f.g. proj  $S$ -mod.

In fact,  $Q$  is a progenerator as a right  $R$ -mod.  
 $P$  ——— " ——— right  $S$ -mod.

(\*)  $(f.g.)$  is invariant under Morita equiv.

$M$  is  $(f.g.)$   $\iff \text{Hom}_{\mathcal{C}}(M, -)$  commutes with filtered direct limit.  
*finitely presented*

$k = k[x_1, \dots, x_n, \dots] / \mathfrak{m}$   
 $\longrightarrow \varinjlim k[x_1, \dots] / (x_1, \dots, x_n)$

$M \in \mathcal{C}$  is f.g. if for any collection of subobjects  $\{M_\alpha\}_{\alpha \in I}$   
 st.  $\sum M_\alpha = M$ .  
 $(\bigoplus M_\alpha \twoheadrightarrow M)$ .  
 then  $\exists$  finite subset  $I' \subset I$   
 st.  $\sum_{\alpha \in I'} M_\alpha = M$ .

Claim:  $Q$  is a progen as right  $R$ -mod.

Pf.

$$P \oplus P' = R^{\oplus n}$$

$$\text{Hom}_R(-, R)$$

$$Q \oplus Q' = R^{\oplus n} \text{ as right } R\text{-mod}$$



$\Rightarrow$   $Q$  is f.g. proj right  $R$ -mod.

$$R \oplus M = P^{\oplus n}$$

$$\text{Hom}_R(-, R)$$

get  $R \oplus \text{Hom}(M, R) = Q^{\oplus n}$   
 as right  $R$ -mod.

$\Rightarrow$   $Q$  is a generator.

$$\begin{array}{ccc} P \otimes_S (-) : S\text{-mod} & \longleftrightarrow & R\text{-mod} : \text{Hom}_R(P, -) \\ \parallel_S & & \parallel_S \\ \text{Hom}_S(Q, -) & & Q \otimes_R (-) \\ & & Q = \text{Hom}_R(P, R) \\ & & P = \text{Hom}_S(Q, S) \end{array}$$

Morita context:

Data:

$$\begin{pmatrix} R & P \\ Q & S \end{pmatrix}$$

$$R \supseteq P S S$$

$$S \supseteq Q S R$$

$$\lambda: P \otimes_S Q \longrightarrow R \quad (\text{as } R\text{-bimod})$$

$$\mu: Q \otimes_R P \longrightarrow S \quad (\text{as } S\text{-bimod})$$

