Lecture 10 — 10/7

- Krull-Schmidt Theorem
- Projective modules

Thm (K-S) $R =$ (left) artinian, $M =$ f.g. R-mod.

$$M = \bigoplus_{\text{finite}} (\text{indecomp } R\text{-mod})$$

and isom classes of indecomp summands (including multiplicity) are invariants of M.

Pf • Existence of decomp.

M has finite length.

$$M = M_1 \oplus M_2$$

\# of summands $\leq$ length(M).

- Uniqueness.

Recall. $M =$ ss module.

S : any simple R-mod.

$\boxed{\mathrm{Hom}_R(S, M)}$.

is D-vector space

$D = \mathrm{End}_R(S)$.

$N =$ indecomp. R-mod

? "multiplicity space" of N in M.

$\underbrace{\mathrm{Hom}_R(N, M)}$ $\circlearrowleft$ $\underbrace{\mathrm{End}_R(N)}$

$\Uparrow$ Fitting's lemma

This is a local ring

with $J = J(\mathrm{End}_R(N))$

and residue ring

$\mathrm{End}_R(N)/J = D$. div. ring

$$\overline{\mathrm{Hom}}_R(N, M) := \mathrm{Hom}_R(N, M) / \mathrm{Hom}_R(N, M) \cdot J$$

$= D$-vector sp.

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_a$$

M_i indecomp.

$$\overline{\mathrm{Hom}}_R(N, M) = \bigoplus_i \overline{\mathrm{Hom}}_R(N, M_i)$$

If $M_i \cong N$. $\quad \overline{\mathrm{Hom}}_R(N, M_i) \cong \overline{\mathrm{End}}_R(N)$

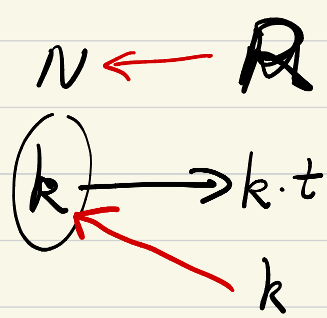
$$= D.$$

If $M_i \not\cong N$. $\quad \overline{\mathrm{Hom}}_R(N, M_i) = 0 \ ?$

e.g. $\quad R = k[t]/t^2. \quad (\sim \mathbb{Z}/p^2).$

$N = k, \quad \mathrm{End}_R(N) = k.$

$M_i = R.$



$\mathrm{Hom}_R(N, M_i) = k$

$$\mathrm{Hom}_R(\underset{\substack{\| \\ R}}{M_i}, N) = k$$

$$\begin{array}{ccccc} \mathrm{Hom}_R(N, M_i) & \times & \mathrm{Hom}_R(M_i, N) & \longrightarrow & \mathrm{Hom}_R(N,N) \\ \| & & \| & & \| \\ k & \times & k & \xrightarrow{\ 0\ } & k \end{array}$$

Look at

$$D \circlearrowright \overline{\mathrm{Hom}}_R(M, N) \times \overset{\circlearrowleft D}{\overline{\mathrm{Hom}}_R(N, M)} \longrightarrow \underset{\substack{\| \\ D}}{\overline{\mathrm{End}}_R(N)}$$

$$(g, f) \longmapsto g \circ f.$$

This is D-bilinear.

$$(dg, f) = d(g, f) \qquad d \in D.$$

$$(g, fd) = (g, f)d.$$

rank of this pairing:

$$D \circlearrowleft V \times W \circlearrowright D \xrightarrow[D\text{-bilinear}]{} D.$$

$$D \circlearrowleft V \xrightarrow[\substack{\text{left} \\ D\text{-linear}}]{} \boxed{\mathrm{Hom}_{D_r}(W, D)} = W^*$$

right D-linear maps $W \to D$.

left D-mod.

rank of pairing = rank of $V \to W^*$.

<u>Claim</u> M_i indecomp. $M_i \not\cong N$.

$$\overline{\mathrm{Hom}}_R(M_i, N) \times \overline{\mathrm{Hom}}_R(N, M_i) \longrightarrow D$$

is identically zero.

<u>Pf</u>. $g : M_i \to N$.

$f : N \to M_i$

$$g \circ f: \begin{tikzcd} N \arrow[r] \arrow[rr, bend right] & M_i \arrow[r] & N \end{tikzcd}$$

if $\bar{g} \circ \bar{f} \neq 0 \in D$.

$g \circ f \in \mathrm{End}_R(N)$ is local.

$g \circ f$ is invertible.

$$\Rightarrow \quad \underbrace{M_i}_{\text{indecomp}} = N \oplus (\;-\;)$$

$$\Rightarrow \quad M_i \cong N. \quad \times$$

∎

Claim $\Rightarrow$ rank of

$$\overline{\mathrm{Hom}_R(M, N)} \times \overline{\mathrm{Hom}_R(N, M)} \longrightarrow D$$

is $=$ (# of indecomp summands in M that are $\cong N$).

∎

Uniqueness part uses $\mathrm{End}_R(\text{indecomp mod}) = \text{local}$.

Without artinian assumption:

A = Dedekind domains $\supsetneq$ PID.

e.g. $\mathcal{O}_K \subset K$. $\quad K$ = number field $[K:\mathbb{Q}] < \infty$

$A = \mathbb{Z}[\sqrt{-5}]$ not PID

$I = (2, 1+\sqrt{-5})$ not principal.

$I^2 = (2)$.

$\det = I \otimes I = I^2 \cong A$

Then: $I \oplus I \cong A \oplus A$

and $\underline{\underline{A, I}}$ are both indecomp.

Fact: $R \circlearrowright R$ is indecomp
$\iff \nexists$ nontrivial idempotents $x \in R$
$(x^2 = x$ but $x \neq 0, 1)$.
$x(1-x) = 0$.

$\implies$ If R has no zero divisor, then $R \circlearrowright R$ is indecomp.

Classification of torsion-free, f.g. A-mods
($\iff$ projective)

$\updownarrow$ (rank, det)

$\mathbb{Z}_{\geq 0} \times \mathrm{Pic}(A)$
$\quad \quad \quad \quad \quad \| $
$\quad \quad \quad \quad \quad Cl(A)$

M rk n

$\downarrow$

$\Lambda^n_A M$. rk 1 proj A-mod (invertible).

Projective modules

Defn/Prop. $R \circlearrowright P$ TFAE:

① $\operatorname{Hom}_R(P, -)$ is exact

② $\operatorname{Hom}_R(P, -)$ is right exact.

③
$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed, "\exists g"'] \\ M \arrow[r, two heads] & M'' \end{tikzcd}$$

③ ⇒ ④
Take $M \twoheadrightarrow M''$ to be $M \twoheadrightarrow P$, $f = id_P$.

$$\begin{tikzcd} & P \arrow[d, equal] \arrow[dl, dashed, "s"'] \\ M \arrow[r, two heads] & P \end{tikzcd}$$

④ Any surjection $M \twoheadrightarrow P$ has a section s.

$$\begin{tikzcd} M \arrow[r, two heads] & P \arrow[l, dashed, bend left, "s"] \end{tikzcd}$$

⑤ P is a direct summand of a free R-mod

④ ⇒ ⑤
$$\begin{tikzcd} R^I \arrow[r, two heads] & P \arrow[l, dashed, bend left, "s"] \end{tikzcd}$$

Exact means

$$0 \to M' \to M \to M'' \to 0$$

$$0 \to \operatorname{Hom}_R(P, M') \to \operatorname{Hom}_R(P, M) \to \operatorname{Hom}_R(P, M'') \to 0$$

(ex. seq of abelian gps)

or $\operatorname{End}_R(P)$-mod

$\mathrm{Hom}_R(\underset{\text{any } R\text{-mod}}{Q}, -)$ is always left exact.

⑤ $\Rightarrow$ ① $\mathrm{Hom}_R(R^{\oplus I}, M) = \prod_I M$.

$e_i \mapsto m_i$ exact in M.

any direct summand of an exact functor is still exact.

$\mathrm{Hom}_R(P, -) \oplus \mathrm{Hom}_R(R^{\oplus I}, -)$.

$\blacksquare$

Ex. • $M_n(k) \circlearrowright k^n$ = projective.

$$M_n(k) = k^n \oplus \cdots \oplus k^n.$$

• For a ss ring R, any module is projective

($\because$ any module $= \bigoplus \underline{S_i}$, $S_i \oplus R$.)

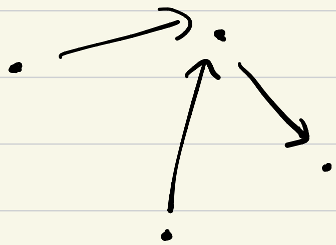
Fact. $P = \bigoplus_{i \in I} P_i$, then P is projective

$\Updownarrow$

each P_i is projective.

$$\mathrm{Hom}_R(P, -) = \prod_{i \in I} \mathrm{Hom}_R(P_i, -)$$

- A = Dedekind domain M = f.g. A-mod
then M is projective $\Longleftrightarrow$ M is torsion-free.

- 

$$\bigoplus k\cdot 1_v$$
$$\|$$
$$Q \rightsquigarrow R_Q = R_0 \oplus R_1 \oplus \cdots$$

v: vertex. idempotent.

as a module
$$\boxed{P_v = R_Q \cdot 1_v}$$

$x^2 = x$ projective

$R \circlearrowright R = \underline{Rx} \oplus \underline{R(1-x)}$.

$ax = b(1-x)$

$a\cdot x^2 = b(1-x)\cdot x$

$\quad \| \qquad\qquad \|$

$\quad ax \qquad\qquad 0$.

$$R_Q = \bigoplus k\cdot p.$$

p: paths in Q (directed).

$$P_v = \bigoplus k\cdot p \quad \text{projective.}$$

p starting from v.

$R_Q \circlearrowright \boxed{V = \bigoplus V_v}$ a repn of the quiver Q as a funct

$$\begin{tikzcd}
 & V_2 \arrow[dr] & \\
V_1 \arrow[ur] \arrow[loop left, "\varepsilon"] & & V_3 \\
 & V_4 \arrow[uu] &
\end{tikzcd}$$

$$\boxed{\mathrm{Hom}_{R_Q}(P_v, V) \Rightarrow V_v.}$$

$$(P_v \xrightarrow{f} V) \mapsto f(1_v).$$

$$\left.\begin{array}{l} a \in V_v. \\ P_v \longrightarrow V. \\ p \longmapsto p(a). \end{array}\right] \text{ gives } V_v \to \operatorname{Hom}_{R_Q}(P_v, V)$$

- For Any projective R-mod P.
  can find two free modules F_1, F_2.
  s.t. $\underbrace{P \oplus F_1 \cong F_2}$.

Pf. $R^{\oplus I} = P \oplus Q$.

$$F_2 = R^{\oplus I} \oplus R^{\oplus I} \oplus R^{\oplus I} \oplus \cdots$$
$$\| $$
$$P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus (Q \oplus \cdots) \cdots$$
$$\|$$
$$P \oplus \underbrace{R^{\oplus I} \oplus R^{\oplus I} \oplus R^{\oplus I} \oplus \cdots}_{F_1}$$

- A = commutative.

$$\left\{\begin{array}{c}\text{finitely gen'd} \\ \text{projective } A\text{-mods}\end{array}\right\} = \left\{\begin{array}{c}\text{vector bundles} \\ \text{on } X = \operatorname{Spec} A\end{array}\right\}$$

(locally free A-mod).

Compact nice Hausdorff

X = usual top space,

$M = \mathrm{Sect}(E)$. $\quad \longleftrightarrow \quad$ E locally $U_i \times \mathbb{R}^n$.

is a projective A-mod.

(of rank n.)

$$\begin{array}{c} E \\ \downarrow \text{v.b.} \\ X = \bigcup U_i \end{array}$$

$$A = C(X, \mathbb{R}).$$