Lecture 1 (Sep 2)

Rings & Examples.

(have 1, associative).

1. Matrix rings.
2. Group rings.
3. $\mathrm{End}_{\mathcal{C}}(X)$.
4. Quaternion algebra.
   $$\mathbb{H} = \mathbb{C} \oplus \mathbb{C} j .$$
5. Tensor algebra
6. $U(\mathfrak{g})$. $\mathfrak{g}$ = Lie alg.

Examples suggested by students

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1. Matrix rings.

$M_n(k)$ k = field.

$M_n(R)$ R = ring.

$\boxed{M_\infty(R)}$?

$$(\cdots a_i \cdots) \begin{pmatrix} \vdots \\ b_i \\ \vdots \end{pmatrix}$$

$\sum a_i b_i$ defined only when almost all $a_i b_i$ are zero.

- column-finite $\infty \times \infty$ matrices. (ring)

  $\because$ each col. is almost all zero.

$$1 = \begin{pmatrix} \ddots & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}$$

- row-finite. (ring).

$$M_n(R^{op}) \cong \mathrm{End}_R(R^n).$$

$$\boxed{\begin{aligned} & R^n = Re_1 \oplus \cdots \oplus Re_n \\ & \varphi : R^n \longrightarrow R^n \\ & \quad e_i \longmapsto \sum a_{ij} e_j \end{aligned}}$$

R^{op} same $(R, +)$.

$$a \bullet b = ba$$

mult. in R^{op}

$n = 1$. $\quad R \curvearrowright R \circlearrowleft R.$

$$\mathrm{End}_R({}_R R) \cong R^{op}$$

$$(a \mapsto ar) \longleftarrow r$$

I set. $|I| = \infty$.

$R^{\oplus I}$ free R-mod. basis $\longleftrightarrow I$.
$\{e_i\}_{i \in I}$

$$\mathrm{End}_R(R^{\oplus I})$$

$$\varphi : e_i \longmapsto \sum_{j \in I} a_{ij} e_j$$

For fixed i
only fin. many j
s.t. $a_{ij} \neq 0$.

row/column-finite $I \times I$ matrices.

Subrings of $M_n(R)$

$$\begin{pmatrix} * & * & \cdots & * \\ & * & & \vdots \\ & & \ddots & \vdots \\ 0 & & & * \end{pmatrix} \qquad \xcancel{\begin{pmatrix} 0 & * & \cdots & * \\ & \ddots & & \vdots \\ & & \ddots & * \\ & & & 0 \end{pmatrix}}$$

no 1

$$\left\{ \begin{pmatrix} a & & & * \\ & a & & \\ & & \ddots & \\ 0 & & & a \end{pmatrix} \;\middle|\; a \in R \right\}$$

Take subrings $R_1, R_2 \subset R$.

$$\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; \begin{matrix} a \in R_1 \\ d \in R_2 \end{matrix}, \; b \in R \right\}$$

subring of $M_2(R)$.

$Y \subset R$. <u>subset</u>

$$\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; \begin{matrix} a \in R_1 \\ d \in R_2 \end{matrix}, \; b \in Y \right\}$$

when is this a ring?

$$\begin{cases} R_1 \curvearrowright Y \curvearrowleft R_2 \\ Y \text{ additive subgp of } R. \end{cases}$$

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2. Gp rings.

A = comm ring

G group.

$A[G]$ free A-mod basis $\longleftrightarrow g \in G$.

$$\left(\sum a_g \cdot g\right)\left(\sum b_g \cdot g\right)$$

$$= \sum_{g \in G} \left(\sum_{h \in G} a_h \cdot b_{h^{-1}g} \right) g$$

finite sum

$A[G]$ comm $\iff$ G comm.

$$\sum_{\substack{(h_1, h_2) \in G \times G \\ h_1 h_2 = g}} a_{h_1} \cdot b_{h_2}$$

$A[G]$ is defined for any semigroup (no 1, no inverses).

$A \rightsquigarrow R$.

$R[G]$.

3. Tensor Algebra / Free Algebra.

comm. $A[x_1, x_2, \dots, x_n]$.

A = comm.

3.1 Free asso alg: $A\langle x_1, x_2, \dots, x_n \rangle$.

- free A-mod. with basis (monomial)

  words in letters $\{x_1, x_2, \dots, x_n\}$.

  $x_1 \, x_n \, x_1$, $x_1^3 \cdot x_n^2 \cdot x_1$

  $1 \longleftrightarrow$ empty word.

- A commutes with everything.
  A is central.

I = set.

$A\langle I \rangle$. free variables $\{x_i\}_{i \in I}$.
words are finite.

Universal property of $A\langle I \rangle$.

R : ~~r~~ring.

$$\left\{ A\langle I\rangle \xrightarrow[\text{ring}]{\varphi} R \right\}$$

$$\downarrow$$

$$\left\{ A \xrightarrow{\varphi_0} R;\ \{ r_i = \varphi(x_i) \}_{i \in I} \right\}$$

- $\{ \varphi_0, (r_i)_{i\in I} \}$ determines φ?
- any conditions $\{ \varphi_0, (r_i) \}$ should satisfy?

$$\boxed{\varphi_0(A) \text{ should comm with all } r_i.}$$

$$\operatorname{Hom}_{\text{ring}}(A\langle I\rangle, R) = \left\{ \begin{array}{l} (\varphi_0 : A \to R; \\ \quad \{r_i\}_{i\in I},\ r_i \in R) \\ \text{s.t. } \varphi_0(A) \text{ comm with all } r_i \end{array} \right\}$$

Work with $\underline{\underline{\text{central } A\text{-algebras}}}$.

i.e. $\left(R, \ \varphi_0 : A \xrightarrow{\text{ring}} R \atop \varphi_0(A) \text{ central} \right)$

R : central A-alg.

$$\boxed{\operatorname{Hom}_{A\text{-alg}}(A\langle I\rangle, R) = R^{I}}$$ ← direct product.

$$\varphi \longmapsto \{ r_i = \varphi(x_i) \}_{i \in I}.$$

Take $R = A\langle I\rangle$

$\operatorname{Aut}(I) \circlearrowright A\langle I\rangle$

$$\operatorname{End}_A(A\langle I\rangle) = (A\langle I\rangle)^{I}$$

$$\left(\begin{array}{ccc} A\langle I\rangle & \longrightarrow & A\langle I\rangle \\ f(\underline{x}) & \longmapsto & f(F_1(\underline{x}), F_2(\underline{x}), \ldots) \end{array} \right) \longleftarrow (F_i(\underline{x}))_{i \in I}$$

Linear substitutions ($\mathbf{F}_i$ are linear in $\underline{x}$).

$$\underbrace{\operatorname{End}_A(A^{\oplus I})}_{\text{linear sub.}} \circlearrowright A\langle I\rangle.$$

3.2 Tensor algebra.

A = comm. $\circlearrowright M$

want a central A-alg R.

with $M \subset R$.

$A \subset R$

$M \subset R$.

$M \otimes_A M = \{\sum m_1 m_2 \mid m_i \in M\} \subset R$

$\{m_1 m_2 \cdots m_n \mid m_i \in M\} \subset R$.

$\quad\quad\quad\quad \mathbin{/\!/}$

$\underbrace{M \otimes_A \cdots \otimes_A M}_{n}$

$$T_A(M) = A \oplus M \oplus (\overset{M^{\otimes 2}}{\overset{\mathbin{/\!/}}{M \otimes_A M}}) \oplus \cdots \oplus (\underbrace{M \otimes_A \cdots \otimes_A M}_{n}) \oplus \cdots$$

$= M^{\otimes n}$

mult. $\quad \underline{\underline{M^{\otimes i}}} \times \underline{\underline{M^{\otimes j}}} \longrightarrow \underline{\underline{M^{\otimes i+j}}}$

concatenation.

Univ. property of $\mathbb{T}_A(M)$?

R = central A-alg.

$$\mathrm{Hom}_{A\text{-alg}}(\mathbb{T}_A(M), R) = \mathrm{Hom}_{A\text{-mod}}(M, R)$$

$$\varphi \longmapsto \left(\varphi_1 = \varphi|_M : M \to R \text{ A-linear.}\right)$$

$$\boxed{\begin{array}{c}\varphi(m_1 \otimes \cdots \otimes m_n) \\ \parallel \\ \varphi_1(m_1)\,\varphi_1(m_2)\cdots\varphi_1(m_n)\end{array}} \longleftarrow (\varphi_1 : M \to R)$$

Take $M = A^{\oplus I}$, basis e_i

$$T_A(M) \cong A\langle I \rangle.$$

dep. choice of basis $\{e_i\}_{i\in I}$ of M.

$$e_i \longmapsto x_i$$

$$\boxed{\mathrm{End}_A(M)} \circlearrowright T_A(\underline{M}) \xrightarrow{\sim} A\langle I\rangle.$$

acting by linear sub.

$\boxed{\underline{k\langle x, y\rangle}}$

4. Quotients of tensor algebra.

R/I two-sided ideal

$$R = \boxed{k\langle x, y\rangle / (xy)}$$

dividing by $I = \left\{ \sum_i f_i(x,y)\, xy\, g_i(x,y) \right\}$

$xy, \; (\cdots)\, xy\, (\cdots, \cdots)$

$$= k\cdot 1 \oplus \underset{n\geq 1}{k\cdot x^n} \oplus \underset{n\geq 1}{k\cdot y^n} \oplus k\cdot \underline{y^a \cdot x^b}$$

$xy = 0$ in R.

$yx \neq 0$ in R.

4.1 Symmetric alg.

$A \circlearrowleft M$. $\qquad S_A(M) := T_A(M) \Big/ \left(m_1 \otimes m_2 - m_2 \otimes m_1 \;\; \forall\, m_1, m_2 \in M \right)$

$M = A^{\oplus I} \rightsquigarrow S_A(M) = $ polynomial ring $A[x_i ; i \in I]$

4.2 Exterior alg. (alternating).

$A \circlearrowright M$. $\Lambda_A(M) = T_A(M) \Big/ \left(\begin{matrix} m \otimes m \\ m \in M \end{matrix} \right)$

$m \otimes m = 0 \in \Lambda(M)$.

$(m_1 + m_2) \otimes (m_1 + m_2) = 0$

$\Downarrow$

$m_1 \otimes m_2 + m_2 \otimes m_1 = 0$

i.e., m_1, m_2 anti-commute.

Only imposing $m_1 \otimes m_2 + m_2 \otimes m_1 = 0$

$\not\Longrightarrow m \otimes m = 0$. (in char 2).

Triangular rings

R, S rings

$R \circlearrowright M \circlearrowleft S$.

$\begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| \begin{matrix} a \in R, d \in S \\ b \in M \end{matrix} \right\}$ is a ring.