18.706 HOMEWORK 9

DUE NOV. 9, 2020

Theorems

In this part you will see theorems that we studied in the class. Please supply a detailed proof of each.

Convention: a central simple algebra over k is assumed to be **finite-dimensional** as a k-vector space.

Theorem 1. *Let R be a central simple algebra over a field k.*

(1) $[R : k] := \dim_k R$ *is a square.*
(2) *Let $L \subset R$ be a subfield that is a maximal commutative subalgebra. Then $[L : k]^2 = [R : k]$.*

Theorem 2. *Let R be a central simple algebra over a field k. Let S be a simple k-algebra and $\varphi_1, \varphi_2 : S \to R$ be two k-linear ring homomorphisms. Then there exists an invertible element $u \in R$ such that $\varphi_2(s) = u\varphi_1(s)u^{-1}$ for all $s \in S$.*

Exercises

Problem 1. Let R be a central simple algebra over k, and $R_1 \subset R$ a central simple subalgebra over k. Let $R_2 = Z_R(R_1)$ (centralizer of R_1 in R).

(1) Show that R_2 is also central simple over k and $Z_R(R_2) = R_1$.
(2) When R is a division algebra, show that $[R_1 : k][R_2 : k] = [R : k]$.
(3) When R is not a division algebra, does the equality in (2) still hold?

Problem 2. Let $k = \mathbb{F}_p((t))$, and $L = \mathbb{F}_{p^n}((t))$. Let $\sigma \in \mathrm{Gal}(L/k)$ be the automorphism of L fixing k that is the Frobenius on $\mathbb{F}_{p^n}$. Form the cyclic algebra $R_{n,d} = L\langle x; \sigma\rangle/(x^n - t^d)$ for $d \in \mathbb{Z}$.

(1) Show that the isomorphism class of $R_{n,d}$ depends only on $d \mod n$.
(2) When d is prime to n, show that $R_{n,d}$ is a central division algebra over k. You may follow the hints below:
   - An $R_{n,d}$-module is the same as an L-vector space V together with a semilinear endomorphism $x : V \to V$ (semilinar in the sense that $x(cv) = \sigma(c)x(v)$, $c \in L, v \in V$).
   - Although $\det(x|V)$ is not well-defined, its valuation is well-defined (independent of the choice of an L-basis).
   - Calculate $\det(x|R_{n,d})$.
   - Write $R_{n,d}$ as a sum of simple $R_{n,d}$-modules, get another calculation of $\det(x|R_{n,d})$. Conclude that $R_{n,d}$ is a simple $R_{n,d}$-module.
(3) (Optional) If $\gcd(d, n) = e$, show that $R_{n,d} \cong M_e(R_{n/e,d/e})$.