18.706 HOMEWORK 8

DUE OCT.28, 2020

Theorems

In this part you will see theorems that we studied in the class. Please supply a detailed proof of each.

Theorem 1. *Let R be a ring and P be a progenerator in R-modules (a f.g. projective R-module such that R is quotient of P^n for some n). Let $S = \mathrm{End}_R(P)^{op}$. Then the functor*

$$F : R-\mathrm{Mod} \to S-\mathrm{Mod} \tag{0.1}$$

given by $\mathrm{Hom}_R(P, -)$ is an equivalence of categories.

Theorem 2. *Suppose $F : R-\mathrm{Mod} \to S-\mathrm{Mod}$ is an equivalence of categories. Then there exists a (R,S)-bimodule P and a natural isomorphism of functors $F \cong \mathrm{Hom}_R(P,-)$. Moreover, $S \cong \mathrm{End}_R(P)^{op}$, P is a progenerator as a left R-module, and a progenerator as a right S-module.*

Exercises

Problem 1. (1) Show that if R and S are Morita equivalent, so are R^{op} and S^{op}.
(2) Is R necessarily Morita equivalent to R^{op}?

Problem 2. Let $(R, S, P, Q, \lambda : P \otimes_S Q \to R, \mu : Q \otimes_R P \to S)$ be a Morita context.

(1) Equip the set
$$R\#S = \left\{ \begin{pmatrix} r & p \\ q & s \end{pmatrix} \mid r \in R, s \in S, p \in P, q \in Q \right\}$$
with a ring structure.

(2) Describe the category of left $R\#S$-modules in terms of the categories $\mathcal{C} = R-\mathrm{Mod}$, $\mathcal{D} = S-\mathrm{Mod}$, the functors $F = Q \otimes_R (-)$ and $G = P \otimes_S (-)$, and the natural transformations $\alpha : FG \to \mathrm{id}_{\mathcal{D}}$ and $\beta : GF \to \mathrm{id}_{\mathcal{C}}$ given by μ and λ. In other words, your should not mention R, S, P, Q, λ or μ in your answer.