Theorems

In this part you will see theorems that we studied in the class. Please supply a detailed proof of each.

**Theorem 1.** Let $R$ be a left artinian ring and $M$ be a finitely generated $R$-module. Then $M$ admits a decomposition $M = \oplus_{i=1}^{n} M_i$ into indecomposable $R$-modules. For a fixed indecomposable $R$-module $N$, the numbers of summands isomorphic to $N$ in any two such decompositions are the same.

**Theorem 2.** Let $R$ be a left artinian ring and $M$ be a finitely generated $R$-module. Then $M$ has a projective cover $P_M$, and $P_M$ is unique up to isomorphism.

Exercises

**Problem 1.** For a finite-dimensional $k$-algebra $R$ with $\{P_i\}_{i \in I}$ the set of indecomposable projective modules (up to isomorphism), we define its Cartan matrix to be $C = (C_{ij})_{i,j \in I}$ where $C_{ij} = \dim_k \text{Hom}_R(P_i, P_j)$.

Let $Q$ be a finite quiver without oriented cycles, and $R_Q$ be its path algebra.

1. Compute the Cartan matrix of $R_Q$ in terms of $Q$.
2. Show that any finite-dimensional $R_Q$-module $M$ admits a two-step projective resolution, i.e., a short exact sequence of $R_Q$-modules

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

with $P_1$ and $P_0$ projective.

**Problem 2.** Let $k$ be an algebraically closed field with $\text{char}(k) = 2$ or 3. Describe indecomposable projective $k[S_3]$-modules.