# 18.706 HOMEWORK 2

DUE SEP.16,2020

## Theorems

In this part you will see theorems that we studied in the class. Please supply a detailed proof of each.

**Theorem 1.** *Let $M$ be an $R$-module equipped with two finite composition series $0 \subset M_1 \subset M_2 \subset \cdots \subset M_a = M$, and $0 \subset M_1' \subset M_2' \subset \cdots \subset M_b' = M$ (namely $M_i/M_{i-1}$ and $M_j'/M_{j-1}'$ are simple $R$-modules for $1 \leq i \leq a, 1 \leq j \leq b$). Then $a = b$. Moreover, there are* canonical *permutation $\sigma : \{1, \cdots, a\} \xrightarrow{\sim} \{1, \cdots, b\}$ and* canonical *isomorphisms $M_i/M_{i-1} \xrightarrow{\sim} M'_{\sigma(i)}/M'_{\sigma(i)-1}$, for all $1 \leq i \leq a$.*

## Exercises

**Problem 1.** (1) Construct a $\mathbb{C}$-algebra $S$ such that an $S$-module is the same as a $\mathbb{C}$-vector space $V$ together with an antilinear involution $\sigma : V \to V$ (antilinear means $\sigma(av) = \overline{a}v$ for $a \in \mathbb{C}, v \in V$).

(2) Show that a $\mathbb{C}$-vector space $V$ together with an antilinear involution $\sigma : V \to V$ is the same structure as a vector space $V_0$ over $\mathbb{R}$.

(3) Combining (1) an (2), we get an equivalence of categories between $S$-modules and $\mathbb{R}$-vector spaces. How do you see this equivalence directly? What is $S$ in more familiar terms?

**Problem 2.** Let $(V, q)$ be an $n$-dimensional quadratic space over a field $k$. Recall the Clifford algebra $\mathrm{Cl}(V, q) = T(V)/I$ where $I$ is the two-sided ideal generated by $v \cdot v - q(v)$ for $v \in V$.

(1) Let $\{v_1, \cdots, v_n\}$ be a basis of $V$. For a subset $I = \{i_1, \cdots, i_r\} \subset \{1, 2 \cdots, n\}$ such that $i_1 < i_2 < \cdots < i_r$, let $v_I$ be the image of $v_{i_1} \cdots v_{i_r}$ in $\mathrm{Cl}(V, q)$. When $I = \varnothing$ set $v_\varnothing = 1$. Show that $\{v_I\}_{I \subset \{1,2\cdots,n\}}$ form a $k$-basis of $\mathrm{Cl}(V, q)$.

(2) Let $k = \mathbb{R}$ and $n = 2$. Assume $q$ is nondegenerate. Describe $\mathrm{Cl}(V, q)$ in more familiar terms according to the signature of $q$.

(3) Suppose $V$ has an orthogonal decomposition $V = V_1 \oplus \langle u, v\rangle$ such that $q(u) = 0 = q(v)$ and $q(u + v) = 1$. Show that $\mathrm{Cl}(V, q)$ is isomorphic to the ring of 2-by-2 matrices with entries in $\mathrm{Cl}(V_1, q_1)$ (where $q_1$ is the restriction of $q$ to $V_1$).

Hint: Find a left ideal $I \subset \mathrm{Cl}(V, q)$ that is a free module of rank two under the right multiplication by $\mathrm{Cl}(V_1, q_1)$.