18.706 HOMEWORK 11

DUE DEC.2, 2020

Theorems

In this part you will see theorems that we studied in the class. Please supply a detailed proof of each.

Theorem 1. *Let R be a ring with a multiplicative subset S satisfying conditions:*

- *For any $s \in S, a \in R$, $sR \cap aS \neq \varnothing$.*
- *If $a \in R, s \in S$ and $sa = 0$, then there exists $s' \in S$ such that $as' = 0$.*

Then there exists a ring R_S with a ring homomorphism

$$\varphi : R \to R_S$$

satisfying

1. *$\varphi(S)$ consists of invertible elements in R_S.*
2. *(R_S, φ) is initial in the category of pairs $(R', \varphi' : R \to R')$ satisfying (1).*
3. *Every element in R_S has the form $\varphi(a)\varphi(s)^{-1}$ for some $a \in R, s \in S$.*
4. $\ker(\varphi) = \{a \in R | as = 0$ *for some* $s \in S\}$.

Exercises

A multiplicative subset S of R is called a *right Ore set* if the two conditions in the above theorem are satisfied. It is a *left Ore set* if it is a right Ore set in R^{op}.

Problem 1. Let R be a ring. For $x \in R$ denote $ad(x) : R \to R$ the map $a \mapsto xa - ax$. An element $x \in R$ is said to be ad locally nilpotent if for any $a \in R$, $\mathrm{ad}(x)^n a = 0$ for some $n \geq 1$ (depending on a). Show that a multiplicative set consisting of ad locally nilpotent elements is both a left and a right Ore set.

Problem 2.

1. Let R_0 be a ring and $\sigma : R_0 \to R_0$ be an injective ring homomorphism. Let $R = R_0\langle x; \sigma\rangle$ be the skew polynomials in one variable x over R_0 with commutation relation $xa = \sigma(a)x$ for $a \in R_0$. Show that the set S of powers of x is a left Ore set in R, and that it is a right Ore set if and only if σ is bijective.
2. (Optional) Suppose R_0 is a commutative domain and σ is bijective. Show that all nonzero elements in R form a right Ore set. (Hint: you may first consider the case where R_0 is a field.)