18.706 HOMEWORK 1

DUE SEP.9,2020

Problem 1. Let k be a field of characteristic zero. Let W be the algebra of differential operators on $k[x]$, i.e., W consists of k-linear maps $P : k[x] \to k[x]$ of the form

$$(Pf)(x) = a_n(x)f^{(n)}(x) + a_{n-1}(x)f^{(n-1)}(x) + \cdots + a_0(x)f(x)$$

where $a_i(x) \in k[x]$, and $f^{(i)}$ is the i-th derivative of $f(x)$.

(1) Show that W is the isomorphic to $k\langle x, y\rangle/(yx - xy - 1)$ (quotient of the free algebra $k\langle x, y\rangle$ by the two-sided ideal generated by $yx - xy - 1$).
(2) Determine the center of W.
(3) Show that $\{0\}$ and W itself are the only two-sided ideals of W.
(4) Show that W does not have a nonzero left module that is finite-dimensional over k.

Problem 2. Consider the triangular ring $B = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$, where R and S are rings, and M is an (R, S)-bimodule.

(1) Show that any left B-module is a direct sum $X \oplus Y$, where X is a left R-module, Y is a left S-module together with an R-linear map $M \otimes_S Y \to X$.
(2) Describe the opposite of B as a triangular ring.
(3) Describe left ideals of B.
(4) Describe two-sided ideals of B.
(5) (Optional) Give an example of a triangular ring which is right noetherian but not left noetherian (Hint: take R and S to be two fields, one contained in the other).