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Abstract

We compare the cohomology of (parabolic) Hitchin fibers for Langlands dual groups G and $G^\vee$. The comparison theorem fits in the framework of the global Springer theory developed by the author. We prove that the stable parts of the parabolic Hitchin complexes for Langlands dual group are naturally isomorphic after passing to the associated graded of the perverse filtration. Moreover, this isomorphism intertwines the global Springer action on one hand and Chern class action on the other. Our result is inspired by the mirror symmetric viewpoint of geometric Langlands duality. Compared to the pioneer work in this subject by T. Hausel and M. Thaddeus, R. Donagi and T. Pantev, and N. Hitchin, our result is valid for more general singular fibers. The proof relies on a variant of Ngô's support theorem, which is a key point in the proof of the Fundamental Lemma.

Contents

1. Introduction

This paper is a revised version of part of the author's PhD thesis [Yun09]. In the first parts of the thesis (published as [Yun11]), we introduced a global analog of Springer representation. This paper studies the behavior of global Springer representations under Langlands duality. The main results in [Yun11] on which this paper relies will be reviewed in § 1.3.

1.1 Motivations

Let G and $G^\vee$ be almost simple algebraic groups over an algebraic closed field k with dual root systems. We call them a pair of Langlands dual groups. Fix an algebraic curve X over k and a sufficiently positive divisor D on X, we can define the Hitchin moduli stacks $\mathcal{M}_G^{\mathrm{Hit}}$ and $\mathcal{M}_{G^\vee}^{\mathrm{Hit}}$ of G and $G^\vee$-Higgs bundles (see [Hit87] for the case D is the canonical divisor of X, and see [Ngo06] for the general case). The Hitchin bases $\mathcal{A}_G^{\mathrm{Hit}}$ and $\mathcal{A}_{G^\vee}^{\mathrm{Hit}}$ can be identified using a Weyl group

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invariant nondegenerate symmetric bilinear form on $\mathfrak{t}$ (a Cartan subalgebra of $\mathfrak{g} = \mathrm{Lie}\, G$). We use $\mathcal{A}^{\mathrm{Hit}}$ to denote this common base. Consider Hitchin fibrations for $\mathcal{M}_G^{\mathrm{Hit}}$ and $\mathcal{M}_{G^\vee}^{\mathrm{Hit}}$.

$$\begin{tikzcd} \mathcal{M}_G^{\mathrm{Hit}} \arrow[dr, "f^{\mathrm{Hit}}"'] & & \mathcal{M}_{G^\vee}^{\mathrm{Hit}} \arrow[dl, "f^{\vee,\mathrm{Hit}}"] \\ & \mathcal{A}^{\mathrm{Hit}} & \end{tikzcd} \tag{1.1}$$

It was observed by Hausel and Thaddeus [HT03] (who worked in the case $G = \mathrm{SL}_n$) that the above diagram fits into the T-duality picture of Strominger, Yau and Zaslow [SYZ96], hence giving an example of mirror symmetry. The mirror symmetry between $\mathcal{M}_G^{\mathrm{Hit}}$ and $\mathcal{M}_{G^\vee}^{\mathrm{Hit}}$ is realized via the fiberwise T-duality, i.e., the generic fibers of $\mathcal{M}_G^{\mathrm{Hit}}$ and $\mathcal{M}_{G^\vee}^{\mathrm{Hit}}$ over $\mathcal{A}^{\mathrm{Hit}}$ are dual abelian varieties (up to components). The T-duality for smooth fibers of this mirror pair has been studied from the Hodge-theoretic point of view by Donagi and Pantev [DP06] for general G; see also Hitchin [Hit07] for a concrete description in the case of G_2.

On the other hand, according to Kapustin and Witten [KW07], the geometric Langlands correspondence may be interpreted in terms of the mirror symmetry between $\mathcal{M}_G^{\mathrm{Hit}}$ and $\mathcal{M}_{G^\vee}^{\mathrm{Hit}}$. The conjectural homological mirror symmetry of Kontsevich implies a quasi-classical limit statement: there should exist a natural equivalence of triangulated categories:

$$D^b_{\mathrm{coh}}(\mathcal{M}_G^{\mathrm{Hit}}) \cong D^b_{\mathrm{coh}}(\mathcal{M}_{G^\vee}^{\mathrm{Hit}}) \tag{1.2}$$

where $D^b_{\mathrm{coh}}(-)$ means the derived category of coherent sheaves. Over the generic locus where the diagram (1.1) realizes dual torus fibrations, the equivalence (1.2) should be given by the Fourier-Mukai transform (see [Ari02, BB07]). Moreover, the equivalence (1.2) is expected to respect symmetries on the two derived categories. More precisely, the quasi-classical limit of the Hecke operators on $D_{\mathrm{coh}}(\mathcal{M}_G^{\mathrm{Hit}})$ (also called 't Hooft operators [KW07]) is expected to transform into Wilson operators on $D_{\mathrm{coh}}(\mathcal{M}_{G^\vee}^{\mathrm{Hit}})$ (tensoring with vector bundles induced from the universal $G^\vee$-torsor on $\mathcal{M}_{G^\vee}^{\mathrm{Hit}} \times X$).

The goal of this paper is to establish a topological shadow of the conjectural equivalence (1.2) (or rather its variant for parabolic Hitchin fibrations), after passing first from the derived categories to K-groups, and then from K-groups to cohomology.

1.2 Notation

The base field k (algebraically closed) will be fixed throughout. Fix a prime ℓ different from $\mathrm{char}(k)$. Throughout this paper, except for Appendix A, let X be a smooth connected projective curve over k of genus g_X. We also fix a divisor $D = 2D'$ on X with $\deg D > 2g_X$.

Let G be a connected almost simple group over k of rank n. We make the assumption that $\mathrm{char}(k) > 2h$ (h is the Coxeter number of G) to ensure the existence of the Kostant section, see [Ngo10, § 1.2]. We fix a Borel subgroup B of G with universal quotient torus T. Let $\mathbb{X}_*(T)$ and $\mathbb{X}^*(T)$ be the cocharacter and character groups of T. The Lie algebras of G, B, T are denoted by $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$. The counterparts on the Langlands dual side are denoted by $G^\vee, T^\vee, T^\vee, \mathfrak{t}^\vee$, etc. Let W be the canonical Weyl group given by (G, B). Let $\widetilde{W} = \mathbb{X}_*(T) \rtimes W$ be the extended affine Weyl group.

For a scheme S of finite type over k, or a Deligne–Mumford stack, let $D^b_c(S, \overline{\mathbb{Q}}_\ell)$ denote the derived category of constructible $\overline{\mathbb{Q}}_\ell$-complexes on S. Let $\mathcal{F} \mapsto \mathcal{F}(1)$ be the Tate twist in $D^b_c(S, \overline{\mathbb{Q}}_\ell)$.

For a morphism $f: S \to T$, we have derived functors $\mathbf{L}f^*, \mathbf{R}f_*, \mathbf{R}f_!, \mathbf{R}f^!$ between $D^b_c(S, \overline{\mathbb{Q}}_\ell)$ and $D^b_c(T, \overline{\mathbb{Q}}_\ell)$. In the rest of the paper, we will simply write them as $f^*, f_*, f_!, f^!$; *all such functors are understood to be derived.*

We use $\mathbb{D}_{S/T}$ or $\mathbb{D}_f$ to denote the relative dualizing complex $f^!\overline{\mathbb{Q}}_\ell$. When $T = \operatorname{Spec} k$, we simply write $\mathbb{D}_S$ for the dualizing complex of S. The homology complex of $f: S \to T$ is defined as

$$\mathbf{H}_*(S/T) := f_! f^! \overline{\mathbb{Q}}_\ell = f_! \mathbb{D}_{S/T}.$$

The homology sheaves $\mathbf{H}_i(S/T)$ are the cohomology sheaves $\mathbf{R}^{-i}\mathbf{H}_*(S/T)$.

We use notations from [BBD82] for perverse sheaves. In particular, for a complex $\mathcal{F} \in D^b_c(S, \overline{\mathbb{Q}}_\ell)$, we have the perverse truncations ${}^p\tau_{\leqslant i}\mathcal{F}$ and perverse cohomology sheaves ${}^p\mathbf{H}^i\mathcal{F}$.

1.3 Review of global Springer theory

To state our main results, we need to consider a parabolic version of the Hitchin fibration, and the symmetries on the its cohomology. This is the content of the global Springer theory, developed by the author in [Yun11]. Let us recall the basic setup and main results in [Yun11].

For notations such as $B, W, \widetilde{W}$ associated with G, we refer to § 1.2. In [Yun11, Definition 2.1.1], we defined the *parabolic Hitchin moduli stack* $\mathcal{M}^{\mathrm{par}} = \mathcal{M}^{\mathrm{par}}_{G,X,D}$ as the moduli stack of quadruples $(x, \mathcal{E}, \varphi, \mathcal{E}^B_x)$ where:

- $\mathcal{E}$ is a G-torsor on X with a B-reduction $\mathcal{E}^B_x$ at $x \in X$;
- $\varphi \in \mathrm{H}^0(X, \mathrm{Ad}(\mathcal{E})(D))$ is a Higgs field compatible with the B-reduction $\mathcal{E}^B_x$ (i.e., the value of φ at x lies in $\mathrm{Ad}(\mathcal{E}^B_x)(D_x)$).

Here $\mathrm{Ad}(\mathcal{E})$ is the vector bundle on X associated with $\mathcal{E}$ and the adjoint representation $\mathrm{Ad}: G \to \mathrm{GL}(\mathfrak{g})$; (D) means tensoring $\mathcal{O}_X(D)$. The stack $\mathcal{M}^{\mathrm{par}}$ is a modification of Hitchin's moduli stack $\mathcal{M}^{\mathrm{Hit}}$, which classifies the data $(\mathcal{E}, \varphi)$ as above.

Let $f_1, \ldots, f_r$ be the canonical homogeneous generators of $k[\mathfrak{g}^*]^G$ of degrees $d_1, \ldots, d_r$. Let $\mathcal{A}^{\mathrm{Hit}} = \bigoplus_{i=1}^r \mathrm{H}^0(X, \mathcal{O}_X(d_i D))$ be the Hitchin base. We have the *parabolic Hitchin fibration* (see [Yun11, Definition 2.2.1]):

$$f^{\mathrm{par}}: \mathcal{M}^{\mathrm{par}} \to \mathcal{A}^{\mathrm{Hit}} \times X$$

which sends $(x, \mathcal{E}, \varphi, \mathcal{E}^B_x)$ to $f_1(\varphi), \ldots, f_r(\varphi)$ and x. The morphism f^{par} is the global analog of the Grothendieck simultaneous resolution $\pi: \widetilde{\mathfrak{g}} \to \mathfrak{g}$ in the classical Springer theory.

Let $\mathcal{A} \subset \mathcal{A}^{\mathrm{Hit}}$ be the *anisotropic locus* (denoted by $\mathcal{A}^{\mathrm{ani}}$ in [Ngo10, § 6.1]). It is the locus over which f^{par} is of finite type.

For the purposes of this paper, we will more often consider the *enhanced parabolic Hitchin fibration* (see [Yun11, Equation (2.2)])

$$\widetilde{f}: \mathcal{M}^{\mathrm{par}} \to \widetilde{\mathcal{A}}$$

where $\widetilde{\mathcal{A}}$ is (the anisotropic part of) the universal cameral cover in [Yun11, Definition 2.2.2]. Note that $q: \widetilde{\mathcal{A}} \to \mathcal{A} \times X$ is a branched W-cover.

In [Yun11, Theorem A], we have constructed an action of the extended affine Weyl group $\widetilde{W}$ on the *parabolic Hitchin complex* $f^{\mathrm{par}}_* \overline{\mathbb{Q}}_\ell$. In this paper, we will need a variant of this action on the *enhanced parabolic Hitchin complex* $\widetilde{f}_* \overline{\mathbb{Q}}_\ell$.

THEOREM [Yun11, Proposition 3.3.5]. *There is a natural* $\widetilde{W}$*-equivariant structure on* $\widetilde{f}_* \overline{\mathbb{Q}}_\ell \in D^b_c(\widetilde{\mathcal{A}}, \overline{\mathbb{Q}}_\ell)$*, compatible with the* W*-action on* $\widetilde{\mathcal{A}}$.

In particular, we have an $\mathbb{X}_*(T)$ action on $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$. This action does *not* come from an action of $\mathbb{X}_*(T)$ on $\mathcal{M}^{\mathrm{par}}$; rather, it comes from Hecke correspondences between $\mathcal{M}^{\mathrm{par}}$ and itself over $\widetilde{\mathcal{A}}$.

In [Yun11, Theorem B], we extended the $\widetilde{W}$-action on $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$ to an action of the graded double affine Hecke algebra $\mathbb{H}$ (graded DAHA) on $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$. The extra ingredient is given by certain line bundles on $\mathcal{M}^{\mathrm{par}}$. For each $\xi \in \mathbb{X}^*(T)$, there is a line bundles $\mathcal{L}(\xi)$ over $\mathcal{M}^{\mathrm{par}}$: its fiber over $(x, \mathcal{E}, \varphi, \mathcal{E}_x^B)$ is the line associated with the B-torsor $\mathcal{E}_x^B$ (over the point $x \in X$) and the homomorphism $B \to T \xrightarrow{\xi} \mathbb{G}_m$. The Chern class $c_1(\mathcal{L}(\xi))$ gives degree-two endomorphisms of $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$ and $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ in the derived category:

$$c_1(\mathcal{L}(\xi)) : \widetilde{f}_*\overline{\mathbb{Q}}_\ell \to \widetilde{f}_*\overline{\mathbb{Q}}_\ell[2](1). \tag{1.3}$$

Now consider the enhanced parabolic Hitchin fibrations of Langlands dual groups G and $G^\vee$.

$$\begin{tikzcd} \mathcal{M}_G^{\mathrm{par}} \arrow[dr, "\widetilde{f}"'] & & \mathcal{M}_{G^\vee}^{\mathrm{par}} \arrow[dl, "\widetilde{f}^\vee"] \\ & \widetilde{\mathcal{A}} & \end{tikzcd} \tag{1.4}$$

Here we have similarly identified the enhanced Hitchin bases $\widetilde{\mathcal{A}}_G$ and $\widetilde{\mathcal{A}}_{G^\vee}$. The global Springer action of $\mathbb{X}_*(T)$ on $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ is the topological shadow of the Hecke operators on $D^b_{\mathrm{coh}}(\mathcal{M}_G^{\mathrm{par}})$; the Chern class cup-product action (1.3) of $\mathbb{X}^*(T^\vee)$ on $\widetilde{f}_*^\vee\overline{\mathbb{Q}}_\ell$ is the topological shadow of the Wilson operators on $D^b_{\mathrm{coh}}(\mathcal{M}_{G^\vee}^{\mathrm{par}})$. The purpose of the paper is to identify (big parts of) the complexes $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ and $\widetilde{f}_*^\vee\overline{\mathbb{Q}}_\ell$ (up to Verdier duality), and show that the two lattice actions get intertwined under this identification (after passing to perverse cohomology). Putting this into the picture of the graded DAHA action, it shows (roughly speaking) that the two lattice pieces of the DAHA action on $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ get interchanged under Langlands duality.

1.4 Main results

1.4.1 *Important notation.* For technical reasons, we sometimes need to restrict our considerations to an open subscheme $\mathcal{A}' \subset \mathcal{A}$ over which the codimension estimate in [Ngo10, Proposition 5.7.2] is satisfied. More precisely, recall from [Ngo10, § 4.9] that we have the global δ-invariant $\delta : \mathcal{A} \to \mathbb{Z}_{\geqslant 0}$, which is upper semicontinuous by [Ngo10, Lemme 5.6.3]. Let $\mathcal{A}_\delta \subset \mathcal{A}$ be the level set of δ. [Ngo10, Proposition 5.7.2] says that there exists an integer δ_D, which goes to infinity when $\deg(D) \to \infty$, such that

$$\operatorname{codim}_{\mathcal{A}}(\mathcal{A}_\delta) \geqslant \delta, \quad \forall \delta \leqslant \delta_D.$$

We let

$$\mathcal{A}' = \bigsqcup_{\delta \leqslant \delta_D} \mathcal{A}_\delta$$

be the open subset of $\mathcal{A}$. When $\mathrm{char}(k) = 0$, the estimate $\operatorname{codim}(\mathcal{A}_\delta) \geqslant \delta$ is always satisfied, see [Ngo11, towards the end of § 2], and therefore we may take $\mathcal{A}' = \mathcal{A}$.

We summarize in the following diagram the various open subschemes of $\mathcal{A}^{\mathrm{Hit}}$ and $\widetilde{\mathcal{A}}^{\mathrm{Hit}}$ that we will use.

$$\begin{tikzcd}
\widetilde{\mathcal{A}}^{\diamondsuit,\mathrm{rs}} \arrow[r, hook] \arrow[d, hook] & \widetilde{\mathcal{A}}'^{\mathrm{rs}} \arrow[r, hook] \arrow[d, hook] & \widetilde{\mathcal{A}}^{\mathrm{rs}} \arrow[d, hook] \\
\widetilde{\mathcal{A}}^{\diamondsuit} \arrow[r, hook] \arrow[d] & \widetilde{\mathcal{A}}' \arrow[r, hook] \arrow[d] & \widetilde{\mathcal{A}} \arrow[d] \\
\mathcal{A}^{\diamondsuit} \arrow[r, hook] & \mathcal{A}' \arrow[r, hook] & \mathcal{A}
\end{tikzcd}$$

The open locus $\mathcal{A}^{\diamondsuit}$ was introduced in [Ngo10, § 4.7], and will be reviewed in § 4.2. Let $(\mathcal{A} \times X)^{\mathrm{rs}}$ be the locus of (a, x) where the value $a(x) \in \mathfrak{t}/W$ is regular semisimple, and $\widetilde{\mathcal{A}}^{\mathrm{rs}}$ is the preimage of $(\mathcal{A} \times X)^{\mathrm{rs}}$ in $\widetilde{\mathcal{A}}$. The other spaces are determined by the fact that all squares in the above diagram are Cartesian.

For the remainder of the paper, the complex $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ will be understood to be its restriction to $\widetilde{\mathcal{A}}'$ without changing notation.

1.4.2 *Langlands duality.* The complex $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ can be decomposed into the direct sum of generalized eigencomplexes under the action of $\mathbb{X}_*(T)$ (see Lemma 2.2.1 and the subsequent discussion). Let $(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}} \subset \widetilde{f}_*\overline{\mathbb{Q}}_\ell$ be the direct summand on which $\mathbb{X}_*(T)$ acts unipotently (see Definition 2.2.3). We call this subcomplex the *stable part*[1] of $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$. Our main result can be viewed as a way to understand the unipotent action of $\mathbb{X}_*(T)$ on $(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ using the Langlands dual parabolic Hitchin complex $(\widetilde{f}^\vee_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$.

Let d be the common dimension of $\mathcal{M}^{\mathrm{par}}_G$ and $\mathcal{M}^{\mathrm{par}}_{G^\vee}$. Consider two complexes on $\widetilde{\mathcal{A}}'$:

$$K = (\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}[d](d/2), \quad L = (\widetilde{f}^\vee_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}[d](d/2).$$

Let K^i, L^i be the ith *perverse* cohomology sheaves of K and L.

On one hand, we have the unipotent global Springer action of $\mathbb{X}_*(T)$ on K. It will be shown (see Lemma 2.2.1) that this action induces the identity action after passing to the perverse cohomology K^i. Hence, for $\lambda \in \mathbb{X}_*(T)$, the endomorphism $\lambda - \mathrm{id}$ on K induces a map

$$\mathrm{Sp}^i(\lambda) := {}^p\mathbf{H}^i(\lambda - \mathrm{id}) : K^i \to K^{i-1}[1],$$

which can be viewed as 'sub-diagonal entries' of the unipotent action of $\mathbb{X}_*(T)$ on K with respect to the perverse filtration.

On the other hand, consider the cup-product action (1.3) for $\mathcal{M}^{\mathrm{par}}_{G^\vee}$ instead of $\mathcal{M}^{\mathrm{par}}_G$. Tautological line bundles $\mathcal{L}(\lambda)$ on $\mathcal{M}^{\mathrm{par}}_{G^\vee}$ are indexed by $\lambda \in \mathbb{X}^*(T^\vee) = \mathbb{X}_*(T)$. Passing to the stable part and the perverse cohomology, the induced map $\cup c_1(\mathcal{L}(\lambda)) : L^i \to L^{i+2}(1)$ is in fact zero (Lemma 3.2.3). Therefore, it makes sense to talk about the 'sub-diagonal entries':

$$\mathrm{Ch}^i(\lambda) := {}^p\mathbf{H}^i(\cup c_1(\lambda)) : L^i \to L^{i+1}[1](1).$$

THEOREM A (See Theorems 3.1.2 and 3.2.4). *For each $i \in \mathbb{Z}$, there are natural isomorphisms of perverse sheaves on $\widetilde{\mathcal{A}}'$ ($\mathbb{D}$ below stands for Verdier dual):*

$$K^{-i} \cong \mathbb{D}K^i \cong L^i(i). \tag{1.5}$$

[1] The term 'stable' comes from the fact that for the usual Hitchin complex, the Frobenius traces of the stable part give stable orbital integrals [Ngo06, § 9]. It has nothing to do with the stability condition of Higgs bundles.

Moreover, for each $\lambda \in \mathbb{X}_*(T) = \mathbb{X}^*(T^\vee)$*, we have a commutative diagram*

$$\begin{tikzcd} K^{-i} \arrow[r, "\sim"] \arrow[d, "\mathrm{Sp}^{-i}(-\lambda)"'] & \mathbb{D}K^i \arrow[r, "\sim"] \arrow[d, "(\mathbb{D}\,\mathrm{Sp}^{i+1}(\lambda))[1]"] & L^i(i) \arrow[d, "\mathrm{Ch}^i(\lambda)(i)"] \\ K^{-i-1}[1] \arrow[r, "\sim"] & \mathbb{D}K^{i+1}[1] \arrow[r, "\sim"] & L^{i+1}[1](i+1) \end{tikzcd} \tag{1.6}$$

where the two rows are the isomorphisms in (1.5).

The concrete implication of Theorem A on the cohomology of parabolic Hitchin fibers is the following.

COROLLARY (See Corollary 3.3.4). *Let* $\widetilde{a} \in \widetilde{\mathcal{A}}'$ *be a geometric point and let* $\mathcal{M}^{\mathrm{par}}_{\widetilde{a}}$ *and* $\mathcal{M}^{\mathrm{par},\vee}_{\widetilde{a}}$ *be the corresponding fibers in* $\mathcal{M}^{\mathrm{par}}_G$ *and* $\mathcal{M}^{\mathrm{par}}_{G^\vee}$*. Let* N *be the common dimension of* $\mathcal{M}^{\mathrm{par}}_{\widetilde{a}}$ *and* $\mathcal{M}^{\mathrm{par},\vee}_{\widetilde{a}}$*. There is a natural filtration* $P_{\leqslant n}$ *on both* $H^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}}$ *and* $H^*(\mathcal{M}^{\mathrm{par},\vee}_{\widetilde{a}})_{\mathrm{st}}$*, induced from the perverse filtration on* K *and* L*, such that the following hold.*

(i) *There is a natural isomorphism* $v_n : \mathrm{Gr}^P_n\, H^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} \cong \mathrm{Gr}^P_{2N-n}\, H^*(\mathcal{M}^{\mathrm{par},\vee}_{\widetilde{a}})_{\mathrm{st}}(N-n)$.

(ii) *For any* $\lambda \in \mathbb{X}_*(T) = \mathbb{X}^*(T^\vee)$*, the following diagram is commutative.*

$$\begin{tikzcd} \mathrm{Gr}^P_n\, H^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} \arrow[r, "v_n"] \arrow[d, "\mathrm{Sp}(-\lambda)"'] & \mathrm{Gr}^P_{2N-n}\, H^*(\mathcal{M}^{\mathrm{par},\vee}_{\widetilde{a}})_{\mathrm{st}}(N-n) \arrow[d, "\mathrm{Ch}(\lambda)"] \\ \mathrm{Gr}^P_{n-1}\, H^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} \arrow[r, "v_{n-1}"] & \mathrm{Gr}^P_{2N-n+1}\, H^*(\mathcal{M}^{\mathrm{par},\vee}_{\widetilde{a}})_{\mathrm{st}}(N-n+1) \end{tikzcd}$$

A few remarks about the proof of Theorem A. One technical result needed in the proof is that K^i and L^i are middle extensions perverse sheaves supported on the whole $\widetilde{\mathcal{A}}'$. This is a consequence of Theorem B that we will state in the next subsection. Recall from [Yun11, Lemma 2.3.3] that there is a Picard stack $\mathcal{P}$ over $\mathcal{A}$ which acts on $\mathcal{M}^{\mathrm{par}}$ fiberwise. The general fibers of $\mathcal{P}$ are abelian varieties (up to components) which acts simply transitively on the generic fibers of $\widetilde{f}$. The construction of the isomorphism (1.5) uses an explicit description of the Picard stack $\mathcal{P}$ and its Tate module, which relies on the result of Donagi and Gaitsgory on the regular centralizer group scheme [DG02]. The proof of the commutativity of (1.6) has two major ingredients. One is the simple observation that Ext^1 between middle extensions of local systems are determined by the Ext^1 between the local systems, so that we can concentrate on a nice locus $\widetilde{\mathcal{A}}^\diamondsuit \subset \widetilde{\mathcal{A}}'$ where the fibers of $\widetilde{f}$ and $\widetilde{f}^\vee$ are abelian varieties (up to components). The other ingredient is an explicit calculation of $\mathrm{Sp}(\lambda)$ and $\mathrm{Ch}(\lambda)$ over this nice locus, which is essentially a manipulation of Abel–Jacobi maps for curves.

1.4.3 *The support theorem.* Recall the following statement in the classical Springer theory: the Springer sheaf $\pi_*\overline{\mathbb{Q}}_\ell$ is the middle extension of its restriction to $\mathfrak{g}^{\mathrm{rs}}$ (the regular semisimple locus of $\mathfrak{g}$). We will prove an analogous result for $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ instead of $\pi_*\overline{\mathbb{Q}}_\ell$.

By the decomposition theorem [BBD82, Théorème 6.2.5], $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ is *noncanonically* a direct sum of shifted perverse sheaves $\bigoplus_i ({}^p\mathbf{H}^i \widetilde{f}_*\overline{\mathbb{Q}}_\ell)[-i]$. We would like to understand the supports of the simple constituents of ${}^p\mathbf{H}^i \widetilde{f}_*\overline{\mathbb{Q}}_\ell$.

THEOREM B (See Theorem 2.1.1 and Corollary 2.2.4). (i) *Let* $\widetilde{j} : \widetilde{\mathcal{A}}'^{\mathrm{rs}} \hookrightarrow \widetilde{\mathcal{A}}'$ *be the open inclusion. For any* i*, any simple constituent of* ${}^p\mathbf{H}^i \widetilde{f}_*\overline{\mathbb{Q}}_\ell$ *is the middle extension of its restriction* *to $\widetilde{\mathcal{A}}'^{\mathrm{rs}}$; i.e.,*

$${}^p\mathbf{H}^i\widetilde{f}_*\overline{\mathbb{Q}}_\ell = \widetilde{j}_{!*}\widetilde{j}^*({}^p\mathbf{H}^i\widetilde{f}_*\overline{\mathbb{Q}}_\ell).$$

Similar result holds for the perverse cohomology sheaves of $f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell$.

(ii) *Recall that K^i are the perverse cohomology sheaves of the stable part $(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}[d](d/2)$. Then the support of K^i is the whole of $\widetilde{\mathcal{A}}'$.*

Note that Theorem B does not imply that the support of every simple constituent $\mathcal{F}$ of ${}^p\mathbf{H}^i\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ is the whole $\widetilde{\mathcal{A}}'$: it only states that the support of $\mathcal{F}$ intersects $\widetilde{\mathcal{A}}'^{\mathrm{rs}}$. The proof of Theorem B uses Ngô's argument for his 'Théorème du support' [Ngo10, Théorème 7.2.1], which is the key geometric ingredient in his proof of the Fundamental lemma.

Using this result, one can show that the action of $\mathbb{X}_*(T)$ on ${}^p\mathbf{H}^i\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ factors through a finite quotient. However, this is no longer true on the level of complexes: we have an example in [Yun11, § 7.2] where the action of $\mathbb{X}_*(T)$ on a 'subregular' stalk of $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ is not semisimple. Nevertheless, one can consider the decomposition of $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ into generalized eigensubcomplexes under $\mathbb{X}_*(T)$:

$$\widetilde{f}_*\overline{\mathbb{Q}}_\ell = \bigoplus_{\kappa\in\widehat{T}(\overline{\mathbb{Q}}_\ell)} (\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_\kappa. \tag{1.7}$$

Here $\widehat{T} = \mathrm{Hom}(\mathbb{X}_*(T), \mathbb{G}_m)$ is a torus over $\overline{\mathbb{Q}}_\ell$. This is the parabolic analog of the endoscopic decomposition considered by Ngô [Ngo06, Théorème 8.5]. The direct summands $(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_\kappa$ have two types.

- When $\kappa \in Z\widehat{G}(\overline{\mathbb{Q}}_\ell)$, then[2] $(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_\kappa$ is naturally isomorphic to $(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$.
- When $\kappa \notin Z\widehat{G}(\overline{\mathbb{Q}}_\ell)$, then $(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_\kappa$ has support on a proper subscheme of $\widetilde{\mathcal{A}}'$. These direct summands can be understood using the endoscopic groups of G. The study of these direct summands is the content of the first half of the preprint [Yun09].

1.5 Organization of the paper

In § 2, we prove Theorem B about the support of simple constituents of ${}^p\mathbf{H}^i\widetilde{f}_*\overline{\mathbb{Q}}_\ell$. We then study the κ-decomposition of $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ in § 2.2.

In § 3, we state Theorem A about Langlands duality and its consequences. In § 3.4, we consider the example of a subregular parabolic Hitchin fiber for $G = \mathrm{SL}(2)$, and verify Theorem A in this case.

In § 4, we prove Theorem A. We first need an explicit description of the Picard stack $\mathcal{P}$ and its Tate module, which we give in § 4.3.

In Appendix A, we collect facts about cap-product action on direct image complexes, partly following [Ngo10, § 7.4].

2. The support theorem and the endoscopic decomposition

2.1 Support theorem

Consider the enhanced parabolic Hitchin fibration $\widetilde{f}: \mathcal{M}^{\mathrm{par}}|_{\widetilde{\mathcal{A}}'} \to \widetilde{\mathcal{A}}'$, which is a proper morphism with source a smooth Deligne–Mumford stack. By the decomposition theorem

[2] The group $\widehat{G}$ is a split group over $\overline{\mathbb{Q}}_\ell$ which is Langlands dual to G; it is not to be confused with $G^\vee$. It contains $\widehat{T}$ as a maximal torus, and $Z\widehat{G}$ is its center.

[BBD82, Théorème 6.2.5], we have a *noncanonical* decomposition

$$\widetilde{f}_*\overline{\mathbb{Q}}_\ell = \bigoplus_{i\in I} \mathcal{F}_i[-n_i] \tag{2.1}$$

where I is a finite index set and $\mathcal{F}_i$ are simple perverse sheaf on $\widetilde{\mathcal{A}}'$.

The support theorem that we will state is the analog of the following statement in classical Springer theory: the direct image complex $\pi_*\overline{\mathbb{Q}}_\ell$ of the Grothendieck simultaneous resolution $\pi:\widetilde{\mathfrak{g}}\to\mathfrak{g}$ is the middle extension from its restriction to $\mathfrak{g}^{\mathrm{rs}}$.

Recall the following maps:

$$\begin{tikzcd} \widetilde{\mathcal{A}}' \arrow[r, "q"'] \arrow[rr, bend left, "\widetilde{p}"] & \mathcal{A}'\times X \arrow[r, "p"'] & \mathcal{A}'. \end{tikzcd} \tag{2.2}$$

Theorem 2.1.1. *For each simple perverse sheaf $\mathcal{F}_i$ appearing in the decomposition (2.1), its support Z is an irreducible component of $\widetilde{p}^{-1}(\widetilde{p}(Z))$. In particular, $\mathcal{F}_i$ is the middle perverse extension of its restriction to $\widetilde{\mathcal{A}}'^{rs}$. Equivalently, for each $i\in\mathbb{Z}$,*

$$^p\mathbf{H}^i\widetilde{f}_*\overline{\mathbb{Q}}_\ell = \widetilde{j}^{\mathrm{rs}}_{!*}\widetilde{j}^{\mathrm{rs},*}(^p\mathbf{H}^i\widetilde{f}_*\overline{\mathbb{Q}}_\ell)$$

where $\widetilde{j}^{\mathrm{rs}}:\widetilde{\mathcal{A}}'^{\mathrm{rs}}\hookrightarrow\widetilde{\mathcal{A}}'$ is the open embedding.

Proof. Let δ_Z be the generic (minimal) value of δ on $\widetilde{p}(Z)$. Let us recall the notion of the amplitude of Z following [Ngo10, § 7.3]. Let $\mathrm{occ}(Z)=\{n_i\mid i\in I,\mathrm{Supp}\,\mathcal{F}_i=Z\}$. Then the amplitude of Z is defined to be the difference between the largest and smallest elements in $\mathrm{occ}(Z)$.

Consider the $\widetilde{\mathcal{P}}$-action on $\mathcal{M}^{\mathrm{par}}$ over $\widetilde{\mathcal{A}}$. In this situation, we can apply [Ngo10, Proposition 7.3.2] to conclude that the amplitude of Z is at least $2(\dim(\mathcal{P}/\mathcal{A})-\delta_Z)$.

Now we can apply the argument of [Ngo10, § 7.3] to show that $\mathrm{codim}_{\widetilde{\mathcal{A}}}(Z)\leqslant\delta_Z$. For completeness, we briefly reproduce the argument here. By Poincaré duality, the set $\mathrm{occ}(Z)$ is symmetric with respect to $\dim(\mathcal{M}^{\mathrm{par}})$. Let n_+ be the largest element in $\mathrm{occ}(Z)$. Since the amplitude of Z is at least $2(\dim(\mathcal{P}/\mathcal{A})-\delta_Z)$, we conclude that $n_+\geqslant\dim(\mathcal{M}^{\mathrm{par}})+\dim(\mathcal{P}/\mathcal{A})-\delta_Z$. Suppose $\mathcal{F}_j$ has support Z and $n_j=n_+$. Let U be an open dense subset of Z over which $\mathcal{F}_j$ is a local system placed in degree $-\dim(Z)$. Pick a point $\widetilde{a}\in U(k)$. Since

$$0\neq i^*_{\widetilde{a}}\mathcal{F}_j\subset \mathrm{H}^{n_+-\dim(Z)}(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}},\overline{\mathbb{Q}}_\ell),$$

we necessarily have

$$\dim(\mathcal{M}^{\mathrm{par}})+\dim(\mathcal{P}/\mathcal{A})-\delta_Z-\dim(Z)\leqslant n_+-\dim(Z)\leqslant 2\dim(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})=2\dim(\mathcal{P}/\mathcal{A}).$$

This implies that $\mathrm{codim}_{\widetilde{\mathcal{A}}}(Z)\leqslant\delta_Z$.

By the definition of $\mathcal{A}'$, we have $\mathrm{codim}_{\mathcal{A}'}(\mathcal{A}'_\delta)\geqslant\delta$. This implies that $\mathrm{codim}_{\widetilde{\mathcal{A}}'}(Z)\geqslant\mathrm{codim}_{\mathcal{A}'}(\widetilde{p}(Z))\geqslant\delta_Z$. Therefore, the inequalities must be equalities; i.e.,

$$\mathrm{codim}_{\widetilde{\mathcal{A}}'}(Z)=\mathrm{codim}_{\mathcal{A}'}(\widetilde{p}(Z))=\delta_Z.$$

This forces the result that $q(Z)=\widetilde{p}(Z)\times X$ and also the result that Z is an irreducible component of $q^{-1}(\widetilde{p}(Z)\times X)=\widetilde{p}^{-1}(\widetilde{p}(Z))$.

Since $(\{a\}\times X)^{\mathrm{rs}}$ is dense in $\{a\}\times X$ for any $a\in\mathcal{A}(k)$, we conclude that $q(Z)^{\mathrm{rs}}$ is dense in $q(Z)$ and therefore Z^{rs} is dense in Z. Therefore, the simple perverse sheaf $\mathcal{F}_i$ is the middle extension of its restriction on Z^{rs}. This completes the proof. $\square$

2.2 Endoscopic decomposition

By the theorem quoted in § 1.3, there is an action of $\mathbb{X}_*(T)$ on the enhanced parabolic Hitchin complex $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$. In the proof of [Yun11, Proposition 3.3.5], we used the Hecke correspondence $\mathcal{H}^\natural_\lambda$ (a self-correspondence of $\mathcal{M}^{\mathrm{par}}$ over $\widetilde{\mathcal{A}}$) to construct the action of $\lambda \in \mathbb{X}_*(T)$. By [Yun11, beginning of the proof of Proposition 3.2.1], there is a morphism

$$s : \mathbb{X}_*(T) \times \widetilde{\mathcal{A}}^{\mathrm{rs}} \subset \widetilde{\mathcal{A}}^0 \to \mathcal{P} \tag{2.3}$$

such that for each $\lambda \in \mathbb{X}_*(T)$, $\mathcal{H}^\natural_\lambda|_{\widetilde{\mathcal{A}}^{\mathrm{rs}}}$ is the graph of the following automorphism of $\mathcal{M}^{\mathrm{par}}|_{\widetilde{\mathcal{A}}^{\mathrm{rs}}}$: the automorphism over $\widetilde{a} \in \widetilde{\mathcal{A}}^{\mathrm{rs}}$ is given by the action of $s(\lambda, \widetilde{a}) \in \mathcal{P}_a$ on $\mathcal{M}^{\mathrm{par}}_{\widetilde{a}}$ (where $a \in \mathcal{A}$ is the image of $\widetilde{a}$).

The morphism s induces a surjective homomorphism of sheaves of abelian groups on $\widetilde{\mathcal{A}}^{\mathrm{rs}}$

$$\pi_0(s) : \mathbb{X}_*(T) \to \pi_0(\widetilde{\mathcal{P}}/\widetilde{\mathcal{A}}^{\mathrm{rs}}) = \widetilde{p}^{\mathrm{rs}*}\pi_0(\mathcal{P}/\mathcal{A}) \tag{2.4}$$

where $\widetilde{p}^{\mathrm{rs}} : \widetilde{\mathcal{A}}^{\mathrm{rs}} \to \mathcal{A}$ is the projection and $\mathbb{X}_*(T)$ stands for the constant sheaf on $\widetilde{\mathcal{A}}^{\mathrm{rs}}$ with stalks $\mathbb{X}_*(T)$.

On the other hand, since $\widetilde{\mathcal{P}}$ acts on $\mathcal{M}^{\mathrm{par}}$ over $\widetilde{\mathcal{A}}$, by homotopy invariance [LN08, Lemme 3.2.3], it induces an action of $\pi_0(\widetilde{\mathcal{P}}/\widetilde{\mathcal{A}})$ on ${}^p\mathbf{H}^i\widetilde{f}_*\overline{\mathbb{Q}}_\ell$.

Lemma 2.2.1. *Over $\widetilde{\mathcal{A}}'$, the global Springer action of $\mathbb{X}_*(T)$ on each perverse cohomology sheaf ${}^p\mathbf{H}^i\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ factors through a finite quotient, and hence is semisimple.*

Proof. By Theorem 2.1.1, it suffices to check this statement over $\widetilde{\mathcal{A}}'^{\mathrm{rs}}$. We actually check this statement over the larger subset $\widetilde{\mathcal{A}}^{\mathrm{rs}}$. By the discussion in the beginning of this subsection, over $\widetilde{\mathcal{A}}^{\mathrm{rs}}$, the action of $\mathbb{X}_*(T)$ comes from the map $s(\lambda, -) : \widetilde{\mathcal{A}}^{\mathrm{rs}} \to \mathcal{P}$ and the action of $\mathcal{P}$ on $\mathcal{M}^{\mathrm{par}}$, and therefore the action of $\mathbb{X}_*(T)$ on $\widetilde{f}_*\overline{\mathbb{Q}}_\ell|_{\widetilde{\mathcal{A}}^{\mathrm{rs}}}$ factors through $\pi_0(s)$ in (2.4). By the definition of the anisotropic locus $\mathcal{A}$, $\pi_0(\mathcal{P}/\mathcal{A})$ is a constructible sheaf of *finite* abelian groups, and hence $\pi_0(s)$ necessarily factors through a finite quotient of $\mathbb{X}_*(T)$. $\square$

Let $\widehat{T} = \mathrm{Hom}(\mathbb{X}_*(T), \mathbb{G}_m)$ be an algebraic torus over $\overline{\mathbb{Q}}_\ell$, such that $\widehat{T}(\overline{\mathbb{Q}}_\ell)$ is the set of characters $\mathbb{X}_*(T) \to \overline{\mathbb{Q}}_\ell^\times$. By Lemma 2.2.1, the action of $\overline{\mathbb{Q}}_\ell[\mathbb{X}_*(T)]$ on $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ factors through a finite length quotient $\overline{\mathbb{Q}}_\ell[\mathbb{X}_*(T)]/I$, whose spectrum is a finite subset $\{\kappa_1, \dots, \kappa_m\} \subset \widehat{T}(\overline{\mathbb{Q}}_\ell)$. By the Chinese remainder theorem, we can find orthogonal idempotents $\iota_1, \dots, \iota_m \in \overline{\mathbb{Q}}_\ell[\mathbb{X}_*(T)]/I$ projecting to the localizations of $\overline{\mathbb{Q}}_\ell[\mathbb{X}_*(T)]/I$ at $\kappa_1, \dots, \kappa_m$. These idempotents give rise to a decomposition of $\widetilde{f}_*\overline{\mathbb{Q}}_\ell \in D^b_c(\widetilde{\mathcal{A}}, \overline{\mathbb{Q}}_\ell)$ into generalized eigensubcomplexes under $\mathbb{X}_*(T)$:

$$\widetilde{f}_*\overline{\mathbb{Q}}_\ell = \bigoplus_{\kappa \in \widehat{T}(\overline{\mathbb{Q}}_\ell)} (\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_\kappa. \tag{2.5}$$

By Lemma 2.2.1, $\mathbb{X}_*(T)$ acts on ${}^p\mathbf{H}^i(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_\kappa$ via the character $\kappa : \mathbb{X}_*(T) \to \overline{\mathbb{Q}}_\ell^\times$. Over $\widetilde{\mathcal{A}}^{\mathrm{rs}}$, the action of $\mathbb{X}_*(T)$ comes from the map s in (2.3), therefore this decomposition is also the decomposition according to generalized eigensubcomplexes of $\pi_0(\widetilde{\mathcal{P}}/\widetilde{\mathcal{A}})$. Hence, passing to perverse cohomology sheaves, the decomposition (2.5) coincides with the κ-decomposition defined by Ngô in [Ngo10, § 6.2.1].

Following [Ngo10, § 6.2.3], for $\kappa \in \widehat{T}(\overline{\mathbb{Q}}_\ell)$, let $\widetilde{\mathcal{A}}^{\mathrm{rs}}_\kappa$ be the locus of $\widetilde{a} \in \widetilde{\mathcal{A}}^{\mathrm{rs}}$ such that $\kappa : \mathbb{X}_*(T) \to \overline{\mathbb{Q}}_\ell^\times$ factors through the homomorphism $s(-, \widetilde{a}) : \mathbb{X}_*(T) \to \pi_0(\mathcal{P}_a)$. Let $\widetilde{\mathcal{A}}_\kappa$ be the closure of $\widetilde{\mathcal{A}}^{\mathrm{rs}}_\kappa$ in $\widetilde{\mathcal{A}}$. Let $\widetilde{\mathcal{A}}'_\kappa = \widetilde{\mathcal{A}}_\kappa \cap \widetilde{\mathcal{A}}'$. Note that in [Ngo10], the notation $\widetilde{\mathcal{A}}$ has a different meaning: it is the preimage of a point $\infty \in X$ in $\widetilde{\mathcal{A}}^{\mathrm{rs}}$ in our notation.

PROPOSITION 2.2.2. *The support of any simple constituent of* ${}^p\mathbf{H}^i(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_\kappa$ *(for any i) is an irreducible component of* $\widetilde{\mathcal{A}}'_\kappa$.

Proof. By Theorem 2.1.1, it suffices to show that for any simple constituent $\mathcal{F}$ of ${}^p\mathbf{H}^i(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_\kappa|_{\widetilde{\mathcal{A}}'^{\mathrm{rs}}}$, the support of $\mathcal{F}$ is an irreducible component of $\widetilde{\mathcal{A}}'^{\mathrm{rs}}_\kappa = \widetilde{\mathcal{A}}' \cap \widetilde{\mathcal{A}}^{\mathrm{rs}}_\kappa$.

We would like to apply the general result [Ngo10, Corollaire 7.2.3] to the fibration $\widetilde{f}^{\mathrm{rs}} : \mathcal{M}^{\mathrm{par}}|_{\widetilde{\mathcal{A}}'^{\mathrm{rs}}} \to \widetilde{\mathcal{A}}'^{\mathrm{rs}}$ together with the action of $\mathcal{P} \times_{\mathcal{A}} \widetilde{\mathcal{A}}'^{\mathrm{rs}}$. Since $\widetilde{f}^{\mathrm{rs}}$ is the base change of the usual Hitchin fibration $f^{\mathrm{Hit}} : \mathcal{M}^{\mathrm{Hit}}|_{\mathcal{A}'} \to \mathcal{A}'$, all the conditions in [Ngo10, § 7.1] are satisfied by the discussion in [Ngo10, § 7.8]. The result given in [Ngo10, Corollaire 7.2.3] implies that Z is an irreducible component of the support of a direct summand of $(\mathbf{R}^{2N}\widetilde{f}^{\mathrm{rs}}_*\overline{\mathbb{Q}}_\ell)_\kappa$ (N is the relative dimension of $\widetilde{f}^{\mathrm{rs}}$). By [Ngo10, Propositions 6.5.1 and 6.3.3], these irreducible components are simply irreducible components of $\widetilde{\mathcal{A}}'^{\mathrm{rs}}_\kappa$. □

DEFINITION 2.2.3. The *stable part* $(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ of the complex $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ is defined as the direct summand in the decomposition (2.5) corresponding to $\kappa = 1$.

In other words, $(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ is the direct summand of $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ on which $\mathbb{X}_*(T)$ acts unipotently. By definition, $\widetilde{\mathcal{A}}_\kappa = \widetilde{\mathcal{A}}$ for $\kappa = 1$, we conclude the following corollary.

COROLLARY 2.2.4 (Of Proposition 2.2.2). *For any $i \in \mathbb{Z}$,* ${}^p\mathbf{H}^i(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ *is the middle extension from its restriction to any nonempty Zariski open subset of* $\widetilde{\mathcal{A}}'$.

Remark 2.2.5. The discussions in this and the previous subsections apply to the parabolic Hitchin complex $f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell$ as well. In particular, we can write $f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell$ as a direct sum of $(f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell)_\kappa$ according to the $\mathbb{X}_*(T)$-action, and we have

$$(f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell)_\kappa = q_*(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_\kappa.$$

We also define the stable part $(f^{\mathrm{par}}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ as the direct summand with $\kappa = 1$.

3. Parabolic Hitchin complexes and Langlands duality

In this section, we will establish a Verdier duality between the stable part $(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ of the enhanced parabolic Hitchin complex to the stable part $\widetilde{f}^\vee_*\overline{\mathbb{Q}}_\ell$ for the Langlands dual group $G^\vee$. We will also relate the global Springer action of $\mathbb{X}_*(T)$ on $(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ and the Chern class action of $\mathbb{X}^*(T^\vee)$ on $(\widetilde{f}^\vee_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$.

3.1 Verdier duality for enhanced parabolic Hitchin complexes

Fix a W-equivariant isomorphism $\iota : \mathfrak{t} \xrightarrow{\sim} \mathfrak{t}^\vee$. This allows us to identify W-invariant polynomials on $\mathfrak{t}$ and on $\mathfrak{t}^\vee$, hence giving isomorphisms between the Hitchin bases of G and $G^\vee$ preserving the anisotropic loci:

$$\iota_{\mathcal{A}} : \mathcal{A}_G \xrightarrow{\sim} \mathcal{A}_{G^\vee},$$
$$\iota_{\widetilde{\mathcal{A}}} : \widetilde{\mathcal{A}}_G \xrightarrow{\sim} \widetilde{\mathcal{A}}_{G^\vee}.$$

Since G is almost simple, the choice of ι is unique up to a scalar. Therefore, the resulting $\iota_{\widetilde{\mathcal{A}}}$ is unique up to the natural action of $\mathbb{G}_m$ on $\widetilde{\mathcal{A}}_G$. Since all the complexes over $\widetilde{\mathcal{A}}_G$ or $\widetilde{\mathcal{A}}_{G^\vee}$ we consider will be $\mathbb{G}_m$-equivariant, this ambiguity is harmless. We therefore fix the identification ι once and for all. We denote the common Hitchin base (respectively enhanced Hitchin base) by $\mathcal{A}$ (respectively $\widetilde{\mathcal{A}}$). We have the enhanced parabolic Hitchin fibrations as in (1.4).

Lemma 3.1.1. *The stacks $\mathcal{M}_G^{\mathrm{par}}$ and $\mathcal{M}_{G^\vee}^{\mathrm{par}}$ have the same dimension.*

Proof. By [Ngo10, § 4.13.4], we have

$$\dim(\mathcal{M}_G^{\mathrm{Hit}}) = \dim(G)\deg(D) = \dim(G^\vee)\deg(D) = \dim(\mathcal{M}_{G^\vee}^{\mathrm{Hit}}).$$

Since the parabolic Hitchin stacks have one dimension more than the usual Hitchin stacks [Yun11, Proposition 3.5.1(2)], we conclude that $\dim(\mathcal{M}_G^{\mathrm{par}}) = \dim(\mathcal{M}_{G^\vee}^{\mathrm{par}})$. □

Let $d = \dim(\mathcal{M}_G^{\mathrm{par}}) = \dim(\mathcal{M}_{G^\vee}^{\mathrm{par}})$. Let

$$K := (\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}[d](d/2), \quad L := (\widetilde{f}^\vee_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}[d](d/2) \tag{3.1}$$

be objects in $D_c^b(\widetilde{\mathcal{A}}', \overline{\mathbb{Q}}_\ell)$. For each $i \in \mathbb{Z}$, let

$$K^i := {}^p\mathbf{H}^iK, \quad L^i := {}^p\mathbf{H}^iL$$

be the perverse cohomology sheaves of K and L.

The first result in this section is a Verdier duality between K and L.

Theorem 3.1.2. *For each $i \in \mathbb{Z}$, there is a natural isomorphism of perverse sheaves on $\widetilde{\mathcal{A}}'$:*

$$\widetilde{\mathrm{VD}}^i : \mathbb{D}K^i \cong L^i(i).$$

The proof will be given in § 4.3.

Remark 3.1.3. Let $d = \dim \mathcal{M}_G^{\mathrm{par}}$. Fix a fundamental class $[\mathcal{M}_G^{\mathrm{par}}]$ of $\mathcal{M}_G^{\mathrm{par}}$, and hence fix an isomorphism $[\mathcal{M}_G^{\mathrm{par}}] : \overline{\mathbb{Q}}_{\ell,\mathcal{M}^{\mathrm{par}}}[d](d/2) \cong \mathbb{D}_{\mathcal{M}_G^{\mathrm{par}}}[d](d/2)$. This induces isomorphisms

$$\widetilde{v} : \widetilde{f}_*\overline{\mathbb{Q}}_\ell[d](d/2) \xrightarrow{\sim} \widetilde{f}_*\mathbb{D}_{\mathcal{M}_G^{\mathrm{par}}}[d](d/2) = \mathbb{D}(\widetilde{f}_*\overline{\mathbb{Q}}_\ell[d](d/2)).$$

In [Yun11, Proposition 4.1.8] we established a relation between global Springer action on $f_*^{\mathrm{par}}\overline{\mathbb{Q}}_\ell$ and Verdier duality. A similar relation for $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ was stated without proof: the isomorphism $\widetilde{v}$ transforms the action of $\lambda \in \mathbb{X}_*(T)$ to the Verdier dual of the action of $-\lambda$. The essential point is that the correspondence $\mathcal{H}_\lambda$ in [Yun11, Definition 3.3.1] (which is used to define the λ-action on $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$) is the same as the transpose of $\mathcal{H}_{-\lambda}$ over $\widetilde{\mathcal{A}}^{\mathrm{rs}}$, and they are both graph-like correspondences with respect to $(\mathcal{A}' \times X)^{\mathrm{rs}} \subset \mathcal{A}' \times X$ (see [Yun11, Lemma 3.1.4]).

By the above discussion, the stable part K is preserved under $\widetilde{v}$. This induces a canonical isomorphism of perverse sheaves on $\widetilde{\mathcal{A}}$:

$$\widetilde{v}^i : K^{-i} \cong \mathbb{D}K^i. \tag{3.2}$$

Therefore, Theorem 3.1.2 can be reformulated as an isomorphism

$$\widetilde{\mathrm{VD}'}^i : K^{-i} \cong L^i(i). \tag{3.3}$$

3.2 Comparison of two actions

3.2.1 *The Springer action.* By the Theorem quoted in § 1.3, we have an action of $\mathbb{X}_*(T)$ on K. By Lemma 2.2.1, the $\mathbb{X}_*(T)$-action on K^i is semisimple. Since K is the stable part, $\mathbb{X}_*(T)$ acts trivially on K^i; i.e., for any $\lambda \in \mathbb{X}_*(T)$, the action of $\lambda - \mathrm{id}$ on K induces the zero map on K^i, therefore ${}^p\tau_{\leqslant i}(\lambda - \mathrm{id})$ factors through

$${}^p\tau_{\leqslant i}(\lambda - \mathrm{id}) : {}^p\tau_{\leqslant i}K \to {}^p\tau_{\leqslant i-1}K.$$

Taking the ith perverse cohomology, we get

$$\mathrm{Sp}^i(\lambda) := {}^p\mathbf{H}^i(\lambda - \mathrm{id}) : K^i \to K^{i-1}[1], \tag{3.4}$$

which is an extension class between the perverse sheaves K^i and K^{i-1}.

3.2.2 *The Chern class action.* Apply the discussion in § 1.3 to $G^\vee$ instead of G, we have tautological line bundles $\mathcal{L}(\lambda)$ on $\mathrm{Bun}^{\mathrm{par}}_{G^\vee}$ indexed by $\lambda \in \mathbb{X}^*(T^\vee)$ (see also [Yun11, Construction 6.1.4]). We denote its pull-back to $\mathcal{M}^{\mathrm{par}}_{G^\vee}$ also by $\mathcal{L}(\lambda)$. The Chern class $c_1(\mathcal{L}(\lambda))$ gives a map $\cup c_1(\mathcal{L}(\lambda)) : \widetilde{f}^\vee_* \overline{\mathbb{Q}}_\ell \to \widetilde{f}^\vee_* \overline{\mathbb{Q}}_\ell[2](1)$ as in (1.3). Taking the stable part of this map,

$$(\cup c_1(\mathcal{L}(\lambda)))_{\mathrm{st}} : L \hookrightarrow \widetilde{f}^\vee_* \overline{\mathbb{Q}}_\ell[d](d/2) \to \widetilde{f}^\vee_* \overline{\mathbb{Q}}_\ell[d+2](d/2+1) \twoheadrightarrow L[2](1).$$

we get a map ${}^p\mathbf{H}^i(\cup c_1(\mathcal{L}(\lambda)))_{\mathrm{st}} : L^i \to L^{i+2}(1)$ on the perverse cohomology.

Lemma 3.2.3. *The map ${}^p\mathbf{H}^i(\cup c_1(\mathcal{L}(\lambda))_{\mathrm{st}}) : L^i \to L^{i+2}(1)$ is zero for all $i \in \mathbb{Z}$.*

We postpone the proof to § 4.6. Admitting this lemma, the map ${}^p\tau_{\leqslant i}(\cup c_1(\mathcal{L}(\lambda)))_{\mathrm{st}}$ factors through ${}^p\tau_{\leqslant i}(\cup c_1(\mathcal{L}(\lambda))_{\mathrm{st}}) : {}^p\tau_{\leqslant i} L \to {}^p\tau_{\leqslant i-1}(L[2](1))$. Taking the ith perverse cohomology, we get

$$\mathrm{Ch}^i(\lambda) = {}^p\mathbf{H}^i(\cup c_1(\mathcal{L}(\lambda))_{\mathrm{st}}) : L^i \to L^{i+1}[1](1). \tag{3.5}$$

The next main result is an identification of the maps $\mathrm{Sp}^i(\lambda)$ and $\mathrm{Ch}^i(\lambda)$ under the Verdier duality given in Theorem 3.1.2.

Theorem 3.2.4. *For each $i \in \mathbb{Z}$ and $\lambda \in \mathbb{X}_*(T) = \mathbb{X}^*(T^\vee)$, we have a commutative diagram.*

$$\begin{tikzcd}
K^{-i} \arrow[r, "\widetilde{v}^{-i}", "\sim"'] \arrow[d, "\mathrm{Sp}^{-i}(-\lambda)"'] & \mathbb{D}K^i \arrow[r, "\widetilde{\mathrm{VD}}^i", "\sim"'] \arrow[d, "(\mathbb{D}\,\mathrm{Sp}^{i+1}(\lambda))[1]"] & L^i(i) \arrow[d, "\mathrm{Ch}^i(\lambda)(i)"] \\
K^{-i-1}[1] \arrow[r, "\widetilde{v}^{-i-1}[1]", "\sim"'] & \mathbb{D}K^{i+1}[1] \arrow[r, "\widetilde{\mathrm{VD}}^{i+1}", "\sim"'] & L^{i+1}[1](i+1)
\end{tikzcd} \tag{3.6}$$

Here the isomorphism $\widetilde{\mathrm{VD}}^i$ is given in Theorem 3.1.2 and the isomorphism $\widetilde{v}^i$ is given in Remark 3.1.3.

The proof will be given in § 4.7.

3.3 The perverse filtration on the cohomology of parabolic Hitchin fibers

In this subsection we spell out the implication of Theorem 3.2.4 on the cohomology of fibers of $\widetilde{f}$ and $\widetilde{f}^\vee$. For any geometric point $\widetilde{a} \in \widetilde{\mathcal{A}}'$, let $\mathcal{M}^{\mathrm{par}}_{\widetilde{a}}$ and $\mathcal{M}^{\mathrm{par},\vee}_{\widetilde{a}}$ be the fibers $\widetilde{f}^{-1}(\widetilde{a})$ and $\widetilde{f}^{\vee,-1}(\widetilde{a})$. Let N be the relative dimension of $\widetilde{f}$ and $\widetilde{f}^\vee$; hence $\dim \mathcal{M}^{\mathrm{par}}_{\widetilde{a}} = \dim \mathcal{M}^{\mathrm{par},\vee}_{\widetilde{a}} = N$.

By the decomposition theorem (see (2.1)), the truncations ${}^p\tau_{\leqslant n}(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ are direct summands of $(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$. Taking stalks at $\widetilde{a}$, they define a filtration on the graded vector space $\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}}) = \oplus \mathrm{H}^m(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}}[-m]$:

$$P_{\leqslant n}\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} := ({}^p\tau_{\leqslant n+\dim \widetilde{\mathcal{A}}}(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}})_{\widetilde{a}}. \tag{3.7}$$

Let $\mathrm{Gr}^P_n \mathrm{H}^m(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} = P_{\leqslant n}\mathrm{H}^m(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} / P_{\leqslant n-1}\mathrm{H}^m(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}}$. Then $\mathrm{Gr}^P_n \mathrm{H}^m(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}}$ is the $(m-n-\dim\widetilde{\mathcal{A}})$th cohomology of the stalk of ${}^p\mathbf{H}^{n+\dim\widetilde{\mathcal{A}}}(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ at $\widetilde{a}$.

Lemma 3.3.1. *Let $\widetilde{a} \in \widetilde{\mathcal{A}}'$ be a geometric point.*

(i) *We have $\mathrm{Gr}^P_n H^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} = 0$ unless $0 \leqslant n \leqslant 2N$.*

(ii) *(Hard Lefschetz) We have $\dim \mathrm{Gr}^P_n H^m(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} = \dim \mathrm{Gr}^P_{2N-n} H^{m+2(N-n)}(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})$, for all $m, n \in \mathbb{Z}$. In particular, $\dim \mathrm{Gr}^P_n H^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}}$ is symmetric about N.*

(iii) *We have $\mathrm{Gr}^P_n H^m(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} = 0$ unless $n \leqslant m \leqslant 2n$.*

Proof. (i) The complex $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ lies in the cohomological range $[0, 2N]$ under the usual t-structure. Over a dense open subset $U \subset \widetilde{\mathcal{A}}$, $\widetilde{f}|_U$ is a disjoint union of families of abelian varieties. Therefore, $(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}|_U$ is a direct sum of shifted local systems, and hence lies in the cohomological range $[\dim \widetilde{\mathcal{A}}, 2N + \dim \widetilde{\mathcal{A}}]$ under the perverse t-structure. By Corollary 2.2.4, $(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ also lies in the cohomological range $[\dim \widetilde{\mathcal{A}}, 2N + \dim \widetilde{\mathcal{A}}]$ under the perverse t-structure.

(ii) Over a dense open subset $U \subset \widetilde{\mathcal{A}}$, the family $\widetilde{f}|_U : \mathcal{M}^{\mathrm{par}}|_U \to U$ can be polarized. This gives a Lefschetz isomorphism of local systems $\mathbf{R}^n(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}|_U \xrightarrow{\sim} \mathbf{R}^{2N-n}(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}(N-n)|_U$. By Corollary 2.2.4, this isomorphism extends to an isomorphism ${}^p\mathbf{H}^{n+\dim \widetilde{\mathcal{A}}}(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}} \xrightarrow{\sim} {}^p\mathbf{H}^{2N-n+\dim \widetilde{\mathcal{A}}}(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}(N-n)$, which implies the required equality of dimensions.

(iii) Any perverse sheaf on $\widetilde{\mathcal{A}}'$, such as ${}^p\mathbf{H}^{n+\dim \widetilde{\mathcal{A}}}(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$, have all stalks lying in cohomological degrees greater than or equal to $-\dim \widetilde{\mathcal{A}}$. Since $\mathrm{Gr}^P_n \mathrm{H}^m(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}}$ is the $(m - n - \dim \widetilde{\mathcal{A}})$th cohomology of the stalk of ${}^p\mathbf{H}^{n+\dim \widetilde{\mathcal{A}}}(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ at $\widetilde{a}$, it is zero for $m < n$.

On the other hand, by (2), we also have $\dim \mathrm{Gr}^P_n \mathrm{H}^m(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}}) = \dim \mathrm{Gr}^P_{2N-n} \mathrm{H}^{m+2(N-n)}(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}}) = 0$ if $m + 2(N-n) > 2N$; i.e., $m > 2n$. $\square$

Remark 3.3.2. If either $k = \mathbb{C}$, or $k = \overline{\mathbb{F}_q}$ and all the relevant stacks are defined over $\mathbb{F}_q$, we have a weight filtration $W_{\leqslant n}\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}}$. Since $\mathcal{M}^{\mathrm{par}}_{\widetilde{a}}$ is proper, we have $\mathrm{H}^n(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} = W_{\leqslant n}\mathrm{H}^n(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}}$. Let $N_{\leqslant n}\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}}$ be the natural filtration on $\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}}$ given by $N_{\leqslant n}\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} = \bigoplus_{m \leqslant n} \mathrm{H}^m(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}}$. We have the following inclusions

$$P_{\leqslant n}\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} \hookleftarrow N_{\leqslant n}\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} \hookrightarrow W_{\leqslant n}\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}}.$$

However, there is no containment relation between $P_{\leqslant n}\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}}$ and $W_{\leqslant n}\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}}$ in either direction, as we will see in the example in § 3.4.

Remark 3.3.3. A similar filtration on the cohomology of stable Higgs bundles (induced from the perverse filtration on the Hitchin complex) is considered by de Cataldo *et al.* [dHM], in which they established an exciting connection between this perverse filtration and the weight filtration of the Betti-counterpart: the character variety (in the case $G = \mathrm{GL}(2), \mathrm{SL}(2)$ or $\mathrm{PGL}(2)$).

Theorems 3.1.2 and 3.2.4 directly translate into the following corollary.

COROLLARY 3.3.4 (Of Theorems 3.1.2 and 3.2.4). *Let $\widetilde{a} \in \widetilde{\mathcal{A}}'$ be a geometric point. Let n, m be integers such that $0 \leqslant n \leqslant m \leqslant \min\{2n, 2N\}$.*

(i) *There is a natural isomorphism*

$$v^m_n : \mathrm{Gr}^P_n H^m(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} \cong \mathrm{Gr}^P_{2N-n} H^{m+2(N-n)}(\mathcal{M}^{\mathrm{par},\vee}_{\widetilde{a}})_{\mathrm{st}}(N-n).$$

(ii) *For any $\lambda \in \mathbb{X}_*(T) = \mathbb{X}^*(T^\vee)$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Gr}^P_n H^m(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} \arrow[r, "v^m_n"] \arrow[d, "\mathrm{Sp}(-\lambda)"'] & \mathrm{Gr}^P_{2N-n} H^{m+2(N-n)}(\mathcal{M}^{\mathrm{par},\vee}_{\widetilde{a}})_{\mathrm{st}}(N-n) \arrow[d, "\mathrm{Ch}(\lambda)"] \\
\mathrm{Gr}^P_{n-1} H^m(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} \arrow[r, "v^m_{n-1}"] & \mathrm{Gr}^P_{2N-n+1} H^{m+2(N-n+1)}(\mathcal{M}^{\mathrm{par},\vee}_{\widetilde{a}})_{\mathrm{st}}(N-n+1)
\end{tikzcd}$$

Proof. By (3.7), we have

$$\mathrm{Gr}^P_n \mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} = {}^p\mathbf{H}^{n+\dim \widetilde{\mathcal{A}}}(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st},\widetilde{a}}[-n - \dim \widetilde{\mathcal{A}}] = K^{n-N}_{\widetilde{a}}[-n - \dim \widetilde{\mathcal{A}}](-d/2).$$

Similarly, we have

$$\mathrm{Gr}^P_{2N-n}\,\mathrm{H}^*(\mathcal{M}^{\mathrm{par},\vee}_{\widetilde{a}})_{\mathrm{st}} = L^{N-n}_{\widetilde{a}}[-2N+n-\dim\widetilde{\mathcal{A}}](-d/2).$$

The isomorphism $\widetilde{\mathrm{VD}'}^i$ in (3.3) then induces

$$\mathrm{Gr}^P_n\,\mathrm{H}^*(\mathcal{M}^{\mathrm{par}}_{\widetilde{a}})_{\mathrm{st}} \cong \mathrm{Gr}^P_{2N-n}\,\mathrm{H}^*(\mathcal{M}^{\mathrm{par},\vee}_{\widetilde{a}})_{\mathrm{st}}[2N-2n](N-n).$$

Taking the degree-m parts of both sides, we get the desired isomorphism v^m_n. The diagram in part (ii) follows from the outer square in (3.6), applied to $i=N-n$. □

3.4 An example

In this subsection, we verify the results in § 3.3 (up to a scalar) on the example we calculated in [Yun11, § 7.2]. More precisely, we take $G=\mathrm{SL}(2)$, $G^\vee=\mathrm{PGL}(2)$. The base curve $X=\mathbb{P}^1$ and the divisor is taken to be $\mathcal{O}(D)\cong\mathcal{O}_{\mathbb{P}^1}(2)$. Fixing an affine coordinate t for $\mathbb{P}^1\backslash\{\infty\}$, we can then identify $\mathcal{A}^{\mathrm{Hit}}=H^0(\mathbb{P}^1,\mathcal{O}(4))$ with polynomials in t of degree at most four. The universal cameral cover $\widetilde{\mathcal{A}}\to\mathcal{A}^{\mathrm{Hit}}\times X$ can be identified with the universal spectral curve Y.

Let $a_0=a_0(t)\in\widetilde{\mathcal{A}}$ be a polynomial with a double root (for example, $a_0(t)=t^2(t-1)$). The spectral curve Y_{a_0} is then a nodal curve of arithmetic genus 1. Let y_0 be the unique node of Y_{a_0} and $x_0\in\mathbb{P}^1$ be its projection.

3.4.1 *The perverse filtration.* We first consider the (enhanced) parabolic Hitchin fiber $\mathcal{M}_{a_0,y_0}=\mathcal{M}^{\mathrm{par}}_{a_0,y_0}$ (we suppress the superscript 'par' from the notations). As we have seen in [Yun11, § 7.1], $\mathcal{M}_{a_0,y_0}$ is topologically a union of two rational curves C_0,C_1 intersecting at two points. In particular, we can compute its Betti numbers:

$$h^0(\mathcal{M}_{a_0,y_0})=1,\quad h^1(\mathcal{M}_{a_0,y_0})=1,\quad h^2(\mathcal{M}_{a_0,y_0})=2.$$

In this example, the stable part $\mathrm{H}^*(\mathcal{M}_{a_0,y_0})_{\mathrm{st}}$ coincides with $\mathrm{H}^*(\mathcal{M}_{a_0,y_0})$.

The decomposition of $(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}}$ into shifted perverse sheaves reads

$$(\widetilde{f}_*\overline{\mathbb{Q}}_\ell)_{\mathrm{st}} = \overline{\mathbb{Q}}_\ell\oplus L[-1]\oplus\overline{\mathbb{Q}}_\ell(-1)[-2].$$

Here the degree-zero and degree-two parts are constant sheaves because they are middle extensions from an open dense subset of $U\subset\widetilde{\mathcal{A}}$ on which they are obviously constant. The degree-one part L is the middle extension of a rank-two local system on U (because the general fibers of $\widetilde{f}$ are elliptic curves).

Since $\mathrm{H}^2(\mathcal{M}_{a_0,y_0})$ is two-dimensional, and $\overline{\mathbb{Q}}_\ell(-1)[-2]$ only contributes one dimension to it, we must have $\dim\mathrm{H}^1(L_{a_0,y_0})=1$. Therefore, the perverse filtration on $\mathrm{H}^*(\mathcal{M}_{a_0,y_0})$ has dimensions,

$$\dim\mathrm{Gr}^P_0\,\mathrm{H}^*(\mathcal{M}_{a_0,y_0})=1,\quad \dim\mathrm{Gr}^P_1\,\mathrm{H}^*(\mathcal{M}_{a_0,y_0})=2,\quad \dim\mathrm{Gr}^P_2\,\mathrm{H}^*(\mathcal{M}_{a_0,y_0})=1.$$

Note that $\mathrm{H}^i(\mathcal{M}_{a_0,y_0})$ has weight $0,0,2$ for $i=0,1,2$ respectively. This shows that there is no containment relation in either direction between the perverse filtration and the weight filtration on $\mathrm{H}^*(\mathcal{M}_{a_0,y_0})$.

3.4.2 *The Springer action.* Since the global Springer action preserves the perverse filtration, the one-dimensional subspace $\mathrm{Gr}^P_1\,\mathrm{H}^2(\mathcal{M}_{a_0,y_0})\subset\mathrm{H}^2(\mathcal{M}_{a_0,y_0})$ is invariant under the unipotent action of the lattice part of $\widetilde{W}$. Let $\{\gamma_0,\gamma_1\}$ be the basis of $\mathrm{H}^2(\mathcal{M}_{a_0,y_0})$ dual to the fundamental classes $\{[C_0],[C_1]\}$. By the calculation in [Yun11, § 7.2], $\gamma_0-\gamma_1$ spans the unique eigenspace of the action of $\alpha^\vee \in \mathbb{X}_*(T) \subset \widetilde{W}$. Hence

$$\mathrm{Gr}_1^P \mathrm{H}^2(\mathcal{M}_{a_0,y_0}) = \mathrm{Span}\{\gamma_0 - \gamma_1\} \subset \mathrm{Span}\{\gamma_0, \gamma_1\}.$$

From the matrix of the $\alpha^\vee$-action in [Yun11, § 7.2], the 'subdiagonal' part of it gives an isomorphism

$$\mathrm{Sp}(\alpha^\vee) : \mathrm{Gr}_2^P \mathrm{H}^2(\mathcal{M}_{a_0,y_0}) \xrightarrow{\sim} \mathrm{Gr}_1^P \mathrm{H}^2(\mathcal{M}_{a_0,y_0}). \tag{3.8}$$

3.4.3 *The Chern class action.* Now we consider the parabolic Hitchin fibers for $G^\vee = \mathrm{PGL}(2)$, with the same choice of $X = \mathbb{P}^1$, D and $(a_0, y_0) \in \widetilde{\mathcal{A}}$. The parabolic Hitchin fiber $\mathcal{M}^\vee_{a_0,y_0}$ (for $\mathrm{PGL}(2)$) classifies pairs $(\mathcal{F} \supset \mathcal{F}')$ up to tensoring line bundles from X. Here $\mathcal{F}, \mathcal{F}'$ are torsion-free rank-one coherent sheaves on Y_{a_0}, such that $\mathcal{F}/\mathcal{F}'$ is of length 1 supported on $y_0 \in Y_{a_0}$. Therefore, $\mathcal{M}^\vee_{a_0,y_0}$ decomposes as the disjoint union of $\mathcal{M}^{\vee,\mathrm{ev}}_{a_0,y_0}$ and $\mathcal{M}^{\vee,\mathrm{od}}_{a_0,y_0}$ according to the parity of $\deg \mathcal{F}$. The stable part $\mathrm{H}^*(\mathcal{M}^\vee_{a_0,y_0})_{\mathrm{st}}$ can be identified with either $\mathrm{H}^*(\mathcal{M}^{\vee,\mathrm{ev}}_{a_0,y_0})$ or $\mathrm{H}^*(\mathcal{M}^{\vee,\mathrm{od}}_{a_0,y_0})$ via the projections to the direct summands:

$$\mathrm{H}^*(\mathcal{M}^\vee_{a_0,y_0})_{\mathrm{st}} \subset \mathrm{H}^*(\mathcal{M}^\vee_{a_0,y_0}) = \mathrm{H}^*(\mathcal{M}^{\vee,\mathrm{ev}}_{a_0,y_0}) \oplus \mathrm{H}^*(\mathcal{M}^{\vee,\mathrm{od}}_{a_0,y_0}).$$

The homomorphism $\mathrm{SL}(2) \to \mathrm{PGL}(2)$ induces a canonical morphism $\mathcal{M}_{a_0,y_0} \to \mathcal{M}^{\vee,\mathrm{ev}}_{a_0,y_0}$, which induces an isomorphism on coarse moduli spaces. Therefore, $\mathcal{M}^{\vee,\mathrm{ev}}_{a_0,y_0}$ also consists of two rational curves C_0, C_1 intersecting at two points.

The line bundle $\mathcal{L}(-\alpha^\vee)$ on $\mathcal{M}^{\vee,\mathrm{ev}}_{a_0,y_0}$ corresponding to the root $-\alpha^\vee \in \mathbb{X}^*(T^\vee)$ assigns to each $(\mathcal{F} \supset \mathcal{F}')$ the line $\mathcal{F}/\mathcal{F}'$. On the C_0 component, $\mathcal{F}'$ is fixed, and hence $\mathcal{L}(-\alpha^\vee)|_{C_0} \cong \mathcal{O}_{C_0}(-1)$. On the C_1 component, $\mathcal{F}$ is fixed, and hence $\mathcal{L}(-\alpha^\vee)|_{C_1} \cong \mathcal{O}_{C_1}(1)$. Here we identify C_0 and C_1 with $\mathbb{P}^1$. Therefore,

$$c_1(\mathcal{L}(-\alpha^\vee)) = \gamma_1 - \gamma_0 \in \mathrm{H}^2(\mathcal{M}^\vee_{a_0,y_0})_{\mathrm{st}} \cong \mathrm{H}^2(\mathcal{M}^{\vee,\mathrm{ev}}_{a_0,y_0}).$$

The Chern class action then gives an isomorphism

$$\mathrm{Ch}(-\alpha^\vee) : \mathrm{Gr}_0^P \mathrm{H}^0(\mathcal{M}^\vee_{a_0,y_0})_{\mathrm{st}} \to \mathrm{Gr}_1^P \mathrm{H}^2(\mathcal{M}^\vee_{a_0,y_0})_{\mathrm{st}}. \tag{3.9}$$

Combining (3.8) and (3.9), we have verified the commutative diagram in Corollary 3.3.4 up to a scalar. The calculation can be summarized into the following diagram, where all the arrows are isomorphisms.

$$\begin{tikzcd}
\mathrm{H}^*(\mathcal{M}_{a_0,y_0})_{\mathrm{st}} : & \mathrm{Gr}_0^P \mathrm{H}^0 \arrow[drrr] & \mathrm{Gr}_1^P \mathrm{H}^1 \arrow[d] & \mathrm{Gr}_1^P \mathrm{H}^2 \arrow[d] & \mathrm{Gr}_2^P \mathrm{H}^2 \arrow[l, bend right, "\mathrm{Sp}(\alpha^\vee)"'] \arrow[dlll] \\
\mathrm{H}^*(\mathcal{M}^\vee_{a_0,y_0})_{\mathrm{st}} : & \mathrm{Gr}_0^P \mathrm{H}^0 \arrow[rr, bend right, "\mathrm{Ch}(-\alpha^\vee)"'] & \mathrm{Gr}_1^P \mathrm{H}^1 & \mathrm{Gr}_1^P \mathrm{H}^2 & \mathrm{Gr}_2^P \mathrm{H}^2
\end{tikzcd}$$

4. Proof of Theorem A

4.1 Preliminaries on the Picard stack

Construction 4.1.1. Let S be a scheme and $a \in \mathcal{A}^{\mathrm{Hit}}(S)$. Recall from [Ngo10, § 4.3.1] that we have the regular centralizer group scheme J_a over $X \times S$. We also have the cameral curve $X_a = S \times_{\mathcal{A}} \widetilde{\mathcal{A}}$, which is a branched W-cover of $X \times S$. For any J_a-torsor Q^J over $S \times X$, the pull-back $q_a^* Q^J$ is a $q_a^* J_a$-torsor over the cameral curve X_a. By [Ngo10, Proposition 2.4.2], we have a natural homomorphism of group schemes over X_a:

$$j_a : q_a^* J_a \to T \times X_a.$$

Therefore, we can form the induced T-torsor $Q^T := q_a^* Q^J \overset{q_a^* J_a, j_a}{\times} T$ over X_a, which carries a strong W-equivariant structure (W acts on both X_a and the group T). For the notion of strong W-equivariance, we refer to [DG02, Definition 5.7]. Let $\mathcal{P}\mathrm{ic}_T(\widetilde{\mathcal{A}}/\mathcal{A}^{\mathrm{Hit}})^W$ be the stack over $\mathcal{A}^{\mathrm{Hit}}$ whose points over $a \in \mathcal{A}^{\mathrm{Hit}}(S)$ comprise the category of strong W-equivariant T-torsors on X_a. The assignment $Q^J \mapsto Q^T$ gives a morphism of Picard stacks over $\mathcal{A}^{\mathrm{Hit}}$ and $\widetilde{\mathcal{A}}$:

$$j_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}\mathrm{ic}_T(\widetilde{\mathcal{A}}/\mathcal{A}^{\mathrm{Hit}})^W. \tag{4.1}$$

Let $j_{\widetilde{\mathcal{P}}}$ be the base change of $j_{\mathcal{P}}$ to from $\mathcal{A}^{\mathrm{Hit}}$ to $\widetilde{\mathcal{A}}$. Over $\widetilde{\mathcal{A}} \times_{\mathcal{A}^{\mathrm{Hit}}} \mathcal{P}\mathrm{ic}_T(\widetilde{\mathcal{A}}/\mathcal{A}^{\mathrm{Hit}})^W$ we have the universal T-torsor Poin^T. Then we define a tautological T-torsor over $\widetilde{\mathcal{P}} = \mathcal{P} \times_{\mathcal{A}^{\mathrm{Hit}}} \widetilde{\mathcal{A}}$:

$$\mathcal{Q}^T := j_{\widetilde{\mathcal{P}}}^* \,\mathrm{Poin}^T .$$

In particular, we get the associated line bundle $\mathcal{Q}(\lambda)$ on $\widetilde{\mathcal{P}}$ for each $\lambda \in \mathbb{X}^*(T)$.

LEMMA 4.1.2. *The restriction of the morphism $j_{\mathcal{P}}$ in (4.1) to $\mathcal{A}$ is an isogeny (i.e., the kernel and cokernel of $j_{\mathcal{P}}$ are finite over $\mathcal{A}$).*

Proof. Fix $a \in \mathcal{A}(S)$. We need a description of $\mathcal{P}_a$ due to Donagi and Gaitsgory in [DG02, § 16.3], which we briefly recall here. For each root $\alpha \in \Phi$, let $D_\alpha \subset X_a$ be the divisor given by the pull-back of the wall $\mathfrak{t}_{\alpha,D} \subset \mathfrak{t}_D$. In other words, D_α is the fixed point locus of the action of the reflection $r_\alpha \in W$ on X_a. For each object $Q^T \in \mathcal{P}\mathrm{ic}_T(X_a)^W(S)$, the r_α-equivariant structure of Q^T gives an isomorphism of T-torsors over D_α:

$$Q^T|_{D_\alpha} \overset{T, r_\alpha}{\times} T \cong Q^T|_{D_\alpha}.$$

Therefore, spelling out the action of r_α on T, we get an isomorphism of T-torsors over D_α:

$$(Q^T|_{D_\alpha} \overset{T,\alpha}{\times} \mathbb{G}_m) \overset{\mathbb{G}_m, \alpha^\vee}{\times} T \cong T \times D_\alpha. \tag{4.2}$$

The result of Donagi and Gaitsgory in [DG02, § 16.3] says that $\mathcal{P}_a(S)$ is the Picard groupoid of tuples $(Q^T, \{\gamma_w\}_{w\in W}, \{\beta_\alpha\}_{\alpha\in\Phi})$ where $(Q^T, \{\gamma_w\}_{w\in W})$ is a strongly W-equivariant T-torsor on X_a and β_α is a trivialization of the $\mathbb{G}_m$-torsor $Q(\alpha)|_{D_\alpha} := Q^T|_{D_\alpha} \overset{T,\alpha}{\times} \mathbb{G}_m$, which is compatible with the trivialization (4.2) and the W-equivariant structure; i.e., γ_w sends the trivialization β_α to the trivialization $\beta_{w(\alpha)}$.

We give a reformulation of their result. For each $\alpha \in \Phi$, let

$$\mu_\alpha := \ker(\mathbb{G}_m \xrightarrow{\alpha^\vee} T).$$

This is either the trivial group or the group μ_2, depending on whether $\alpha^\vee$ is a primitive element of $\mathbb{X}_*(T)$ or not. For $Q^T \in \mathcal{P}\mathrm{ic}_T(X_a)^W(S)$, by the trivialization (4.2), the $\mathbb{G}_m$-torsor $Q(\alpha)|_{D_\alpha}$ in fact comes from a unique μ_α-torsor Q^{μ_α} over D_α (if $\mathrm{char}(k) \neq 2$). An object in $\mathcal{P}_a(S)$ is just an object $(Q^T, \{\gamma_w\}_{w\in W})$ in $\mathcal{P}\mathrm{ic}_T(X_a)^W(S)$ together with a trivialization of the μ_α torsor Q^{μ_α} over D_α for each $\alpha \in \Phi$, compatible with the W-equivariant structure of Q^T. Since the above discussion works for any test scheme S, we get an exact sequence of Picard stacks

$$\left(\prod_{\alpha\in\Phi} \mathrm{Res}_{\widetilde{\mathcal{A}}_\alpha/\mathcal{A}}(\mu_\alpha \times \widetilde{\mathcal{A}}_\alpha)\right)^W \to \mathcal{P} \xrightarrow{j_{\mathcal{P}}} \mathcal{P}\mathrm{ic}_T(\widetilde{\mathcal{A}}/\mathcal{A})^W \to \left(\prod_{\alpha\in\Phi} \mathcal{P}\mathrm{ic}_{\mu_\alpha}(\widetilde{\mathcal{A}}_\alpha/\mathcal{A})\right)^W. \tag{4.3}$$

Here $\widetilde{\mathcal{A}}_\alpha \subset \widetilde{\mathcal{A}}$ is the pull-back of $\mathfrak{t}_{\alpha,D}$, and the last arrow in (4.3) sends $Q^T \in \mathcal{P}\mathrm{ic}_T(X_a)^W(S)$ to the μ_α-torsor Q^{μ_α} over $D_\alpha = S \times_{\mathcal{A}} \widetilde{\mathcal{A}}_\alpha$.

Since $\widetilde{\mathcal{A}}_\alpha$ is finite over $\mathcal{A}$, and μ_α is a finite group scheme, the two ends of the sequence (4.3) are finite Picard stacks. Therefore, $j_{\mathcal{P}}$ is an isogeny. $\square$

By Construction 4.1.1, $\mathcal{Q}^T$ gives a classifying morphism $\widetilde{\mathcal{P}} \to \mathbb{B}T$; we also have a natural map $\mathcal{M}^{\mathrm{par}} \to \mathbb{B}T$ given by the tautological T-torsor $\mathcal{L}^T$ over $\mathcal{M}^{\mathrm{par}}$.

LEMMA 4.1.3. *There is a natural 2-morphism making the following diagram commutative.*

$$\begin{tikzcd} \widetilde{\mathcal{P}} \times_{\widetilde{\mathcal{A}}} \mathcal{M}^{\mathrm{par}} \arrow[r, "\mathrm{act}"] \arrow[d, "\mathcal{Q}^T \times \mathcal{L}^T"] & \mathcal{M}^{\mathrm{par}} \arrow[d, "\mathcal{L}^T"] \\ \mathbb{B}T \times \mathbb{B}T \arrow[r, "\mathrm{mult}"] & \mathbb{B}T \end{tikzcd} \tag{4.4}$$

Here 'act' *is the action map and* 'mult' *stands for the multiplication on the Picard stack* $\mathbb{B}T$ *induced from the multiplication on* T.

Proof. Let $a \in \mathcal{A}^{\mathrm{Hit}}(S)$, $Q^J \in \mathcal{P}_a(S)$ be a J_a-torsor over $S \times X$ and $(x, \mathcal{E}, \varphi, \mathcal{E}^B_x) \in \mathcal{M}^{\mathrm{par}}(S)$ be a point over a, which also gives a point $\widetilde{x} \in X_a(S)$. By Construction 4.1.1, the fiber of $\mathcal{Q}^T$ over the point $(\widetilde{x}, Q^J) \in \widetilde{\mathcal{P}}(S)$ is the T-torsor

$$Q^T_{\widetilde{x}} := \widetilde{x}^* Q^J \overset{\widetilde{x}^* J_a, j}{\times} T$$

over $S = \Gamma(\widetilde{x}) \subset X_a$. Here $j : \widetilde{x}^* J_a \to T$ is induced from j_T.

On the other hand, the fiber of $\mathcal{L}^T$ over the point $(x, \mathcal{E}, \varphi, \mathcal{E}^B_x)$ is the T-torsor $\mathcal{E}^T_x := \mathcal{E}^B_x \overset{B}{\times} T$ over $\Gamma(x)$. By the $\mathcal{P}$-action on $\mathcal{M}^{\mathrm{par}}$ given in [Yun11, Lemma 2.3.3], after twisting by Q^J, the T-torsor $\mathcal{E}^T_x$ becomes the T-torsor,

$$(\widetilde{x}^* Q^J \overset{\widetilde{x}^* J_a, j_B}{\times} \mathcal{E}^B_x) \overset{B}{\times} T \cong \widetilde{x}^* Q^J \overset{\widetilde{x}^* J_a, j}{\times} \mathcal{E}^T_x = Q^T_{\widetilde{x}} \overset{T}{\times} \mathcal{E}^T_x,$$

which is precisely the product of $Q^T_{\widetilde{x}}$ (the fiber of $\mathcal{Q}^T$ at $(\widetilde{x}, Q^J)$) and $\mathcal{E}^T_x$ (the fiber of $\mathcal{L}^T$ at $(x, \mathcal{E}, \varphi, \mathcal{E}^B_x)$) under the multiplication on $\mathbb{B}T$. This completes the proof. $\square$

4.2 The locus $\mathcal{A}^\diamondsuit$

Let $\mathcal{A}^\diamondsuit \subset \mathcal{A}$ be the open dense locus where the cameral curves X_a are transversal to the discriminant divisor in $\mathfrak{t}_D$ (see [Ngo10, § 4.7]). Let $\widetilde{\mathcal{A}}^\diamondsuit = \widetilde{\mathcal{A}} \times_{\mathcal{A}} \mathcal{A}^\diamondsuit$. We collect a few facts about $\mathcal{A}^\diamondsuit$.

- By [Ngo10, Corollaire 4.7.4], for $a \in \mathcal{A}^\diamondsuit$, the cameral curve X_a is smooth and connected.
- By [Ngo10, Proposition 4.7.7], the neutral component $\mathcal{P}^0|_{\mathcal{A}^\diamondsuit}$ is the quotient of an abelian scheme over $\mathcal{A}^\diamondsuit$ by ZG via the trivial action. The Hitchin moduli $\mathcal{M}^{\mathrm{Hit}}|_{\mathcal{A}^\diamondsuit}$ is a torsor under $\mathcal{P}|_{\mathcal{A}^\diamondsuit}$.
- For $a \in \mathcal{A}^\diamondsuit$, the global delta invariant $\delta(a) = 0$ (because $\delta(a)$ is the dimension of the affine part of $\mathcal{P}_a$); in particular, $\mathcal{A}^\diamondsuit \subset \mathcal{A}_0 \subset \mathcal{A}'$. This also implies that the local delta invariants $\delta(a, x) = 0$ for any $x \in X$.
- By [Yun11, Lemma 2.6.1] and the fact that $\delta(a, x) = 0$, we have

$$\mathcal{M}^{\mathrm{par}}|_{\widetilde{\mathcal{A}}^\diamondsuit} \xrightarrow{\sim} \mathcal{M}^{\mathrm{Hit}}|_{\mathcal{A}} \times_{\mathcal{A}} \widetilde{\mathcal{A}}^\diamondsuit.$$

Recall the notion of the Tate module $V_\ell(\mathcal{P}^0/\mathcal{A}^\diamondsuit)$ from § A.1. By Lemma A.3.4, to understand the stable parts (see Definition A.3.1) of the homology $\mathbf{H}_*(\mathcal{P}/\mathcal{A}^\diamondsuit)$, we need only to describe $V_\ell(\mathcal{P}^0/\mathcal{A}^\diamondsuit)$.

Lemma 4.2.1. *There are natural isomorphisms of local systems on $\mathcal{A}^\diamondsuit$:*

$$V_\ell(\mathcal{P}^0/\mathcal{A}^\diamondsuit) \cong (\mathbf{H}_1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \otimes_{\mathbb{Z}} \mathbb{X}_*(T))^W, \tag{4.5}$$

$$V_\ell(\mathcal{P}^{\vee,0}/\mathcal{A}^\diamondsuit)^* \cong (\mathbf{H}^1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \otimes_{\mathbb{Z}} \mathbb{X}^*(T^\vee))_W, \tag{4.6}$$

where the right-hand side are the invariants and coinvariants of the diagonal W-action.

Proof. We prove the first isomorphism only. By Lemma 4.1.2, $j_{\mathcal{P}}$ is an isogeny over $\mathcal{A}$ and hence, in particular, over $\mathcal{A}^\diamondsuit$. Over $\mathcal{A}^\diamondsuit$, the neutral components of both $\mathcal{P}$ and $\mathcal{P}ic_T(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)$ are isogenous abelian schemes, therefore $j_{\mathcal{P}}$ induces an isomorphisms on the $\overline{\mathbb{Q}}_\ell$-Tate modules of the neutral components:

$$V_\ell(j_{\mathcal{P}}) : V_\ell(\mathcal{P}^0/\mathcal{A}^\diamondsuit) \cong V_\ell(\mathcal{P}ic^0_T(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)^W/\mathcal{A}^\diamondsuit). \tag{4.7}$$

Here $\mathcal{P}ic^0_T(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)$ is the neutral component of $\mathcal{P}ic_T(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)$. Note however that the fibers of $\mathcal{P}ic^0_T(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)^W$ are not necessarily connected; by Tate modules we mean the Tate modules of their neutral components.

For any geometric point $a \in \mathcal{A}$, the automorphisms of objects in $\mathcal{P}ic_T(X_a)^W$ are $T(X_a)^W = T^W$ which is finite (here we use the fact that X_a is connected, see [Ngo10, Proposition 4.6.1]). Therefore, $\mathcal{P}ic_T(\widetilde{\mathcal{A}}/\mathcal{A})^W$ is a Deligne–Mumford stack, and it is harmless to replace $\mathcal{P}ic(\widetilde{\mathcal{A}}/\mathcal{A})^W$ by its coarse moduli space $\mathrm{Pic}(\widetilde{\mathcal{A}}/\mathcal{A})^W$ in calculating its Tate module.

For the abelian scheme $\mathrm{Pic}^0_T(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)$, taking W-invariants commutes with taking $\overline{\mathbb{Q}}_\ell$-Tate modules:

$$V_\ell(\mathrm{Pic}^0_T(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)^W/\mathcal{A}^\diamondsuit) \cong (V_\ell(\mathrm{Pic}^0(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)) \otimes_{\mathbb{Z}} \mathbb{X}_*(T))^W. \tag{4.8}$$

For $a \in \mathcal{A}^\diamondsuit$, the cameral curve X_a is smooth, and the Abel–Jacobi map $\widetilde{\mathcal{A}}^\diamondsuit \to \mathrm{Pic}^1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)$ gives a canonical isomorphism

$$\mathbf{H}_1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \cong \mathbf{H}_1(\mathrm{Pic}^1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)).$$

Moreover, we can canonically identify $\mathbf{H}_1(\mathrm{Pic}^1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit))$ with $\mathbf{H}_1(\mathrm{Pic}^0(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)) = V_\ell(\mathrm{Pic}^0(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit))$. In fact, for any étale morphism $S \to \mathcal{A}^\diamondsuit$ and any lifting $\widetilde{s} : S \to \widetilde{\mathcal{A}}^\diamondsuit$ of s, we can use $\widetilde{s}$ to identify $\mathrm{Pic}^0(\widetilde{\mathcal{A}}^\diamondsuit/S)$ and $\mathrm{Pic}^1(\widetilde{\mathcal{A}}^\diamondsuit/S)$. The induced isomorphism on $\mathbf{H}_1(-)$ is independent of the choice of the lifting $\widetilde{s}$ because any two such liftings differ by a translation of $\mathrm{Pic}^0(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)(S)$ and $\mathrm{Pic}^0(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)$ has connected fibers (cf. the argument in Lemma 4.3.1). Therefore, the local identifications between $\mathbf{H}_1(\mathrm{Pic}^1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit))$ and $V_\ell(\mathrm{Pic}^0(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit))$ glue to give a global canonical identification

$$\mathbf{H}_1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \cong V_\ell(\mathrm{Pic}^0(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)). \tag{4.9}$$

The isomorphisms (4.8) and (4.9) give a canonical isomorphism of local systems on $\mathcal{A}^\diamondsuit$:

$$V_\ell(\mathcal{P}ic^0_T(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)^W/\mathcal{A}^\diamondsuit) \cong (\mathbf{H}_1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \otimes_{\mathbb{Z}} \mathbb{X}_*(T))^W.$$

This, together with the isomorphism (4.7), implies the isomorphism (4.5). □

4.3 Proof of Theorem 3.1.2

Using the global Kostant section (see [Ngo10, § 4.2.4]) $\epsilon : \mathcal{A} \to \mathcal{M}^{\mathrm{Hit}}_G$ (respectively $\epsilon^\vee : \mathcal{A} \to \mathcal{M}^{\mathrm{Hit}}_{G^\vee}$), we get a morphism $\tau : \mathcal{P} \to \mathcal{M}^{\mathrm{Hit}}_G$ (respectively $\tau^\vee : \mathcal{P}^\vee \to \mathcal{M}^{\mathrm{Hit}}_{G^\vee}$). Although τ is an isomorphism over $\mathcal{A}^{\diamondsuit}$, this isomorphism depends on the choice of the section ϵ. Fortunately, if we restrict ourselves to the stable parts of the homology or cohomology, and passing to cohomology sheaves, we will get a *canonical* isomorphism.

Lemma 4.3.1. *For each $i \in \mathbb{Z}$, there are* canonical *isomorphisms*

$$\mathbf{H}_i(\mathcal{M}^{\mathrm{Hit}}_G/\mathcal{A}^{\diamondsuit})_{\mathrm{st}} \cong \mathbf{H}_i(\mathcal{P}/\mathcal{A}^{\diamondsuit})_{\mathrm{st}},$$
$$\mathbf{H}^i(\mathcal{M}^{\mathrm{Hit}}_{G^\vee}/\mathcal{A}^{\diamondsuit})_{\mathrm{st}} \cong \mathbf{H}^i(\mathcal{P}^\vee/\mathcal{A}^{\diamondsuit})_{\mathrm{st}}.$$

Proof. We prove the first isomorphism. For any étale map $a: S \to \mathcal{A}^{\diamondsuit}$, and any lifting $m: S \to \mathcal{M}^{\mathrm{Hit}}_G$ of a, we get a trivialization of the $\mathcal{P}_a$-torsor:

$$\widetilde{m}: \mathcal{P}_a \xrightarrow{\sim} \mathcal{M}^{\mathrm{Hit}}_{G,a}.$$

Hence we get an isomorphism

$$\widetilde{m}_{!,\mathrm{st}}: \mathrm{H}_i(\mathcal{P}_a)_{\mathrm{st}} \cong \mathrm{H}_i(\mathcal{M}^{\mathrm{Hit}}_{G,a})_{\mathrm{st}}. \tag{4.10}$$

If we choose another lifting $m': S \to \mathcal{M}^{\mathrm{Hit}}_G$, then m and m' differ by the translation of a section $\varpi: S \to \mathcal{P}$, and the two isomorphisms $\widetilde{m}_{!,\mathrm{st}}$ and $\widetilde{m}'_{!,\mathrm{st}}$ differ by the action of $\varpi \in \mathcal{P}_a(S)$ on $\mathrm{H}_i(\mathcal{M}^{\mathrm{Hit}}_{G,a})_{\mathrm{st}}$. By the homotopy invariance of actions on cohomology (see [LN08, Lemme 3.2.3]), the action of $\mathcal{P}_a(S)$ on $\mathrm{H}_i(\mathcal{M}^{\mathrm{Hit}}_{G,a})$ factors through $\pi_0(\mathcal{P}_a)(S)$. By definition, $\pi_0(\mathcal{P}_a)$ acts trivially on the stable part of $\mathrm{H}_*(\mathcal{M}^{\mathrm{Hit}}_{G,a})$, and the action of ϖ on $\mathrm{H}_i(\mathcal{M}^{\mathrm{Hit}}_{G,a})_{\mathrm{st}}$ is trivial. Therefore, different choices of local sections give rise to the same isomorphism as in (4.10), hence the canonicity. □

Proof of Theorem 3.1.2. Recall that $\widetilde{\mathcal{A}}^{\diamondsuit}$ is the preimage of $\mathcal{A}^{\diamondsuit}$ in $\widetilde{\mathcal{A}}$, and let $K_{\diamondsuit}$ and $L_{\diamondsuit}$ be the restrictions of K and L to $\widetilde{\mathcal{A}}^{\diamondsuit}$. By Lemma 2.2.4, both $\mathbb{D}K^i$ and L^i are middle extensions from $\widetilde{\mathcal{A}}^{\diamondsuit}$, and therefore it suffices to establish a canonical isomorphism $\mathbb{D}K^i_{\diamondsuit} \cong L^i_{\diamondsuit}(i)$.

Let $\widetilde{p}: \widetilde{\mathcal{A}}^{\diamondsuit} \to \mathcal{A}^{\diamondsuit}$ be the natural projection, which is smooth and proper. Since $\mathcal{M}^{\mathrm{par}}_G|_{\widetilde{\mathcal{A}}^{\diamondsuit}} \xrightarrow{\sim} \mathcal{M}^{\mathrm{Hit}}|_{\mathcal{A}^{\diamondsuit}} \times_{\mathcal{A}^{\diamondsuit}} \widetilde{\mathcal{A}}^{\diamondsuit}$ and $\mathcal{M}^{\mathrm{Hit}}_G|_{\mathcal{A}^{\diamondsuit}}$ is a torsor under the abelian stack $\mathcal{P}|_{\mathcal{A}^{\diamondsuit}}$ (see the beginning of § 4.2), the complex $K_{\diamondsuit}$ is a sum of shifted local systems. In particular, $K^i_{\diamondsuit} = \mathbf{H}^{i-\dim\widetilde{\mathcal{A}}}(\widetilde{f}_*\overline{\mathbb{Q}}_\ell[d](d/2)|_{\widetilde{\mathcal{A}}^{\diamondsuit}})_{\mathrm{st}} = \mathbf{H}^{i+N}(\mathcal{M}^{\mathrm{par}}_G/\widetilde{\mathcal{A}}^{\diamondsuit})_{\mathrm{st}}(d/2)$, where $N = d - \dim\widetilde{\mathcal{A}}$ is the relative dimension of $\widetilde{f}$ (and $\widetilde{f}^\vee$). Dualizing the above statement, we get a canonical isomorphism

$$\mathbb{D}K^i_{\diamondsuit} \cong \mathbf{H}_{i+N}(\mathcal{M}^{\mathrm{par}}_G/\widetilde{\mathcal{A}}^{\diamondsuit})_{\mathrm{st}}(d/2-N) = \widetilde{p}^*\mathbf{H}_{i+N}(\mathcal{M}^{\mathrm{Hit}}_G/\mathcal{A}^{\diamondsuit})_{\mathrm{st}}(d/2-N). \tag{4.11}$$

A similar remark applies to $\mathcal{M}^{\mathrm{par}}_{G^\vee}|_{\mathcal{A}^{\diamondsuit}}$, and we get

$$L^i_{\diamondsuit} \cong \mathbf{H}^{i+N}(\mathcal{M}^{\mathrm{par}}_{G^\vee}/\widetilde{\mathcal{A}}^{\diamondsuit})_{\mathrm{st}}(d/2) = \widetilde{p}^*\mathbf{H}^{i+N}(\mathcal{M}^{\mathrm{Hit}}_{G^\vee}/\mathcal{A}^{\diamondsuit})_{\mathrm{st}}(d/2). \tag{4.12}$$

By the isomorphisms in Lemmas A.3.4, and 4.3.1, we have natural isomorphisms

$$\mathbf{H}_*(\mathcal{M}^{\mathrm{Hit}}_G/\mathcal{A}^{\diamondsuit})_{\mathrm{st}} \cong \mathbf{H}_*(\mathcal{P}/\mathcal{A}^{\diamondsuit})_{\mathrm{st}} \cong \bigwedge(V_\ell(\mathcal{P}^0/\mathcal{A}^{\diamondsuit})[1]), \tag{4.13}$$

$$\mathbf{H}^*(\mathcal{M}^{\mathrm{Hit}}_{G^\vee}/\mathcal{A}^{\diamondsuit})_{\mathrm{st}} \cong \mathbf{H}^*(\mathcal{P}^\vee/\mathcal{A}^{\diamondsuit})_{\mathrm{st}} \cong \bigwedge(V_\ell(\mathcal{P}^{\vee,0}/\mathcal{A}^{\diamondsuit})^*[-1]). \tag{4.14}$$

Combining these with the isomorphisms (4.11) and (4.12), we get

$$\mathbb{D}K^i_{\diamondsuit} \cong \widetilde{p}^* \bigwedge^{i+N}(V_\ell(\mathcal{P}^0/\mathcal{A}^{\diamondsuit}))(d/2-N), \tag{4.15}$$

$$L^i_{\diamondsuit} \cong \widetilde{p}^* \bigwedge^{i+N}(V_\ell(\mathcal{P}^{\vee,0}/\mathcal{A}^{\diamondsuit})^*)(d/2). \tag{4.16}$$

Therefore, to prove the theorem, it suffices to give a natural isomorphism of local systems on $\mathcal{A}^\diamondsuit$:

$$\beta : V_\ell(\mathcal{P}^0/\mathcal{A}^\diamondsuit) \cong V_\ell(\mathcal{P}^{\vee,0}/\mathcal{A}^\diamondsuit)^*(1). \tag{4.17}$$

Recall from [Ngo10, Proposition 4.6.1] that the cameral curves X_a are connected provided $\deg(D) > 2g_X$. The cup product for the smooth connected projective family of curves $\widetilde{\mathcal{A}}^\diamondsuit \to \mathcal{A}^\diamondsuit$ gives a perfect pairing

$$\mathbf{H}^1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \otimes \mathbf{H}^1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \xrightarrow{\cup} \mathbf{H}^2(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \cong \overline{\mathbb{Q}}_\ell(-1).$$

Therefore, we have a natural isomorphism of local systems

$$\mathrm{PD} : \mathbf{H}_1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \cong \mathbf{H}^1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)(1). \tag{4.18}$$

Since $\mathbb{X}_*(T) = \mathbb{X}^*(T^\vee)$, we can define a natural isomorphism

$$\begin{aligned} (\mathbf{H}^1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)(1) \otimes_{\mathbb{Z}} \mathbb{X}^*(T^\vee))_W &\xrightarrow{\sim} (\mathbf{H}_1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \otimes_{\mathbb{Z}} \mathbb{X}_*(T))^W \\ h \otimes \lambda &\mapsto \sum_{w \in W} w_* \mathrm{PD}^{-1}(h) \otimes {}^w\lambda. \end{aligned} \tag{4.19}$$

Using the isomorphisms in Lemma 4.2.1, this isomorphism induces the desired isomorphism (4.17). This completes the proof of the theorem. $\square$

4.4 Some reductions towards the Proof of Theorem 3.2.4

The commutativity of the left square follows from Remark 3.1.3. Therefore, it remains to prove the commutativity of the right square. We make the following simple observation about extensions of perverse sheaves.

Lemma 4.4.1. *Suppose $j : U \hookrightarrow Y$ is the inclusion of a Zariski open subset of the scheme Y and $\mathcal{F}_1, \mathcal{F}_2$ are perverse sheaves on U. Let $j_{!*}\mathcal{F}_1$ and $j_{!*}\mathcal{F}_2$ be the middle extension perverse sheaves on Y. Then the restriction map*

$$\mathrm{Ext}^1_Y(j_{!*}\mathcal{F}_1, j_{!*}\mathcal{F}_2) \to \mathrm{Ext}^1_U(\mathcal{F}_1, \mathcal{F}_2)$$

is injective.

Proof. Let $i : Z = Y - U \hookrightarrow Y$ be the closed embedding of the complement of U into Y. We have a long exact sequence

$$\to \mathrm{Hom}_Z(i^* j_{!*}\mathcal{F}_1, i^! j_{!*}\mathcal{F}_2[1]) \to \mathrm{Hom}_Y(j_{!*}\mathcal{F}_1, j_{!*}\mathcal{F}_2[1]) \xrightarrow{j^*} \mathrm{Hom}_U(\mathcal{F}_1, \mathcal{F}_2[1]) \to . \tag{4.20}$$

By the definition of $j_{!*}$, we have $i^* j_{!*}\mathcal{F}_1 \in {}^pD^{\leqslant -1}(Z)$ and $i^! j_{!*}\mathcal{F}_2[1] \in {}^pD^{\geqslant 1}(Z)$. Hence the first term in (4.20) vanishes; i.e., j^* is injective. $\square$

By Lemma 2.2.4, the perverse sheaves K^i, L^i are middle extensions from the open dense subset $\widetilde{\mathcal{A}}^\diamondsuit$ of $\widetilde{\mathcal{A}}'$. By Lemma 4.4.1, in order to prove the commutativity of the diagram (3.6), it suffices to prove the commutativity of its restriction to $\widetilde{\mathcal{A}}^\diamondsuit$. From now on, we will change the indexing scheme of $\mathrm{Sp}(\lambda)$ and $\mathrm{Ch}(\lambda)$ from perverse degrees to the ordinary homological and cohomological degrees respectively:

$$\mathrm{Sp}_i(\lambda) : \mathbf{H}_i(\mathcal{M}^{\mathrm{par}}_G/\widetilde{\mathcal{A}}^\diamondsuit) \to \mathbf{H}_{i+1}(\mathcal{M}^{\mathrm{par}}_G/\widetilde{\mathcal{A}}^\diamondsuit)[1], \tag{4.21}$$

$$\mathrm{Ch}^i(\lambda) : \mathbf{H}^i(\mathcal{M}^{\mathrm{par}}_{G^\vee}/\widetilde{\mathcal{A}}^\diamondsuit) \to \mathbf{H}^{i+1}(\mathcal{M}^{\mathrm{par}}_{G^\vee}/\widetilde{\mathcal{A}}^\diamondsuit)[1](1). \tag{4.22}$$

Here we are using the isomorphisms (4.15) and (4.16).

Remark 4.4.2. We need the following fact about adjunction. Since $\widetilde{p}:\widetilde{\mathcal{A}}^{\diamondsuit}\to\mathcal{A}^{\diamondsuit}$ is smooth of relative dimension one, we have a natural isomorphism of functors $\widetilde{p}^{!}\cong\widetilde{p}^{*}[2](1)$. For any two objects $\mathcal{F}_1,\mathcal{F}_2\in D^b_c(\mathcal{A}^{\diamondsuit},\overline{\mathbb{Q}}_\ell)$, we have

$$\begin{aligned}\mathrm{Hom}_{\widetilde{\mathcal{A}}^{\diamondsuit}}(\widetilde{p}^{*}\mathcal{F}_1,\widetilde{p}^{*}\mathcal{F}_2)&\cong\mathrm{Hom}_{\widetilde{\mathcal{A}}^{\diamondsuit}}(\widetilde{p}^{!}\mathcal{F}_1,\widetilde{p}^{!}\mathcal{F}_2)\\&\cong\mathrm{Hom}_{\mathcal{A}^{\diamondsuit}}(\widetilde{p}_!\widetilde{p}^{!}\mathcal{F}_1,\mathcal{F}_2)=\mathrm{Hom}_{\mathcal{A}^{\diamondsuit}}(\mathbf{H}_*(\widetilde{\mathcal{A}}^{\diamondsuit}/\mathcal{A}^{\diamondsuit})\otimes\mathcal{F}_1,\mathcal{F}_2)\\&\cong\mathrm{Hom}_{\mathcal{A}^{\diamondsuit}}(\mathbf{H}_*(\widetilde{\mathcal{A}}^{\diamondsuit}/\mathcal{A}^{\diamondsuit}),\underline{\mathrm{Hom}}_{\mathcal{A}^{\diamondsuit}}(\mathcal{F}_1,\mathcal{F}_2)).\end{aligned}$$

In other words, any map $\phi:\widetilde{p}^{*}\mathcal{F}_1\to\widetilde{p}^{*}\mathcal{F}_2$ should induce a map

$$\phi^{\natural}:\mathbf{H}_*(\widetilde{\mathcal{A}}^{\diamondsuit}/\mathcal{A}^{\diamondsuit})\to\underline{\mathrm{Hom}}_{\mathcal{A}^{\diamondsuit}}(\mathcal{F}_1,\mathcal{F}_2),$$

and *vice versa*.

Applying the above remark to the maps $\mathrm{Sp}_i(\lambda)$ in (4.21) and $\mathrm{Ch}^i(\lambda)$ in (4.22) (noticing that $\mathbf{H}_i(\mathcal{M}^{\mathrm{par}}_G/\widetilde{\mathcal{A}}^{\diamondsuit})\cong\widetilde{p}^{*}\mathbf{H}_i(\mathcal{M}^{\mathrm{Hit}}_G/\mathcal{A}^{\diamondsuit})$ and $\mathbf{H}^i(\mathcal{M}^{\mathrm{par}}_{G^\vee}/\widetilde{\mathcal{A}}^{\diamondsuit})\cong\widetilde{p}^{*}\mathbf{H}^i(\mathcal{M}^{\mathrm{Hit}}_{G^\vee}/\mathcal{A}^{\diamondsuit})$) the commutativity of the right square in (3.6) over $\widetilde{\mathcal{A}}^{\diamondsuit}$ is equivalent to the commutativity of the following diagram.

$$\begin{tikzcd} & \underline{\mathrm{Hom}}\Big(\bigwedge^{i}(V_\ell(\mathcal{P}^0/\mathcal{A}^{\diamondsuit})),\bigwedge^{i+1}(V_\ell(\mathcal{P}^0/\mathcal{A}^{\diamondsuit}))[1]\Big) \arrow[dd] \\ \mathbf{H}_*(\widetilde{\mathcal{A}}^{\diamondsuit}/\mathcal{A}^{\diamondsuit}) \arrow[ur, "\mathrm{Sp}_i(\lambda)^{\natural}"'] \arrow[dr, "\mathrm{Ch}^i(\lambda)^{\natural}"] & \\ & \underline{\mathrm{Hom}}\Big(\bigwedge^{i}(V_\ell(\mathcal{P}^{\vee,0}/\mathcal{A}^{\diamondsuit})^*(1)),\bigwedge^{i+1}(V_\ell(\mathcal{P}^{\vee,0}/\mathcal{A}^{\diamondsuit})^*(1))[1]\Big)\end{tikzcd}\tag{4.23}$$

Here the vertical arrow is induced from the isomorphism β in (4.17).

4.5 The Springer action by $\mathbb{X}_*(T)$

Above we have put the restriction of $\mathrm{Sp}^i(\lambda)$ to $\widetilde{\mathcal{A}}^{\diamondsuit}$ into the form

$$\mathrm{Sp}_i(\lambda)^{\natural}:\mathbf{H}_*(\widetilde{\mathcal{A}}^{\diamondsuit}/\mathcal{A}^{\diamondsuit})\to\underline{\mathrm{Hom}}\Big(\bigwedge^{i}(V_\ell(\mathcal{P}^0/\mathcal{A}^{\diamondsuit})),\bigwedge^{i+1}(V_\ell(\mathcal{P}^0/\mathcal{A}^{\diamondsuit}))[1]\Big).$$

In this subsection, we write the map $\mathrm{Sp}_i(\lambda)^{\natural}$ more explicitly.

Since $\mathcal{M}^{\mathrm{par}}|_{\widetilde{\mathcal{A}}^{\diamondsuit}}=\widetilde{\mathcal{A}}^{\diamondsuit}\times_{\mathcal{A}}\mathcal{M}^{\mathrm{Hit}}$ (see § 4.2), $\mathbf{H}_*(\mathcal{M}^{\mathrm{par}}/\widetilde{\mathcal{A}}^{\diamondsuit})=\widetilde{p}^{*}\mathbf{H}_*(\mathcal{M}^{\mathrm{Hit}}/\mathcal{A}^{\diamondsuit})$. The Springer action on $\mathbf{H}_*(\mathcal{M}^{\mathrm{par}}/\widetilde{\mathcal{A}}^{\diamondsuit})$ is adjoint to (in the sense of Remark 4.4.2)

$$[\lambda]^{\natural}:\mathbf{H}_*(\widetilde{\mathcal{A}}^{\diamondsuit}/\mathcal{A}^{\diamondsuit})\to\underline{\mathrm{End}}(\mathbf{H}_*(\mathcal{M}^{\mathrm{Hit}}_G/\mathcal{A}^{\diamondsuit})).\tag{4.24}$$

We can rewrite this map in terms of the cap product action of $\mathbf{H}_*(\mathcal{P}/\mathcal{A}^{\diamondsuit})$ on $\mathbf{H}_*(\mathcal{M}^{\mathrm{Hit}}/\mathcal{A}^{\diamondsuit})$ as follows. Recall from [Yun11, beginning of proof of Proposition 3.2.1] that we have a morphism over $\mathcal{A}$,

$$s_\lambda:\widetilde{\mathcal{A}}^{0}\to\mathcal{P},$$

for each $\lambda\in\mathbb{X}_*(T)$. Here $\widetilde{\mathcal{A}}^{0}$ is the locus where the local delta invariant $\delta(a,x)=0$. As discussed in § 4.2, $\widetilde{\mathcal{A}}^{\diamondsuit}\subset\widetilde{\mathcal{A}}^{0}$. By the construction in [Yun11, § 3.3] and the discussion in § 2.2, the action of λ on $\widetilde{f}_*\overline{\mathbb{Q}}_\ell$ is induced from an automorphism of $\mathcal{M}^{\mathrm{par}}|_{\widetilde{\mathcal{A}}^{\diamondsuit}}$, coming from the morphism s_λ and the $\mathcal{P}$-action on $\mathcal{M}^{\mathrm{par}}$. We can rewrite this automorphism of $\mathcal{M}^{\mathrm{par}}|_{\widetilde{\mathcal{A}}^{\diamondsuit}}$ as a morphism

$$\mathcal{M}_G^{\mathrm{par}}|_{\widetilde{\mathcal{A}}^{\diamondsuit}} = \widetilde{\mathcal{A}}^{\diamondsuit} \times_{\mathcal{A}^{\diamondsuit}} \mathcal{M}_G^{\mathrm{Hit}}|_{\mathcal{A}^{\diamondsuit}} \xrightarrow{(s_\lambda,\mathrm{id})} \mathcal{P}|_{\mathcal{A}^{\diamondsuit}} \times_{\mathcal{A}^{\diamondsuit}} \mathcal{M}_G^{\mathrm{Hit}}|_{\mathcal{A}^{\diamondsuit}} \xrightarrow{\mathrm{act}} \mathcal{M}_G^{\mathrm{Hit}}|_{\mathcal{A}^{\diamondsuit}}.$$

Passing to the level of homology, we get

$$\begin{aligned}\mathbf{H}_*(\widetilde{\mathcal{A}}^{\diamondsuit}/\mathcal{A}^{\diamondsuit}) \otimes_{\mathcal{A}^{\diamondsuit}} \mathbf{H}_*(\mathcal{M}_G^{\mathrm{Hit}}/\mathcal{A}^{\diamondsuit}) &\xrightarrow{s_{\lambda,*}\otimes\mathrm{id}} \mathbf{H}_*(\mathcal{P}/\mathcal{A}^{\diamondsuit}) \otimes_{\mathcal{A}^{\diamondsuit}} \mathbf{H}_*(\mathcal{M}_G^{\mathrm{Hit}}/\mathcal{A}^{\diamondsuit})\\ &\xrightarrow{\mathrm{act}_*} \mathbf{H}_*(\mathcal{M}_G^{\mathrm{Hit}}/\mathcal{A}^{\diamondsuit}).\end{aligned}$$

By adjunction, we get

$$\mathbf{H}_*(\widetilde{\mathcal{A}}^{\diamondsuit}/\mathcal{A}^{\diamondsuit}) \xrightarrow{s_{\lambda,*}} \mathbf{H}_*(\mathcal{P}/\mathcal{A}^{\diamondsuit}) \xrightarrow{\cap} \underline{\mathrm{End}}(\mathbf{H}_*(\mathcal{M}_G^{\mathrm{Hit}}/\mathcal{A}^{\diamondsuit})), \tag{4.25}$$

where $\cap$ is (the dual of) the cap product defined in Appendix A.4. By the above discussion, the map (4.25) is the same as $[\lambda]^\natural$ in (4.24). Taking the stable part of $s_{\lambda,*}$, and using the isomorphism (4.13), we can rewrite (4.25) as

$$\mathbf{H}_*(\widetilde{\mathcal{A}}^{\diamondsuit}/\mathcal{A}^{\diamondsuit}) \xrightarrow{s_{\lambda,*}} \bigwedge(V_\ell(\mathcal{P}^0/\mathcal{A}^{\diamondsuit})[1]) \xrightarrow{\wedge} \underline{\mathrm{End}}\Big(\bigwedge(V_\ell(\mathcal{P}^0/\mathcal{A}^{\diamondsuit})[1])\Big).$$

Here the cap product action becomes the wedge product in $\bigwedge(V_\ell(\mathcal{P}^0/\mathcal{A}^{\diamondsuit})[1])$.

We decompose the map $s_{\lambda,*}$ into $\bigoplus_i s_{\lambda,i}$, where

$$s_{\lambda,i} : \mathbf{H}_*(\widetilde{\mathcal{A}}^{\diamondsuit}/\mathcal{A}^{\diamondsuit}) \to \bigwedge^i(V_\ell(\mathcal{P}^0/\mathcal{A}^{\diamondsuit})[1]).$$

Note that

$$\mathbf{H}_*(\widetilde{\mathcal{A}}^{\diamondsuit}/\mathcal{A}^{\diamondsuit}) \xrightarrow{s_{\lambda,0}} \overline{\mathbb{Q}}_\ell \to \underline{\mathrm{End}}\Big(\bigwedge(V_\ell(\mathcal{P}^0/\mathcal{A}^{\diamondsuit})[1])\Big)$$

corresponds to the identity map of $\widetilde{p}^* \bigwedge(V_\ell(\mathcal{P}^0/\mathcal{A}^{\diamondsuit})[1]) \cong \widetilde{p}^*\mathbf{H}_*(\mathcal{M}_G^{\mathrm{par}}/\mathcal{A}^{\diamondsuit})_{\mathrm{st}}$ under the adjunction in Remark 4.4.2. Therefore, the action of $\lambda - \mathrm{id}$ on $\mathbf{H}_*(\mathcal{M}_G^{\mathrm{par}}/\widetilde{\mathcal{A}}^{\diamondsuit})_{\mathrm{st}}$ is adjoint to

$$\mathbf{H}_*(\widetilde{\mathcal{A}}^{\diamondsuit}/\mathcal{A}^{\diamondsuit}) \xrightarrow{\sum_{i\geqslant 1} s_{\lambda,i}} \bigoplus_{i\geqslant 1}\bigwedge^i(V_\ell(\mathcal{P}^0/\mathcal{A}^{\diamondsuit})[1]) \xrightarrow{\wedge} \underline{\mathrm{End}}\Big(\bigwedge(V_\ell(\mathcal{P}^0/\mathcal{A}^{\diamondsuit})[1])\Big).$$

Restricting to the degree -1 part and denoting $s_{\lambda,1}$ by Φ_λ, we conclude that $\mathrm{Sp}_i(\lambda)^\natural$ can be written as

$$\mathrm{Sp}_i(\lambda)^\natural \quad : \quad \mathbf{H}_*(\widetilde{\mathcal{A}}^{\diamondsuit}/\mathcal{A}^{\diamondsuit}) \xrightarrow{\Phi_\lambda} V_\ell(\mathcal{P}^0/\mathcal{A}^{\diamondsuit})[1] \tag{4.26}$$

$$\xrightarrow{\wedge} \underline{\mathrm{Hom}}\Big(\bigwedge^i(V_\ell(\mathcal{P}^0/\mathcal{A}^{\diamondsuit})), \bigwedge^{i+1}(V_\ell(\mathcal{P}^0/\mathcal{A}^{\diamondsuit}))[1]\Big). \tag{4.27}$$

Now we need to understand the map Φ_λ more explicitly. For this, we first describe the morphism $s_\lambda : \widetilde{\mathcal{A}}^{\diamondsuit} \to \mathcal{P}$ in more concrete terms. Consider the composition

$$\widetilde{\mathcal{A}}^{\diamondsuit} \xrightarrow{s_\lambda} \mathcal{P} \xrightarrow{j_{\mathcal{P}}} \mathcal{P}\mathrm{ic}_T(\widetilde{\mathcal{A}}/\mathcal{A})^W \to \mathcal{P}\mathrm{ic}_T(\widetilde{\mathcal{A}}/\mathcal{A}).$$

The pull-back of the universal T-torsor on $\widetilde{\mathcal{A}} \times_{\mathcal{A}} \mathcal{P}\mathrm{ic}_T(\widetilde{\mathcal{A}}/\mathcal{A})$ gives a T-torsor $\mathcal{Q}_\lambda^T$ on $\widetilde{\mathcal{A}}^{\diamondsuit} \times_{\mathcal{A}} \widetilde{\mathcal{A}}$ via the above map, hence a line bundle $\mathcal{Q}_\lambda(\xi)$ on $\widetilde{\mathcal{A}}^{\diamondsuit} \times_{\mathcal{A}} \widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}^{\diamondsuit} \times_{\mathcal{A}^{\diamondsuit}} \widetilde{\mathcal{A}}^{\diamondsuit}$ for each $\xi \in \mathbb{X}^*(T)$.

Lemma 4.5.1. *Let* $\Gamma_w = \{(\widetilde{x}, w\widetilde{x}) \mid \widetilde{x} \in \widetilde{\mathcal{A}}^\diamondsuit\} \subset \widetilde{\mathcal{A}}^\diamondsuit \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit$ *be the graph of the left* w*-action, viewed as a divisor of* $\widetilde{\mathcal{A}}^\diamondsuit \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit$*. Then there is a canonical isomorphism*

$$\mathcal{Q}_\lambda(\xi) \cong \mathcal{O}\Bigg(\sum_{w\in W} \langle {}^w\lambda, \xi\rangle \Gamma_w\Bigg). \tag{4.28}$$

Proof. By [Yun11, Beginning of proof of Proposition 3.2.1], the morphism s_λ comes from a section $\widetilde{s}_\lambda : \widetilde{\mathcal{A}}^\diamondsuit \to \widetilde{\mathcal{G}r}_J \to \mathrm{Gr}_J$. By the definition of the affine Grassmannian $\mathcal{G}r_J$, we obtain a J-torsor $\mathcal{Q}^J_\lambda$ on $\widetilde{\mathcal{A}}^\diamondsuit \times X$ with a canonical trivialization away from the graph Γ of $\widetilde{\mathcal{A}}^\diamondsuit \to X$. By construction, we have

$$\mathcal{Q}^T_\lambda = (\mathrm{id} \times q)^* \mathcal{Q}^J_\lambda \overset{q^*J}{\times} T$$

where $\mathrm{id} \times q : \widetilde{\mathcal{A}}^\diamondsuit \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit \to \widetilde{\mathcal{A}}^\diamondsuit \times_{\mathcal{A}^\diamondsuit} (\mathcal{A}^\diamondsuit \times X) = \widetilde{\mathcal{A}}^\diamondsuit \times X$. Therefore, $\mathcal{Q}^T_\lambda$ has a canonical trivialization τ over $\widetilde{\mathcal{A}}^\diamondsuit \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit - (\mathrm{id} \times q)^{-1}(\Gamma) = \widetilde{\mathcal{A}}^\diamondsuit \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit - \bigcup_{w\in W} \Gamma_w$.

On the other hand, the section $\widetilde{s}_\lambda$ over $\widetilde{\mathcal{A}}^{\mathrm{rs}}$ is defined by the composition

$$\widetilde{\mathcal{A}}^{\mathrm{rs}} \xrightarrow{\mathrm{id} \times \{\lambda\}} \widetilde{\mathcal{A}}^{\mathrm{rs}} \times \mathbb{X}_*(T) \xrightarrow{\sim} (\widetilde{\mathcal{G}r}^{\mathrm{rs}}_T)^{\mathrm{red}} \hookrightarrow \widetilde{\mathcal{G}r}^{\mathrm{rs}}_T \xrightarrow{\sim} \widetilde{\mathcal{G}r}^{\mathrm{rs}}_J.$$

Here $\widetilde{\mathcal{G}r}^{\mathrm{rs}}_T \to \widetilde{\mathcal{A}}^{\mathrm{rs}}$ is the Beilinson–Drinfeld Grassmannian of T-torsors on the family of cameral curves $\widetilde{\mathcal{A}} \to \mathcal{A}$: over $(a, \widetilde{x}) \in \widetilde{\mathcal{A}}^{\mathrm{rs}}$, it classifies a T-torsor on X_a together with a trivialization of it over $X_a - \{\widetilde{x}\}$. The last isomorphism above is the inverse of the one defined in [Yun11, Lemma 3.2.5]. We first look at the morphism $\widetilde{\mathcal{A}}^{\mathrm{rs}} \xrightarrow{\mathrm{id} \times \{\lambda\}} \widetilde{\mathcal{A}}^{\mathrm{rs}} \times \mathbb{X}_*(T) \to \widetilde{\mathcal{G}r}^{\mathrm{rs}}_T$. This amounts to giving a T-torsor $\mathcal{G}^T_\lambda$ on $\widetilde{\mathcal{A}}^{\mathrm{rs}} \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit$ with a trivialization outside the diagonal (the graph Γ^{rs}_e). The associated line bundle $\mathcal{G}_\lambda(\xi)$ with the induced trivialization on $\widetilde{\mathcal{A}}^{\diamondsuit,\mathrm{rs}} \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit - \Gamma^{\mathrm{rs}}_e$ has the form

$$\mathcal{G}_\lambda(\xi) = \mathcal{O}(\langle \lambda, \xi\rangle \Gamma^{\mathrm{rs}}_e).$$

Here, we define $\Gamma^{\mathrm{rs}}_w = \Gamma_w \cap (\widetilde{\mathcal{A}}^{\diamondsuit,\mathrm{rs}} \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit)$.

Next, by the construction of the isomorphism $\widetilde{\mathcal{G}r}^{\mathrm{rs}}_T \xrightarrow{\sim} \widetilde{\mathcal{G}r}^{\mathrm{rs}}_J$ in [Yun11, Lemma 3.2.5], the line bundles $\mathcal{Q}_\lambda(\xi)$ and $\mathcal{G}_\lambda(\xi)$ are canonically isomorphic over the open subset

$$\widetilde{\mathcal{A}}^{\diamondsuit,\mathrm{rs}} \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit - \bigsqcup_{w\in W, w\neq e} \Gamma^{\mathrm{rs}}_w.$$

In view of the W-invariance of $\mathcal{Q}_\lambda(\xi)$, we must have a canonical isomorphism

$$\mathcal{Q}_\lambda(\xi)|_{\widetilde{\mathcal{A}}^{\diamondsuit,\mathrm{rs}} \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit} \cong \mathcal{O}\Bigg(\sum_{w\in W} \langle {}^w\lambda, \xi\rangle \Gamma^{\mathrm{rs}}_w\Bigg). \tag{4.29}$$

Moreover, the trivialization of $\mathcal{Q}^T_\lambda|_{\widetilde{\mathcal{A}}^{\diamondsuit,\mathrm{rs}} \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit}$ on $(\widetilde{\mathcal{A}}^{\diamondsuit,\mathrm{rs}} \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit - \bigcup_{w\in W} \Gamma^{\mathrm{rs}}_w)$ given in the isomorphism (4.29) is the same as the trivialization τ of $\mathcal{Q}^T_\lambda$ on $(\widetilde{\mathcal{A}}^\diamondsuit \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit - \bigcup_{w\in W} \Gamma_w)$ given by $\widetilde{s}_\lambda$ (in the beginning of the proof). Therefore, the expression (4.28) holds over

$$\Bigg(\widetilde{\mathcal{A}}^\diamondsuit \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit - \bigcup_{w\in W} \Gamma_w\Bigg) \cup (\widetilde{\mathcal{A}}^{\mathrm{rs}} \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit) = \widetilde{\mathcal{A}}^\diamondsuit \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit - \bigcup_{w\in W} (\Gamma_w - \Gamma^{\mathrm{rs}}_w).$$

Since $\bigcup_{w\in W}(\Gamma_w - \Gamma^{\mathrm{rs}}_w)$ has codimension at least two in the smooth variety $\widetilde{\mathcal{A}}^\diamondsuit \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit$, the expression (4.28) must hold on the whole $\widetilde{\mathcal{A}}^\diamondsuit \times_{\mathcal{A}^\diamondsuit} \widetilde{\mathcal{A}}^\diamondsuit$. □

Lemma 4.5.2. *Under the isomorphism (4.5), the degree -1 part of the map Φ_λ is given by*

$$\Phi_{\lambda,1}: \mathbf{H}_1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \to V_\ell(\mathcal{P}^0/\mathcal{A}^\diamondsuit) \to (\mathbf{H}_1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \otimes_{\mathbb{Z}} \mathbb{X}_*(T))^W$$
$$h \longmapsto \sum_{w\in W} w_* h \otimes {}^w\lambda.$$

Proof. This is a statement about a map between local systems, hence it suffices to check it on the stalks of geometric points. We fix a geometric point $a \in \mathcal{A}^\diamondsuit$. For each $\xi \in \mathbb{X}^*(T)$, consider the morphism

$$X_a \xrightarrow{s_\lambda} \mathcal{P}_a \to \mathcal{P}\mathrm{ic}_T(X_a) \xrightarrow{I_\xi} \mathcal{P}\mathrm{ic}(X_a) \tag{4.30}$$

where I_ξ sends a T-torsor to the line bundle associated with the character ξ. Since $\pi_0(\mathcal{P}_a)$ is torsion, the map (4.30) must land in $\mathcal{P}\mathrm{ic}^0(X_a)$. By Lemma 4.5.1, the map (4.30) takes $\widetilde{x} \in X_a$ to the line bundle $\mathcal{O}(\sum_{w\in W}\langle {}^w\lambda, \xi\rangle w\widetilde{x}) \in \mathrm{Pic}^0(X_a)$. Therefore, it induces the following map on homology:

$$\mathrm{H}_1(X_a) \to \mathrm{H}_1(\mathrm{Pic}^0(X_a)) \cong \mathrm{H}_1(X_a)$$
$$h \mapsto \sum_{w\in W}\langle {}^w\lambda, \xi\rangle w_* h.$$

Here we use the Picard scheme Pic rather than the Picard stack $\mathcal{P}\mathrm{ic}$ without affecting the Tate modules. Since $\xi \in \mathbb{X}^*(T)$ is arbitrary, this proves the lemma. □

4.6 The Chern class action by $\mathbb{X}^*(T^\vee)$

Let $\lambda \in \mathbb{X}^*(T^\vee)$. The Chern class of $\mathcal{Q}(\lambda)$ gives a map

$$c_1(\mathcal{Q}(\lambda)): \overline{\mathbb{Q}}_{\ell,\widetilde{\mathcal{A}}} = \widetilde{p}^*\overline{\mathbb{Q}}_{\ell,\mathcal{A}} \to \mathbf{H}^*(\widetilde{\mathcal{P}}/\widetilde{\mathcal{A}})[2](1) = \widetilde{p}^*\mathbf{H}^*(\mathcal{P}/\mathcal{A})[2](1). \tag{4.31}$$

By adjunction from Remark 4.4.2, this gives a map

$$c_1(\mathcal{Q}(\lambda))^\natural: \mathbf{H}_*(\widetilde{\mathcal{A}}/\mathcal{A}) \to \mathbf{H}^*(\mathcal{P}/\mathcal{A})[2](1).$$

Lemma 4.6.1. *Under the natural decomposition (A4), the map $c_1(\mathcal{Q}(\lambda))^\natural$ factors through*

$$\Psi_\lambda: \mathbf{H}_*(\widetilde{\mathcal{A}}/\mathcal{A}) \to \mathbf{H}^1(\mathcal{P}/\mathcal{A})_{\mathrm{st}}[1](1) \subset \mathbf{H}^*(\mathcal{P}/\mathcal{A})[2](1). \tag{4.32}$$

Using the isomorphism (4.6), the degree -1 part of $\Psi_\lambda|_{\mathcal{A}^\diamondsuit}$ takes the form

$$\begin{array}{ccccc} \Psi_{\lambda,1}: \mathbf{H}_1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) & \to & V_\ell(\mathcal{P}^{\vee,0}/\mathcal{A}^\diamondsuit)^*(1) & \xrightarrow{\sim} & (\mathbf{H}^1(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)(1) \otimes_{\mathbb{Z}} \mathbb{X}^*(T^\vee))_W \\ h & & \longmapsto & & \mathrm{PD}(h) \otimes \lambda \end{array} \tag{4.33}$$

where PD *is the Poincaré duality isomorphism defined in (4.18).*

Proof. By construction, the line bundle $\mathcal{Q}(\lambda)$ on $\widetilde{\mathcal{A}} \times_{\mathcal{A}} \mathcal{P}$ is the pull-back of the Poincaré (universal) line bundle on $\widetilde{\mathcal{A}} \times_{\mathcal{A}} \mathcal{P}\mathrm{ic}(\widetilde{\mathcal{A}}/\mathcal{A})$ using the morphism

$$j_\lambda: \mathcal{P} \xrightarrow{j_{\mathcal{P}}} \mathcal{P}\mathrm{ic}_T(\widetilde{\mathcal{A}}/\mathcal{A}) \xrightarrow{I_\lambda} \mathcal{P}\mathrm{ic}(\widetilde{\mathcal{A}}/\mathcal{A}) \tag{4.34}$$

where I_λ sends a T-torsor to the line bundle induced by the character λ. For any geometric point $a \in \mathcal{A}$, $\pi_0(\mathcal{P}\mathrm{ic}(X_a))$ is always a free abelian group while $\pi_0(\mathcal{P}_a)$ is finite, and the morphism (4.34) necessarily factors through the neutral component $\mathcal{P}\mathrm{ic}(\widetilde{\mathcal{A}}/\mathcal{A})^0$ of $\mathcal{P}\mathrm{ic}(\widetilde{\mathcal{A}}/\mathcal{A})$. Therefore, $c_1(\mathcal{Q}(\lambda))^\natural$ factors through

$$\mathbf{H}_*(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}) \to \mathbf{H}^*(\mathcal{P}\mathrm{ic}(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A})^0/\mathcal{A})[2](1) \to \mathbf{H}^*(\mathcal{P}/\mathcal{A})[2](1).$$

Since $\mathcal{P}\mathrm{ic}(\widetilde{\mathcal{A}}/\mathcal{A})^0 \to \mathcal{A}$ has connected fibers, the last map above has to land in the stable part of $\mathbf{H}^*(\mathcal{P}/\mathcal{A})[2](1)$ by Remark A.3.3.

By the definition of the tautological line bundle $\mathcal{Q}(\lambda)$, for each integer $N \in \mathbb{Z}$, we have

$$(\mathrm{id}_{\widetilde{\mathcal{A}}} \times [N])^* \mathcal{Q}(\lambda) \cong \mathcal{Q}(N\lambda) \cong \mathcal{Q}(\lambda)^{\otimes N}.$$

Therefore, we have a commutative diagram.

$$\begin{tikzcd} \mathbf{H}_*(\widetilde{\mathcal{A}}/\mathcal{A}) \arrow[r, "c_1(\mathcal{Q}(\lambda))"] \arrow[rr, bend right, "Nc_1(\mathcal{Q}(\lambda))"'] & \mathbf{H}^*(\mathcal{P}/\mathcal{A})[2](1) \arrow[r, "{[N]^*}"] & \mathbf{H}^*(\mathcal{P}/\mathcal{A})[2](1) \end{tikzcd}$$

This implies that $c_1(\mathcal{Q}(\lambda))$ factors through the eigensubcomplex of $\mathbf{H}^*(\mathcal{P}/\mathcal{A})_{\mathrm{st}}[2](1)$ with eigenvalue N under the endomorphism $[N]^*$, i.e., $\mathbf{H}^1(\mathcal{P}/\mathcal{A})_{\mathrm{st}}[1](1)$ (see Remark A.2.2). This proves the first half of the lemma.

To prove (4.33), it suffices to fix a geometric point $a \in \mathcal{A}^{\diamondsuit}$ and base change the maps in (4.34) to the fibers over a. Let $\mathcal{L}^{\mathrm{Poin}}$ be the universal line bundle on $X_a \times \mathcal{P}\mathrm{ic}^0(X_a)$. Then $\mathcal{Q}(\lambda) \cong (\mathrm{id} \times j_\lambda)^* \mathcal{L}^{\mathrm{Poin}}$.

We first calculate $c_1(\mathcal{L}^{\mathrm{Poin}})$. For an abelian variety A and the Poincaré line bundle $\mathcal{L}_A$ on $A \times \widehat{A}$, the Chern class $c_1(\mathcal{L}_A) \in \mathrm{H}^2(A \times \widehat{A})(1) = \wedge^2 V_\ell(A \times \widehat{A})^*(1)$ is dual to the (alternating) Weil pairing:

$$e^{\mathcal{L}_A}(-,-) : V_\ell(A \times \widehat{A}) \otimes V_\ell(A \times \widehat{A}) \to \overline{\mathbb{Q}}_\ell(1).$$

According to [Mum74, p.188], the Weil pairing takes the form $e^{\mathcal{L}_A}(x, \widehat{x}; y, \widehat{y}) = (x, \widehat{y}) - (y, \widehat{x})$, where $(-,-) : V_\ell(A) \otimes V_\ell(\widehat{A}) \to \overline{\mathbb{Q}}_\ell(1)$ is the canonical Weil pairing.

We apply the above discussion to the self-dual abelian variety $\mathrm{Pic}^0(X_a)$. The Riemann form is given by the principal polarization $V_\ell(\mathrm{Pic}^0(X_a)) \otimes V_\ell(\mathrm{Pic}^0(X_a)) \cong \mathrm{H}_1(X_a) \otimes \mathrm{H}_1(X_a) \to \overline{\mathbb{Q}}_\ell(1)$, which in turn is dual to the cup product on $\mathrm{H}^1(X_a)$. Therefore,

$$c_1(\mathcal{L}_{\mathrm{Pic}^0(X_a)}) = \sum_i h^i \otimes \mathrm{PD}(h_i) \in \mathrm{H}^1(X_a) \otimes \mathrm{H}^1(\mathrm{Pic}^0(X_a))(1) \cong \mathrm{H}^1(X_a) \otimes \mathrm{H}^1(X_a)(1)$$

where $\{h^i\}$ and $\{h_i\}$ are dual bases of $\mathrm{H}^1(X_a)$ and $\mathrm{H}_1(X_a)$. Therefore, $c_1(\mathcal{Q}(\lambda)) = (\mathrm{id} \times j_\lambda)^* c_1(\mathcal{L}^{\mathrm{Poin}})$ is the image of $\sum_i h^i \otimes \mathrm{PD}(h_i) \otimes \lambda \in \mathrm{H}^1(X_a) \otimes \mathrm{H}^1(X_a)(1) \otimes_{\mathbb{Z}} \mathbb{X}^*(T^\vee)$ in $\mathrm{H}^1(X_a) \otimes (\mathrm{H}^1(X_a)(1) \otimes_{\mathbb{Z}} \mathbb{X}^*(T^\vee))_W \cong \mathrm{H}^1(X_a) \otimes \mathrm{H}^1(\mathcal{P}_a)_{\mathrm{st}}(1)$. This immediately implies (4.33). □

Using the Kostant section $\epsilon^\vee : \mathcal{A} \to \mathcal{M}^{\mathrm{Hit,reg}}_{G^\vee}$, we get a section $\widetilde{\epsilon}^\vee : \widetilde{\mathcal{A}} \to \mathcal{M}^{\mathrm{Hit,reg}}_{G^\vee} \times_{\mathcal{A}} \widetilde{\mathcal{A}} \cong \mathcal{M}^{\mathrm{par,reg}}_{G^\vee} \subset \mathcal{M}^{\mathrm{par}}_{G^\vee}$ and a morphism

$$\widetilde{\tau}^\vee : \widetilde{\mathcal{P}}^\vee = \widetilde{\mathcal{P}}^\vee \times_{\widetilde{\mathcal{A}}} \widetilde{\mathcal{A}} \xrightarrow{(\mathrm{id}, \widetilde{\epsilon}^\vee)} \widetilde{\mathcal{P}}^\vee \times_{\widetilde{\mathcal{A}}} \mathcal{M}^{\mathrm{par}}_{G^\vee} \xrightarrow{\mathrm{act}} \mathcal{M}^{\mathrm{par}}_{G^\vee}$$

which is an isomorphism over $\widetilde{\mathcal{A}}^{\diamondsuit}$. By diagram (4.4), we get

$$\begin{aligned} \widetilde{\tau}^{\vee *}\mathcal{L}(\lambda) &= (\mathrm{id} \times \widetilde{\epsilon}^\vee)^*(\mathcal{Q}(\lambda) \boxtimes_{\widetilde{\mathcal{A}}} \mathcal{L}(\lambda)) \cong \mathcal{Q}(\lambda) \boxtimes_{\widetilde{\mathcal{A}}} \widetilde{\epsilon}^{\vee *}\mathcal{L}(\lambda) \\ &= \mathcal{Q}(\lambda) \otimes \widetilde{g}^{\vee *}\widetilde{\epsilon}^{\vee *}\mathcal{L}(\lambda) \end{aligned} \tag{4.35}$$

where $\widetilde{g}^\vee : \widetilde{\mathcal{P}}^\vee \to \widetilde{\mathcal{A}}$ is the projection.

By Lemma 4.6.1, the Chern class of the line bundle $\mathcal{Q}(\lambda)$ can be written as

$$c_1(\mathcal{Q}(\lambda))^\natural : \mathbf{H}_*(\widetilde{\mathcal{A}}/\mathcal{A}) \to \mathbf{H}^1(\mathcal{P}^\vee/\mathcal{A})_{\mathrm{st}}[1](1) \subset \mathbf{H}^*(\mathcal{P}^\vee/\mathcal{A})[2](1).$$

The line bundle $\widetilde{\epsilon}^{\vee *}\mathcal{L}(\lambda)$ on $\widetilde{\mathcal{A}}$ also induces a map

$$c_1(\widetilde{\epsilon}^{\vee *}\mathcal{L}(\lambda))^\natural : \mathbf{H}_*(\widetilde{\mathcal{A}}/\mathcal{A}) \to \overline{\mathbb{Q}}_\ell[2](1) \cong \mathbf{H}^0(\mathcal{P}^\vee/\mathcal{A})_{\mathrm{st}}[2](1) \subset \mathbf{H}^*(\mathcal{P}^\vee/\mathcal{A})[2](1).$$

Putting these results together, using (4.35), we can write the Chern class of $\widetilde{\tau}^{\vee *}\mathcal{L}(\lambda)$ as

$$c_1(\widetilde{\tau}^{\vee *}\mathcal{L}(\lambda))^\natural = c_1(\widetilde{\epsilon}^{\vee *}\mathcal{L}(\lambda))^\natural \oplus c_1(\mathcal{Q}(\lambda))^\natural : \quad \mathbf{H}_*(\widetilde{\mathcal{A}}/\mathcal{A}) \to \mathbf{H}^0(\mathcal{P}^\vee/\mathcal{A})_{\mathrm{st}}[2](1) \oplus \mathbf{H}^1(\mathcal{P}^\vee/\mathcal{A})_{\mathrm{st}}[1](1). \tag{4.36}$$

4.6.2 *Proof of Lemma 3.2.3.*

Proof. Since both L^i and L^{i+2} are middle extensions from $\widetilde{\mathcal{A}}^\diamondsuit$, it is enough to check this statement over $\widetilde{\mathcal{A}}^\diamondsuit$. Using the adjunction in Remark 4.4.2 and the isomorphism (4.12), we can write the action of $c_1(\mathcal{L}(\lambda))$ as

$$c_1(\mathcal{L}(\lambda))^\natural : \mathbf{H}_*(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \to \mathbf{H}^*(\mathcal{M}^{\mathrm{Hit}}_{G^\vee}/\mathcal{A}^\diamondsuit)[2](1) \xrightarrow{\cup} \underline{\mathrm{End}}(\mathbf{H}^*(\mathcal{M}^{\mathrm{Hit}}_{G^\vee}/\mathcal{A}^\diamondsuit))[2](1) \tag{4.37}$$

where $\cup$ is the cup product on $\mathbf{H}^*(\mathcal{M}^{\mathrm{Hit}}_{G^\vee}/\mathcal{A}^\diamondsuit)$. Using the trivialization $\tau^\vee : \mathcal{P}^\vee|_{\mathcal{A}^\diamondsuit} \xrightarrow{\sim} \mathcal{M}^{\mathrm{Hit}}_{G^\vee}|_{\mathcal{A}^\diamondsuit}$ to identify $\mathbf{H}^*(\mathcal{M}^{\mathrm{Hit}}_{G^\vee}/\mathcal{A}^\diamondsuit)$ with $\mathbf{H}^*(\mathcal{P}^\vee/\mathcal{A}^\diamondsuit)$, and using (4.36), the effect of $c_1(\widetilde{\tau}^{\vee *}\mathcal{L}(\lambda))^\natural$ on the stable part is

$$\begin{aligned}\mathbf{H}_*(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) &\xrightarrow{c_1(\widetilde{\epsilon}^{\vee *}\mathcal{L}(\lambda))^\natural \oplus c_1(\mathcal{Q}(\lambda))^\natural} \mathbf{H}^0(\mathcal{P}^\vee/\mathcal{A}^\diamondsuit)_{\mathrm{st}}[2](1) \oplus \mathbf{H}^1(\mathcal{P}^\vee/\mathcal{A}^\diamondsuit)_{\mathrm{st}}[1](1)\\ &\xrightarrow{\cup} \underline{\mathrm{End}}(\mathbf{H}^*(\mathcal{P}^\vee/\mathcal{A}^\diamondsuit)_{\mathrm{st}})[2](1).\end{aligned} \tag{4.38}$$

Since the image of $c_1(\widetilde{\epsilon}^{\vee *}\mathcal{L}(\lambda))^\natural \oplus c_1(\mathcal{Q}(\lambda))^\natural$ only involves cohomology sheaves in degree less than or equal to 1, using Remark 4.4.2 backwards, we see that $\cup c_1(\mathcal{L}(\lambda))_{\mathrm{st}}$ sends ${}^p\tau_{\leqslant i}L_\diamondsuit$ to ${}^p\tau_{\leqslant i+1}L_\diamondsuit$. This proves the lemma. □

Since we will be concentrating on the degree-one part of $c_1(\mathcal{L}(\lambda))$, we can ignore the contribution of $c_1(\widetilde{\epsilon}^{\vee *}\mathcal{L}(\lambda))$ in (4.38). Using (4.38) and (A7), we can finally write $\mathrm{Ch}^i(\lambda)^\natural$ as

$$\begin{aligned}\mathrm{Ch}^i(\lambda)^\natural : \mathbf{H}_*(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) &\xrightarrow{\Psi_\lambda} V_\ell(\mathcal{P}^{\vee,0}/\mathcal{A}^\diamondsuit)^*(1)[1]\\ &\xrightarrow{\wedge} \underline{\mathrm{Hom}}\left(\bigwedge^i(V_\ell(\mathcal{P}^{\vee,0}/\mathcal{A}^\diamondsuit)^*(1)), \bigwedge^{i+1}(V_\ell(\mathcal{P}^{\vee,0}/\mathcal{A}^\diamondsuit)^*(1))[1]\right).\end{aligned} \tag{4.39}$$

4.7 Proof of Theorem 3.2.4

In this subsection we finish the proof of Theorem 3.2.4. By the reduction at the end of § 4.3 (see (4.23)), the expression (4.26) for $\mathrm{Sp}_i(\lambda)^\natural$ and the expression (4.39) for $\mathrm{Ch}^i(\lambda)^\natural$, it remains to prove the commutativity of the following diagram.

$$\begin{tikzcd} & V_\ell(\mathcal{P}^0/\mathcal{A}^\diamondsuit)[1] \arrow[dd, "\beta[1]"] \\ \mathbf{H}_*(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \arrow[ur, "\Phi_\lambda"] \arrow[dr, "\Psi_\lambda"'] & \\ & V_\ell(\mathcal{P}^{\vee,0}/\mathcal{A}^\diamondsuit)^*[1](1) \end{tikzcd}$$

Both maps Φ_λ and Ψ_λ necessarily factor through $\tau_{\geqslant -1}\mathbf{H}_*(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)$. We computed $\Phi_{\lambda,1}$ in Lemma 4.5.2 and $\Psi_{\lambda,1}$ in Lemma 4.6.1. Comparing the two results with the way we defined the isomorphism β in (4.19), we conclude that, for all $\lambda \in \mathbb{X}_*(T) = \mathbb{X}^*(T^\vee)$,

$$\beta \circ \Phi_{\lambda,1} = \Psi_{\lambda,1}.$$

Therefore, the difference $\beta \circ \Phi_\lambda - \Psi_\lambda$ must factor through a map

$$\Delta_\lambda : \mathbf{H}_0(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \cong \overline{\mathbb{Q}}_\ell \to V_\ell(\mathcal{P}^{\vee,0}/\mathcal{A}^\diamondsuit)^*[1](1).$$

All we need to show is $\Delta_\lambda = 0$.

Lemma 4.7.1. *The maps Φ_λ and Ψ_λ are additive in λ.*

Proof. For Ψ_λ, since $\mathcal{Q}(\lambda+\mu) \cong \mathcal{Q}(\lambda) \otimes \mathcal{Q}(\mu)$, we have

$$c_1(\mathcal{Q}(\lambda+\mu)) = c_1(\mathcal{Q}(\lambda)) + c_1(\mathcal{Q}(\mu)),$$

which implies the additivity of Ψ_λ.

For Φ_λ, recall that it comes from the morphism $s_\lambda : \widetilde{\mathcal{A}}^\diamondsuit \to \mathcal{P}$. These morphisms are additive in λ (using the multiplication of $\mathcal{P}$).

$$\begin{tikzcd} \widetilde{\mathcal{A}}^\diamondsuit \arrow[r, "(s_\lambda, s_\mu)"] \arrow[rr, bend right, "s_{\lambda+\mu}"'] & \mathcal{P} \times_{\mathcal{A}} \mathcal{P} \arrow[r, "\text{mult}"] & \mathcal{P} \end{tikzcd}$$

Therefore, the induced maps on homology satisfy the following commutative diagram.

$$\begin{tikzcd} \mathbf{H}_*(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \arrow[r, "s_{\lambda,*} \otimes s_{\mu,*}"] \arrow[rr, bend right, "s_{\lambda+\mu,*}"'] & \mathbf{H}_*(\mathcal{P}/\mathcal{A}^\diamondsuit) \otimes \mathbf{H}_*(\mathcal{P}/\mathcal{A}^\diamondsuit) \arrow[r, "\text{Pontryagin}"] & \mathbf{H}_*(\mathcal{P}/\mathcal{A}^\diamondsuit) \end{tikzcd}$$

Taking the degree -1 stable parts, we get the commutative diagram

$$\begin{tikzcd} & V_\ell(\mathcal{P}^0/\mathcal{A}^\diamondsuit)[1] \otimes \overline{\mathbb{Q}}_\ell \oplus \overline{\mathbb{Q}}_\ell \otimes V_\ell(\mathcal{P}^0/\mathcal{A}^\diamondsuit)[1] \arrow[d, "\wedge"] \\ \mathbf{H}_*(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \arrow[ur, "\Phi_\lambda \otimes \widetilde{p}_! + \widetilde{p}_! \otimes \Phi_\mu"'] \arrow[dr, "\Phi_{\lambda+\mu}"] & \\ & V_\ell(\mathcal{P}^0/\mathcal{A}^\diamondsuit)[1] \end{tikzcd} \tag{4.40}$$

where $\widetilde{p}_! : \mathbf{H}_*(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \to \mathbf{H}_*(\mathcal{A}^\diamondsuit/\mathcal{A}^\diamondsuit) = \overline{\mathbb{Q}}_\ell$ is the push-forward along $\widetilde{p} : \widetilde{\mathcal{A}}^\diamondsuit \to \mathcal{A}^\diamondsuit$. The diagram (4.40) implies that $\Phi_{\lambda+\mu} = \Phi_\lambda + \Phi_\mu$. □

Using Lemma 4.7.1, and observing that the right-hand side of Δ_λ is torsion-free, we see that in order to show $\Delta_\lambda = 0$ for all $\lambda \in \mathbb{X}_*(T)$, it suffices to show it for a $\mathbb{Q}$-basis of $\mathbb{X}_*(T)_{\mathbb{Q}}$. Hence we can reduce the problem to the following lemma.

Lemma 4.7.2. *For each coroot $\alpha^\vee \in \Phi^\vee$, the map $\Delta_{\alpha^\vee} = 0$.*

Proof. Let $\mathfrak{t}_\alpha$ be the wall corresponding to the simple root α in $\mathfrak{t}$. The Killing form $\mathfrak{t} \xrightarrow{\sim} \mathfrak{t}^\vee$ identifies $\mathfrak{t}_\alpha$ with $\mathfrak{t}^\vee_{\alpha^\vee}$. Let $\widetilde{\mathcal{A}}^\diamondsuit_\alpha \subset \widetilde{\mathcal{A}}^\diamondsuit$ be the preimage of $\mathfrak{t}_{\alpha,D}$ under the evaluation morphism $\widetilde{\mathcal{A}}^\diamondsuit \to \mathfrak{t}_D$. Since $\widetilde{\mathcal{A}}^\diamondsuit \to \mathfrak{t}_D$ is smooth by the proof of [Yun11, Lemma 2.2.3], $\widetilde{\mathcal{A}}^\diamondsuit_\alpha$ is a smooth variety. Moreover, the morphism $\widetilde{\mathcal{A}}^\diamondsuit_\alpha \to \mathcal{A}^\diamondsuit$ is finite and surjective. Therefore, the natural map $\mathbf{H}_0(\widetilde{\mathcal{A}}^\diamondsuit_\alpha/\mathcal{A}^\diamondsuit) \to \mathbf{H}_0(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) = \overline{\mathbb{Q}}_\ell$ has a section; i.e., $\mathbf{H}_0(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)$ is a direct summand of $\mathbf{H}_0(\widetilde{\mathcal{A}}^\diamondsuit_\alpha/\mathcal{A}^\diamondsuit)$. Therefore, in order to show that $\Delta_{\alpha^\vee} = 0$, it suffices to show that the composition

$$\mathbf{H}_0(\widetilde{\mathcal{A}}^\diamondsuit_\alpha/\mathcal{A}^\diamondsuit) \to \mathbf{H}_0(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit) \xrightarrow{\Delta_{\alpha^\vee}} V_\ell(\mathcal{P}^{\vee,0}/\mathcal{A}^\diamondsuit)^*[1](1) \tag{4.41}$$

is zero. This composition is given by the difference of the restrictions of $\beta \circ \Phi_{\alpha^\vee}$ and $\Psi_{\alpha^\vee}$ on $\mathbf{H}_*(\widetilde{\mathcal{A}}^\diamondsuit_\alpha/\mathcal{A}^\diamondsuit)$. In view of the definitions of $\beta \circ \Phi_{\alpha^\vee}$ and $\Psi_{\alpha^\vee}$, the vanishing of (4.41) follows from the next two lemmas. $\square$

LEMMA 4.7.3. *The map induced by $s_{\alpha^\vee}$ (see (2.3))*

$$s_{\alpha^\vee,*} : \mathbf{H}_*(\widetilde{\mathcal{A}}^\diamondsuit_\alpha/\mathcal{A}^\diamondsuit) \to \mathbf{H}_*(\mathcal{P}/\mathcal{A}^\diamondsuit) \tag{4.42}$$

factors through the direct summand $\mathbf{H}_0(\mathcal{P}/\mathcal{A}^\diamondsuit) \subset \mathbf{H}_*(\mathcal{P}/\mathcal{A}^\diamondsuit)$ defined by the canonical decomposition in Lemma A.2.1.

Proof. We first claim that the morphism

$$\widetilde{\mathcal{A}}^\diamondsuit_\alpha \xrightarrow{s_{\alpha^\vee}} \mathcal{P} \xrightarrow{j_\mathcal{P}} \mathcal{P}\mathrm{ic}_T(\widetilde{\mathcal{A}}^\diamondsuit/\mathcal{A}^\diamondsuit)^W \tag{4.43}$$

is trivial (i.e., factors through the identity section). In other words, the line bundles $\mathcal{Q}_{\alpha^\vee}(\xi)$ are canonically trivialized on $\widetilde{\mathcal{A}}^\diamondsuit_\alpha \times_\mathcal{A} \widetilde{\mathcal{A}}$ for all $\xi \in \mathbb{X}^*(T)$. By Lemma 4.5.1, we have

$$\mathcal{Q}_{\alpha^\vee}(\xi)|_{\widetilde{\mathcal{A}}^\diamondsuit_\alpha \times_\mathcal{A} \widetilde{\mathcal{A}}} \cong \mathcal{O}\Bigg(\sum_{w\in W} \langle {}^w\alpha^\vee, \xi\rangle \Gamma_w(\widetilde{\mathcal{A}}^\diamondsuit_\alpha)\Bigg)$$

where $\Gamma_w(\widetilde{\mathcal{A}}^\diamondsuit_\alpha) = \Gamma_w \cap (\widetilde{\mathcal{A}}^\diamondsuit_\alpha \times_\mathcal{A} \widetilde{\mathcal{A}})$. Since the reflection r_α defined by α fixes $\widetilde{\mathcal{A}}^\diamondsuit_\alpha$, we have $\Gamma_w(\widetilde{\mathcal{A}}^\diamondsuit_\alpha) = \Gamma_{wr_\alpha}(\widetilde{\mathcal{A}}^\diamondsuit_\alpha)$. Therefore, we have an equality of divisors:

$$\begin{aligned}\sum_{w\in W} \langle {}^w\alpha^\vee, \xi\rangle \Gamma_w(\widetilde{\mathcal{A}}^\diamondsuit_\alpha) &= \sum_{w\in W/\langle r_\alpha\rangle} \langle {}^w\alpha^\vee + {}^{wr_\alpha}\alpha^\vee, \xi\rangle \Gamma_w(\widetilde{\mathcal{A}}^\diamondsuit_\alpha) \\ &= \sum_{w\in W/\langle r_\alpha\rangle} \langle {}^w\alpha^\vee - {}^w\alpha^\vee, \xi\rangle \Gamma_w(\widetilde{\mathcal{A}}^\diamondsuit_\alpha) = 0.\end{aligned}$$

Here $\sum_{w\in W/\langle r_\alpha\rangle}$ means summing over the representatives of the cosets $W/\langle r_\alpha\rangle$. Hence $\mathcal{Q}_{\alpha^\vee}(\xi)$ is canonically trivialized over $\widetilde{\mathcal{A}}^\diamondsuit_\alpha \times_\mathcal{A} \widetilde{\mathcal{A}}$; i.e., the map (4.43) is zero.

Let $\mathcal{K} = \ker(j_\mathcal{P})$. Then the morphism $s_{\alpha^\vee}$ factors through $\widetilde{\mathcal{A}}^\diamondsuit_\alpha \to \mathcal{K}$. Hence $s_{\alpha^\vee,*}$ factors through

$$\mathbf{H}_*(\widetilde{\mathcal{A}}^\diamondsuit_\alpha/\mathcal{A}^\diamondsuit) \to \mathbf{H}_*(\mathcal{K}/\mathcal{A}^\diamondsuit) \to \mathbf{H}_*(\mathcal{P}/\mathcal{A}^\diamondsuit).$$

By Lemma 4.1.2, $\mathcal{K}$ is finite over $\mathcal{A}^\diamondsuit$, and therefore $\mathbf{H}_*(\mathcal{K}/\mathcal{A}^\diamondsuit) = \mathbf{H}_0(\mathcal{K}/\mathcal{A}^\diamondsuit)$. Since the decomposition in Lemma A.2.1 is functorial for group stacks, the homomorphism $\mathcal{K} \to \mathcal{P}$ induces

$$\mathbf{H}_*(\mathcal{K}/\mathcal{A}^\diamondsuit) = \mathbf{H}_0(\mathcal{K}/\mathcal{A}^\diamondsuit) \to \mathbf{H}_0(\mathcal{P}/\mathcal{A}^\diamondsuit) \subset \mathbf{H}_*(\mathcal{P}/\mathcal{A}^\diamondsuit).$$

This proves the lemma. $\square$

LEMMA 4.7.4. *The map*

$$c_1(\mathcal{Q}(\alpha^\vee))^\natural : \mathbf{H}_*(\widetilde{\mathcal{A}}^\diamondsuit_\alpha/\mathcal{A}^\diamondsuit) \to \mathbf{H}^*(\mathcal{P}^\vee/\mathcal{A}^\diamondsuit)[2](1) \tag{4.44}$$

is zero.

Proof. It suffices to show that $c_1(\mathcal{Q}(\alpha^\vee)) = 0$, or, even stronger, the tautological line bundle $\mathcal{Q}(\alpha^\vee)$ is trivial on $\widetilde{\mathcal{A}}_\alpha \times_\mathcal{A} \mathcal{P}^\vee$. However, this follows from the description of $\mathcal{P}^\vee$ given in [DG02, § 16.3], as we recalled in the proof of Lemma 4.1.2. Therefore, the map (4.44) is also zero. $\square$

Tracing the above reductions backwards, we have already completed the proof of Theorem 3.2.4.

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Appendix A. Generalities on the Pontryagin and the cap product

In this appendix, we recall the formalism of cap product by the homology sheaf of a commutative smooth group scheme, partly following [Ngo10, § 7.4].

A.1 The group stack and its Tate module

Let $P^\sharp$ be a commutative smooth group scheme of finite type over a scheme S. Let $P=[P^\sharp/F]$ where F is a finite étale group scheme over S which acts trivially on $P^\sharp$. Let $P^0=[P^{\sharp,0}/F]$ be the neutral component of P.

Let ℓ be an invertible prime in S. For each $n\in\mathbb{Z}_{\geqslant 1}$, the ℓ^n-torsion $P^{\sharp,0}[\ell^n]$ is a sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules on S for the étale topology. The projective system $\{P^{\sharp,0}[\ell^n]\}_n$ gives a $\mathbb{Z}_\ell$-sheaf on S, which is called the *Tate module of $P^{\sharp,0}$ (or of P^0) over S*

$$T_\ell(P^0/S):=\varprojlim_n P^{\sharp,0}[\ell^n].$$

The $\overline{\mathbb{Q}}_\ell$-*Tate module* $V_\ell(P^0/S)$ is the same object $T_\ell(P^0/S)$, viewed as an object in $D_c^b(S,\overline{\mathbb{Q}}_\ell)$. The stalk of $V_\ell(P^0/S)$ at a geometric point $s\in S$ is the usual $\overline{\mathbb{Q}}_\ell$-Tate module of the group scheme $P_s^{\sharp,0}$.

A.2 The Pontryagin product on homology

Let $g:P\to S$ be the structure map and let $\mathbf{H}_*(P/S):=g_!g^!\overline{\mathbb{Q}}_{\ell,S}$ be the homology complex of P on S.

LEMMA A.2.1. *There is a canonical decomposition in $D_c^b(S,\overline{\mathbb{Q}}_\ell)$:*

$$\mathbf{H}_*(P/S)\cong\bigoplus_{i\geqslant 0}\mathbf{H}_i(P/S)[i]. \tag{A1}$$

Proof. This follows from Lieberman's trick [Kle68, 2A11]. Take any $N\in\mathbb{Z}$ which is coprime to the cardinalities of $\pi_0(P_s)$ for all $s\in S$ (such an N exists because there are only finitely many isomorphism types of $\pi_0(P_s)$). The Nth power map $[N]:P\to P$ induces an endomorphism $[N]_*$ on $\mathbf{H}_*(P/S)$. Let $\mathcal{H}_i$ be the direct summand of $\mathbf{H}_*(P/S)$ on which the eigenvalues of $[N]_*$ have norm N^i for any embedding $\overline{\mathbb{Q}}_\ell\hookrightarrow\mathbb{C}$. It is easy to see that $\mathcal{H}_i$ is independent of the choice of such N. By construction we have

$$\bigoplus_{i\geqslant 0}\mathcal{H}_i\subset\mathbf{H}_*(P/S). \tag{A2}$$

It remains to check that each $\mathcal{H}_i$ is a sheaf in degree $-i$, and that this inclusion is an equality. For each geometric point $s\in S$, we have $\mathrm{H}_i(P_s)=\overline{\mathbb{Q}}_\ell[\pi_0(P_s)]\otimes\mathrm{H}_i(P_s^0)=\overline{\mathbb{Q}}_\ell[\pi_0(P_s)]\otimes\bigwedge^i(V_\ell(P_s^0))$. Since N is prime to $\#\pi_0(P_s)$, $[N]_*$ induces an automorphism of $\pi_0(P_s)$; moreover, $[N]_*$ acts on $V_\ell(P_s^0)$ by N. Therefore, the eigenvalues of $[N]_*$ on $\mathrm{H}_i(P_s)$ are N^i times roots of unity. From this, we conclude that the stalk $\mathcal{H}_{i,s}$ is equal to $\mathrm{H}_i(P_s)[i]$. Therefore, (A2) is an equality and it gives the desired decomposition. □

The multiplication map $\mathrm{mult}: P \times_S P \to P$ induces a map on homology complexes, which is called the *Pontryagin product*,

$$\mathbf{H}_*(P/S) \otimes \mathbf{H}_*(P/S) \cong \mathbf{H}_*(P \times_S P/S) \xrightarrow{\mathrm{mult}_*} \mathbf{H}_*(P/S),$$

which, in turn, induces a Pontryagin product on the homology sheaves $\mathbf{H}_i(P/S)$. Since the multiplication map is compatible with the Nth power map in the obvious sense, the decomposition (A1) intertwines the Pontryagin product on the homology *complex* (left-hand side) and the Pontryagin product on the homology *sheaves* (right-hand side).

We have the following facts about the homology sheaves of P/S.

- There is a canonical isomorphism $\mathbf{H}_0(P/S) \cong \overline{\mathbb{Q}}_\ell[\pi_0(P/S)]$. Recall from [Ngo06, Proposition 6.2] that there is a sheaf of abelian groups $\pi_0(P/S)$ on S for the étale topology whose fiber at $s \in S$ is the finite group of connected components of P_s. Therefore, the group algebra $\overline{\mathbb{Q}}_\ell[\pi_0(P/S)]$ is a $\overline{\mathbb{Q}}_\ell$-sheaf of algebras on S whose fiber at $s \in S$ is the 0th homology of P_s. This algebra structure is the same as the one induced from the Pontryagin product.
- If P_s is *connected* for some geometric point $s \in S$, the stalk of $\mathbf{H}_1(P/S)$ at s is the $\overline{\mathbb{Q}}_\ell$-Tate module $V_\ell(P_s) = T_\ell(P_s) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell$ of P_s. Moreover, the Pontryagin product induces an isomorphism
$$\bigwedge^i V_\ell(P_s) = \bigwedge^i \mathbf{H}_1(P_s) \cong \mathbf{H}_i(P_s). \tag{A3}$$

Remark A.2.2. If we work with cohomology rather than homology, the Nth power map also gives a natural decomposition

$$\mathbf{H}^*(P/S) \cong \bigoplus_i \mathbf{H}^i(P/S)[-i]. \tag{A4}$$

This decomposition intertwines the cup product on the cohomology complex and the cup product on the cohomology sheaves.

A.3 The stable part

The Pontryagin product gives an action of $\mathbf{H}_0(P/S) = \overline{\mathbb{Q}}_\ell[\pi_0(P/S)]$ on $\mathbf{H}_*(P/S)$ and $\mathrm{H}_*(P/S)$.

Definition A.3.1. The *stable part* of $\mathbf{H}_*(P/S)$ (respectively $\mathbf{H}^*(P/S)$) is the maximal direct summand on which the action of $\pi_0(P/S)$ is trivial. We denote the stable parts by $\mathbf{H}_*(P/S)_{\mathrm{st}}$ and $\mathbf{H}^*(P/S)_{\mathrm{st}}$. They have decompositions

$$\mathbf{H}_*(P/S)_{\mathrm{st}} = \bigoplus_i \mathbf{H}_i(P/S)_{\mathrm{st}}[i]; \quad \mathbf{H}^*(P/S)_{\mathrm{st}} = \bigoplus_i \mathbf{H}^i(P/S)_{\mathrm{st}}[-i]$$

induced from (A1) and (A4).

Remark A.3.2. To make sense of the invariants of a sheaf under the action of another sheaf of finite abelian groups, we refer to [Ngo06, Proposition 8.3].

Remark A.3.3. The stable part construction is functorial with respect to pull-back. Suppose we have a homomorphism $\phi: P \to Q$ of smooth commutative group stacks over S. The pull-back $\phi^*: \mathbf{H}^*(Q/S) \to \mathbf{H}^*(P/S)$ is $\pi_0(P/S)$-equivariant (where $\pi_0(P/S)$ acts on $\mathbf{H}^*(Q/S)$ via $\phi_*: \pi_0(P/S) \to \pi_0(Q/S)$). Therefore, ϕ^* sends $\mathbf{H}^*(Q/S)_{\mathrm{st}}$ to $\mathbf{H}^*(P/S)_{\mathrm{st}}$. However, ϕ_* does not send $\mathbf{H}_*(P/S)_{\mathrm{st}}$ to $\mathbf{H}_*(Q/S)_{\mathrm{st}}$.

It is clear that the stable part $\mathbf{H}_*(P/S)_{\mathrm{st}}$ (respectively $\mathbf{H}^*(P/S)_{\mathrm{st}}$) inherits a Pontryagin product (respectively a cup product) from that of $\mathbf{H}_*(P/S)$ (respectively $\mathbf{H}^*(P/S)$).

Lemma A.3.4. (i) *The embedding $P^0 \subset P$ and the Pontryagin product gives a natural isomorphism of $\overline{\mathbb{Q}}_\ell[\pi_0(P/S)]$-algebra objects in $D^b_c(S, \overline{\mathbb{Q}}_\ell)$:*

$$\overline{\mathbb{Q}}_\ell[\pi_0(P/S)] \otimes \mathbf{H}_*(P^0/S) \xrightarrow{\sim} \mathbf{H}_*(P/S). \tag{A5}$$

The natural embedding $P^0 \subset P$ followed by the projection onto the stable part gives a natural isomorphism of algebra objects in $D^b_c(S, \overline{\mathbb{Q}}_\ell)$:

$$\bigwedge(V_\ell(P^0/S)[1]) \cong \mathbf{H}_*(P^0/S) \to \mathbf{H}_*(P/S) \twoheadrightarrow \mathbf{H}_*(P/S)_{\mathrm{st}}. \tag{A6}$$

(ii) *Similar statements hold for $H^*(P/S)$. In particular, we have an isomorphism of algebra objects (under the cup product and the exterior product)*

$$\mathbf{H}^*(P/S)_{\mathrm{st}} \cong \mathbf{H}^*(P^0/S) \cong \bigwedge(V_\ell(P^0/S)^*[-1]). \tag{A7}$$

Proof. We give the proof for part (i). To check (A5) and (A6) are isomorphisms, it suffices to check on the stalks. Fix a geometric point $s \in S$. Since all connected components of P_s are isomorphic to P^0_s, we have a $\pi_0(P_s)$-equivariant isomorphism

$$\mathrm{H}_*(P_s) \cong \mathrm{H}_*(P^0_s) \otimes \overline{\mathbb{Q}}_\ell[\pi_0(P_s)] \tag{A8}$$

on which $\pi_0(P_s)$ acts via the regular representation on $\overline{\mathbb{Q}}_\ell[\pi_0(P_s)]$. This proves (A5). Using (A8), the natural embedding $P^0_s \subset P_s$ followed by the projection onto the stable part

$$\mathrm{H}_*(P^0_s) \hookrightarrow \mathrm{H}_*(P_s) \twoheadrightarrow \mathrm{H}_*(P_s)_{\mathrm{st}} \tag{A9}$$

becomes the tensor product of the identity map on $\mathrm{H}_*(P^0_s)$ with the map

$$\overline{\mathbb{Q}}_\ell \cdot e \hookrightarrow \overline{\mathbb{Q}}_\ell[\pi_0(P_s)] \twoheadrightarrow \overline{\mathbb{Q}}_\ell[\pi_0(P_s)]^{\pi_0(P_s)} \tag{A10}$$

where $e \in \pi_0(P_s)$ is the identity element, and the second map is the projector onto the invariants. Since the composition of the maps in (A10) is an isomorphism, so is the composition of the maps in (A9). To obtain the first isomorphism in (A6), we need only to apply the isomorphism (A3) to the Picard stack P^0/S. □

A.4 The cap product

Suppose P acts on a Deligne–Mumford stack M over S, with the action and projection morphisms

$$P \times_S M \underset{\mathrm{proj}}{\overset{\mathrm{act}}{\rightrightarrows}} M.$$

Suppose $\mathcal{F}$ is a P-equivariant complex on M, then in particular we are given an isomorphism

$$\phi : \mathrm{act}^! \mathcal{F} \xrightarrow{\sim} \mathrm{proj}^! \mathcal{F}.$$

Therefore, we have a map

$$\mathrm{act}_! \mathrm{proj}^! \mathcal{F} \xrightarrow{\mathrm{act}_! \phi^{-1}} \mathrm{act}_! \mathrm{act}^! \mathcal{F} \xrightarrow{\mathrm{ad.}} \mathcal{F}. \tag{A11}$$

Let $f : M \to S$ be the structure map. Using Künneth formula (P is smooth over S), we get

$$\mathbf{H}_*(P/S) \otimes f_! \mathcal{F} = g_! \mathbb{D}_{P/S} \otimes f_! \mathcal{F} \cong (g \times f)_! \mathrm{proj}^! \mathcal{F} = f_! \mathrm{act}_! \mathrm{proj}^! \mathcal{F}. \tag{A12}$$

Applying $f_!$ to the map (A11) and combining it with the isomorphism (A12), we get the *cap product*

$$\cap : \mathbf{H}_*(P/S) \otimes f_!\mathcal{F} \to f_!\mathcal{F} \tag{A13}$$

such that $f_!\mathcal{F}$ becomes a module over the algebra $\mathbf{H}_*(P/S)$ under the Pontryagin product. Using the decomposition (A1) we get the actions

$$\bigcap_i : \mathbf{H}_i(P/S) \otimes f_!\mathcal{F} \to f_!\mathcal{F}[-i],$$
$$\bigcap_i^m : \mathbf{H}_i(P/S) \otimes \mathbf{R}^m f_!\mathcal{F} \to \mathbf{R}^{m-i} f_!\mathcal{F}.$$

When $i=0$, the cap product $\bigcap_0$ gives an action of $\overline{\mathbb{Q}}_\ell[\pi_0(P/S)]$ on $f_!\mathcal{F}$. By the isomorphism (A5), to understand the cap product, we need only to understand $\bigcap_0$ and $\bigcap_1$.

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