NOTES ON $\mathcal{D}$-MODULES (FOR TALBOT 2008)

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These are the notes prepared for an introductory lecture on $\mathcal{D}$-modules presented during the Talbot workshop (April 2008). The biblical reference is [4]. I would like to thank D.Gaitsgory for answering a lot of my questions and other participants of the Talbot for helpful feedbacks.

1. Definition of algebraic $\mathcal{D}$-modules

For any scheme X, $D^b(\mathcal{O}_X)$ will denote the bounded derived category of quasi-coherent complexes on X and $\mathrm{qcoh}(\mathcal{O}_X)$ the abelian category of quasi-coherent sheaves. The dualizing *complex* will be denoted by ω_X.

In this section, the ambient scheme X is *smooth* of equidimension n over an algebraically closed field k of characteristic 0. The tangent sheaf and the sheaf of i-forms will be denoted by Θ_X and Ω^i_X respectively. Note that $\omega_X = \Omega^n_X[n]$.

1.1. **The definitions.** We only need to define the sheaf of differential operators $\mathcal{D}_X$ and $\mathcal{D}_X$-modules will be sheaves on the Zariski site of X with a left module structure under $\mathcal{D}_X$. We give several equivalent definitions

- This is a quasi-coherent $\mathcal{O}_X$-module defined as the quotient of the tensor algebra $\bigotimes^*_{\mathcal{O}_X} \Theta_X$ be the two-sided ideal generated by $\xi f - f\xi = \mathrm{Lie}_\xi(f)$ and $\xi\eta - \eta\xi = [\xi, \eta]$ for $\xi, \eta \in \Theta_X$ and $f \in \mathcal{O}_X$. (Looks like a "universal enveloping algebra")
- For any Zariski open subset $U \subset X$, $\mathcal{D}_X(U) \subset \mathrm{End}_k(\mathcal{O}_X(U))$ is the subalgebra generated by multiplication by functions $\mathcal{O}_X(U)$ and derivations $\Theta_X(U)$.
- For any Zariski open subset $U \subset X$, $\mathcal{D}^{\leq i}_X(U) \subset \mathrm{End}_k(\mathcal{O}_X(U))$ consists of those operators P such that $[[P, f_0], \cdots, f_i] = 0$ for any $f_0, \cdots, f_1 \in \mathcal{O}_X(U)$ (these are differential operators of degree $\leq i$). $\mathcal{D}_X(U) = \cup_i \mathcal{D}^{\leq i}_X(U)$.
- Consider the formal completion $\mathfrak{X}$ of the diagonal $X \subset X \times X$, it has an $\mathcal{O}_X$-bimodule structure. We define $\mathcal{D}_X$ as $\underline{\mathrm{Hom}}_{cont,\mathcal{O}_X}(\mathcal{O}_\mathfrak{X}, \mathcal{O}_X)$ (using one $\mathcal{O}_X$ to define $\underline{\mathrm{Hom}}$ but the result is still an $\mathcal{O}_X$-bimodule). Note that we take continuous dual.

A $\mathcal{D}_X$-module is *quasi-coherent* if its quasi-coherent as an $\mathcal{O}_X$-module. They form an abelian category $\mathrm{qcoh}(\mathcal{D}_X)$. We will let $D^b_{\mathrm{qcoh}}(\mathcal{D}_X)$ to denote the bounded derived category of $\mathcal{D}_X$-modules with quasi-coherent cohomologies. We have canonically

$$D^b(\mathrm{qcoh}(\mathcal{D}_X)) \cong D^b_{\mathrm{qcoh}}(\mathcal{D}_X).$$

We suppress "qcoh" from the notation since we never need to consider larger categories.

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1.1.1. **Example.** Suppose $X = \mathbb{A}^n$ and we are given an algebraic PDE: $P\underline{f} = 0$ where $\underline{f} = (f_1, \cdots, f_m)^t$ and P is a $\ell \times m$ matrix of differential operators with polynomial coefficients. We can form the $\mathcal{D}_X$-module M associated to this PDE as the cokernel of $\mathcal{D}_X^{\ell} \xrightarrow{P^t} \mathcal{D}_X^m$. A $\mathcal{D}_X$-module morphism $M \to \mathcal{O}_X$ gives a solution to the PDE. This also makes sense for analytic $\mathcal{D}$-modules where we have more chance to get solutions (cf section 4.2)

- Let $X = \mathbb{A}^1$, the $\mathcal{D}_X$-module $\mathcal{O}_X e^x$ *generated by* e^x is the cokernel of $\mathcal{D}_X \xrightarrow{\partial_x - 1} \mathcal{D}_X$.
- Let $X = \mathbb{A}^1$, $\lambda \in k$, the $\mathcal{D}_X$-module $\mathcal{O}_X x^\lambda$ *generated by* x^λ is the cokernel of $\mathcal{D}_X \xrightarrow{x\partial_x - \lambda} \mathcal{D}_X$.

Here are some alternative ways to think of $\mathcal{D}_X$-modules.

- As $\mathcal{O}_X$-modules with flat connections. For a $\mathcal{D}_X$-module M, the action of Θ_X gives a map (which is not $\mathcal{O}_X$-linear)
$$\nabla : M \to \Omega_X^1 \otimes_{\mathcal{O}_X} M.$$
The defining relations of $\mathcal{D}_X$ ensures that ∇ is a flat connection.
- As deformations of quasi-coherent sheaves on the cotangent bundle. If we consider λ-connections, $\lambda \in \mathbb{A}^1$, we get a $\mathbb{G}_m$-equivariant family of filtered $\mathcal{O}_X$-algebras over $\mathbb{A}^1$ whose fiber at 1 is $\mathcal{D}_X$ and fiber at 0 is $\mathrm{Sym}^*(\Theta_X) = \mathcal{O}_{T^*X}$. Therefore the category of $\mathcal{D}_X$-modules can be thought of as a deformation of the category of $\mathcal{O}_{T^*X}$-modules. In particular, the associated graded of this family is canonically trivialized. The singular support (support of the classical limit) makes sense as a conical cycle in T^*X.
- As $\mathcal{O}$-modules on the crystalline site (or $\mathcal{D}$-crystals, cf [2]). A $\mathcal{D}_X$-module can be viewed as a quasi-coherent $\mathcal{O}_X$-module M with the following data: whenever two maps $\mathrm{Spec}\, R \rightrightarrows X$ coincide on $\mathrm{Spec}\, R^{\mathrm{red}}$, the pull-backs of M to $\mathrm{Spec}\, R$ are canonically identified.

  More precisely, we consider the crystalline site X_{crys}. Objects in this site are pairs $(U, \hat{U})$ consisting of a Zariski open set $U \subset X$ with a thickening $U \hookrightarrow \hat{U}$. For $p : (U, \hat{U}) \to (V, \hat{V})$, we can define $p^!$ or $p^* : D^b(\mathcal{O}_{\hat{V}}) \to D^b(\mathcal{O}_{\hat{U}})$ and form the categories $\mathfrak{D}(\mathcal{O}^!_{\mathrm{crys}})$ or $\mathfrak{D}(\mathcal{O}^*_{\mathrm{crys}})$ fibered over X_{crys}. An $\mathcal{O}^!_{\mathrm{crys}}$ (resp. $\mathcal{O}^*_{\mathrm{crys}}$)-complex is a Cartesian section of $\mathfrak{D}(\mathcal{O}^!_{\mathrm{crys}})$ (resp. $\mathfrak{D}(\mathcal{O}^*_{\mathrm{crys}})$). For X smooth, it is enough to consider the hyper-covering
$$\cdots \mathfrak{X}_3 \mathrel{\substack{\longrightarrow\\[-0.6ex]\longrightarrow\\[-0.6ex]\longrightarrow}} \mathfrak{X}_2 \underset{p_2}{\overset{p_1}{\rightrightarrows}} X .$$
  where $\mathfrak{X}_n$ is the formal completion of X^n along the small diagonal. Therefore an $\mathcal{O}_{\mathrm{crys}}$-module is a descent datum for this hyper-covering. Note that by the last definition of $\mathcal{D}_X$, we have $\mathcal{D}_X = p_{2,*}p_1^!\mathcal{O}_X$. We can use this to identify $\mathcal{O}^!_{\mathrm{crys}}$ (resp. $\mathcal{O}^*_{\mathrm{crys}}$)-complexes with complexes of right (resp. left) $\mathcal{D}_X$-modules (see section 1.3). It is this point of view that generalizes to singular and ind-schemes.

1.2. **Finiteness conditions.** A $\mathcal{D}_X$-module is *coherent* if it is finitely generated over $\mathcal{D}_X$. They form an abelian category $\mathrm{coh}(\mathcal{D}_X)$. As above, we have canonically
$$D^b(\mathrm{coh}(\mathcal{D}_X)) \cong D^b_{\mathrm{coh}}(\mathcal{D}_X).$$

Remark. Although this seems to be a reasonable finiteness condition, it is not stable under standard functors as we will see in example 2.1.1. We need a stronger finiteness condition which is the following.

A coherent $\mathcal{D}_X$-module is *holonomic* if its singular support has minimal dimension (=$\dim X$). They form an abelian category $\mathrm{hol}(\mathcal{D}_X)$. As above, we have canonically (by J.Bernstein)

$$D^b(\mathrm{hol}(\mathcal{D}_X)) \cong D^b_{\mathrm{hol}}(\mathcal{D}_X).$$

1.2.1. **Example.** If $\dim X > 0$, $\mathcal{D}_X$ is *not* holonomic. However, a coherent $\mathcal{O}_X$-module with a flat connection is holonomic.

1.3. **The left-right issue and Verdier duality.** We write "$\mathcal{D}_X^{op}$-modules" for right $\mathcal{D}_X$-modules. We have an equivalence of categories given by

$$D^b(\mathcal{D}_X) \underset{\overleftarrow{\Omega}}{\overset{\overrightarrow{\Omega}}{\rightleftarrows}} D^b(\mathcal{D}_X^{op})$$

where

$$\overrightarrow{\Omega} = \omega_X \otimes_{\mathcal{O}_X} ; \overleftarrow{\Omega} = \otimes_{\mathcal{O}_X} \omega_X^{-1}.$$

Remark. The sheaf ω_X has a natural $\mathcal{D}_X^{op}$-module structure given by Lie derivative. The action of $\xi \in \Theta_X$ on $\omega \otimes_{\mathcal{O}_X} M$ is given by $-\mathrm{Lie}_\xi \otimes 1 - 1 \otimes \xi$.

From the crystalline point of view, the transition from $\mathcal{O}^*_{\mathrm{crys}}$-complexes to $\mathcal{O}^!_{\mathrm{crys}}$-complexes (view as Cartesian sections of fibered categories over X_{crys}) are given by $\omega_{\hat{U}} \otimes (-)$ for each $(U, \hat{U}) \in X_{\mathrm{crys}}$. As we will see in the case of singular and ind-schemes (section 3.2), it is more natural to identify left and right $\mathcal{D}_X$-modules and view the left-right issue as different forgetful functors $D^b(\mathcal{D}_X) \to D^b(\mathcal{O}_X)$. We usually prefer working with *right* $\mathcal{D}$-modules since the Riemann-Hilbert correspondence (see section 4) works better for them.

We define Verdier duality for coherent left $\mathcal{D}_X$-modules by

$$\mathbb{D}_X : D^b(\mathcal{D}_X) \xrightarrow{\underline{\mathrm{Hom}}_{\mathcal{D}_X}(-,\mathcal{D}_X)} D^b(\mathcal{D}_X^{op}) \xrightarrow{\overleftarrow{\Omega}} D^b(\mathcal{D}_X).$$

1.3.1. **Proposition.** *Verdier duality is a contravariant auto-equivalence of $D^b_{\mathrm{coh}}(\mathcal{D}_X)$. It is t-exact under the natural t-structure.*

2. The six-functor formalism for $\mathcal{D}$-modules

All functors are derived. For a continuous map f, push-forward and pull-back of plain sheaves are denoted by $f_\bullet$ and $f^\bullet$. Suppose $f : X \to Y$ is a morphism between two smooth equidimensional schemes over k.

2.1. **†-pullback.** We define

$$f^\dagger : D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X) \tag{2.1.1}$$
$$M \mapsto f^* M. \tag{2.1.2}$$

(pull-back as $\mathcal{O}_X$-complexes) The Θ_X-action on f^*M is induced by the tangent map $\Theta_X \to f^\bullet \Theta_Y$.

Similarly, we define

$$f^\dagger : D^b(\mathcal{D}_Y^{op}) \to D^b(\mathcal{D}_X^{op}) \tag{2.1.3}$$
$$M \mapsto f^! M. \tag{2.1.4}$$

It is easy to check that the two definitions are compatible with the identification in section 1.3.

2.1.1. **Example.** Suppose $f : X = \{0\} \hookrightarrow \mathbb{A}^n = Y$. Then $f^\dagger \mathcal{D}_Y$ is $k[\partial_1, \cdots, \partial_n]$, which is not a coherent $\mathcal{D}_X$-module.

2.1.2. **Example.** Suppose Y is a point, then $f^\dagger \omega_Y = \omega_X$ as right $\mathcal{D}_X$-modules.

2.2. **$\dagger$-pushforward.** We define

$$f_\dagger : D^b(\mathcal{D}_X^{op}) \rightarrow D^b(\mathcal{D}_Y^{op}) \tag{2.2.1}$$
$$M \mapsto f_\bullet(M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\rightarrow Y}). \tag{2.2.2}$$

where $\mathcal{D}_{X\rightarrow Y} = f^\dagger \mathcal{D}_Y$ is naturally a $(\mathcal{D}_X, f^\bullet \mathcal{D}_Y)$-bimodule.

2.2.1. **Example.** Suppose $f : X = \{0\} \hookrightarrow \mathbb{A}^n = Y$. Then $f_\dagger \mathcal{O}_X$ is $k[\partial_1, \cdots, \partial_n]$ (the Dirac distribution supported at the origin), which is a holonomic $\mathcal{D}_Y$-module.

2.2.2. **Example.** Suppose f is an open immersion. Then $f_\dagger M = f_* M$ as $\mathcal{O}_Y$-complexes.

Remark. We see from definition that $f_\dagger$ is a composite of a left exact functor and right exact functor, hence it is neither left nor right exact. When f is an affine morphism, $f_\bullet$ is exact, hence $f_\dagger$ is right exact; when f is a closed embedding, $f_\dagger$ is t-exact (see theorem 3.1).

2.2.3. **Example.** Suppose X is affine and Y is a point. Then $f_\dagger$ is the left derived functor of $M \mapsto M/M\Theta_X$ (de-Rham cohomology). Therefore $f_\dagger \mathcal{D}_X = \Gamma(X, \mathcal{O}_X)$, which is not coherent in general. However, coherence is preserved by $f_\dagger$ if f is proper.

2.3. **Other functors.** For right $\mathcal{D}$-modules, we define $f^! = f^\dagger$ and $f_* = f_\dagger$. As in the topological situation, we define $f^* := \mathbb{D}_X \circ f^\dagger \circ \mathbb{D}_Y$ and $f_! := \mathbb{D}_Y \circ f_\dagger \circ \mathbb{D}_X$. We have

2.3.1. **Proposition.** *D^b_{hol} is preserved by these functors, and the usual adjunctions hold.*

2.4. **Tensor and inner Hom.** Exterior tensor product $\boxtimes$ is easy to define. We can define tensor product for right $\mathcal{D}$-modules to be:

$$\begin{aligned} \otimes^! : D^b(\mathcal{D}_X^{op}) \times D^b(\mathcal{D}_X^{op}) &\rightarrow D^b(\mathcal{D}_X^{op}) \\ (M, N) &\rightarrow \Delta^!(M \boxtimes N). \end{aligned}$$

This endows $D^b(\mathcal{D}_X^{op})$ with a monoidal structure with unit object ω_X.

We can define $\otimes^*$ for left $\mathcal{D}$-modules by using $*$ restriction of $M \boxtimes N$ to the diagonal. The underlying $\mathcal{O}_X$-complex is the same as the usual tensor product over $\mathcal{O}_X$. The unit object is $\mathcal{O}_X$.

Inner $\underline{\mathrm{Hom}}$ is defined as a right adjoint of $\otimes^!$ or $\otimes^*$. For left $\mathcal{D}$-modules, the underlying $\mathcal{O}_X$-complex is the same as the usual $\underline{\mathrm{Hom}}_{\mathcal{O}_X}$.

3. Kashiwara's theorem and applications

Suppose $i : Z \hookrightarrow X$ is a closed embedding and $j : U \hookrightarrow X$ is the complement. Let $D^b_Z(\mathcal{D}_X^{op}) \subset D^b(\mathcal{D}_X^{op})$ be the full triangulated subcategory consisting of complexes with *set-theoretical* support in Z (or only require this cohomologically). In other words, we have an exact sequence of triangulated categories

$$D^b_Z(\mathcal{D}_X^{op}) \underset{\Gamma_{|Z|}}{\overset{}{\leftrightarrows}} D^b(\mathcal{D}_X^{op}) \underset{j_\dagger}{\overset{j^\dagger}{\rightleftarrows}} D^b(\mathcal{D}_U^{op})$$

3.1. **Theorem** (Kashiwara)**.** *We have an equivalence of categories given by*

$$D^b(\mathcal{D}_Z^{op}) \underset{i^\dagger}{\overset{i_\dagger}{\rightleftarrows}} D^b_Z(\mathcal{D}_X^{op}) .$$

which is also t-exact with respect to the natural t-structures.

3.2. **$\mathcal{D}$-modules on singular and ind-schemes.**

3.2.1. **Example.** For X singular, $\mathcal{D}_X$ is bad behaved. Take $X \subset \mathbb{A}^2$ to be the cusp curve $y^2 = x^3$. Then the global sections of $\mathcal{D}_X$ is not a Noetherian ring.

To remedy, we define *right* $\mathcal{D}_X$-modules instead using Kashiwara's theorem: taking (local) embedding of X into a smooth X', and let

$$D^b(\mathcal{D}_X^{op}) := D^b_X(\mathcal{D}_{X'}^{op}).$$

where now the LHS is merely a symbol, but it coincides with the old notion for X smooth. One checks that $D^b(\mathcal{D}_X^{op})$ is canonically independent of the choice of X' and Verdier duality and six functors still make sense and work well.

For a strict ind-scheme X of ind-finite type $X = \bigcup X_n$, we can define

$$D^b(\mathcal{D}_X^{op}) := \varinjlim D^b(\mathcal{D}_{X_n}^{op}).$$

A more intrinsic way to define right $\mathcal{D}$-modules on singular or ind-schemes is to define them as Cartesian sections of $\mathfrak{D}(\mathcal{O}^!_{\text{crys}})$-modules on the crystalline site (see the last paragraph of section 1.1). To work with ind-schemes, we have to modify the crystalline site by considering $(X \overset{j}{\hookleftarrow} U \hookrightarrow \hat{U})$ where j can be any locally closed embedding into some X_n. In particular, by forgetting all the sections except the section over X, we get

$$\text{Forget}^{op} : D^b(\mathcal{D}_X^{op}) \to D^b(\mathcal{O}_X)$$

If X is a singular scheme with an embedding $i : X \hookrightarrow X'$ into a smooth one, it is easy to see that $\text{Forget}(M) = i^! M$ where $i^!$ is taken in the $\mathcal{O}$-module sense (right derived functor of sections *scheme-theoretically* supported on X).

If X is an ind-scheme, we have to make sense of $\mathcal{O}_X$-modules first. This is defined as a Cartesian section of the category $\mathfrak{D}(\mathcal{O}^!_{\text{Zar}})$ fibered over the Zariski site X_{Zar}. Concretely, an $\mathcal{O}_X$-module M is a collection of M_n on X_n with isomorphisms $i^!_{n-1} M_n \cong M_{n-1}$. The global section can be defined as $\Gamma(X, M) := \varinjlim \Gamma(X_n, M_n)$.

Similarly, we can define *left* $\mathcal{D}_X$-modules as Cartesian sections of $\mathfrak{D}(\mathcal{O}^*_{\text{crys}})$.

3.2.2. **Example.** For the affine Grassmannian $X = \mathcal{G}r_G = G(F)/G(\mathcal{O}_F)$ (where $F = k((z))$ and $\mathcal{O}_F = k[[z]]$), let δ be the Dirac distribution at the base point. Then the global sections of δ as a quasi-coherent $\mathcal{O}_X$-module is $\Gamma(\mathcal{G}r_G, \delta) = U(\mathfrak{g} \otimes F)/$(the right ideal generated by $\mathfrak{g} \otimes \mathcal{O}_F$).

4. The Riemann-Hilbert correspondence

In this section, X is a *smooth* equidimensional scheme over $\mathbb{C}$.

4.1. **Regularity.** A holonomic $\mathcal{D}_X$-module is *regular* (or has *regular singularity*) if its !-pullback to any smooth curve is. For X a smooth curve, let $\overline{X}$ be a compactification and $Z = \overline{X} - X$. A $\mathcal{D}_X$-module M (viewed as a quasi-coherent $\mathcal{O}_X$-module with connection ∇) is *regular* if there exists an extension $(\tilde{M}, \tilde{\nabla})$ of (M, ∇) to $\overline{X}$ such that $\tilde{\nabla}(\tilde{M}) \subset \Omega^1_{\overline{X}}(\log Z) \otimes_{\mathcal{O}_{\overline{X}}} \tilde{M}$.

Remark. Unlike holonomicity, regularity is an algebraic notion, which does not pass to analytic $\mathcal{D}_{X^{an}}$-modules. Consider the case $X = \mathbb{A}^1$ and the left $\mathcal{D}_X$-modules M generated by e^x. Then M is not regular at ∞. We have $M^{an} \cong \mathcal{O}_{X^{an}}$ but $M \ncong \mathcal{O}_X$.

As in section 1.2, we define $\mathrm{rh}(\mathcal{D}_X)$ and $D^b_{\mathrm{rh}}(\mathcal{D}_X)$.

4.2. **De-Rham functor.** We define the *de-Rham* functor

$$\begin{aligned} \mathrm{dR} : D^b(\mathcal{D}_X) &\to D^b(X^{an}; \mathbb{C}) \\ M &\mapsto (\omega_X \otimes_{\mathcal{D}_X} M)^{an} \end{aligned}$$

$$\begin{aligned} \mathrm{dR} : D^b(\mathcal{D}_X^{op}) &\to D^b(X^{an}; \mathbb{C}) \\ M &\mapsto (M \otimes_{\mathcal{D}_X} \mathcal{O}_X)^{an} \end{aligned}$$

Using the Koszul resolution of ω_X by locally free $\mathcal{D}_X$-modules, we recover the usual de-Rham complex for left $\mathcal{D}_X$-modules:

$$\mathrm{dR}(M) \xrightarrow{\mathrm{qis}} ((\Omega^*_X \otimes_{\mathcal{O}_X} M[\dim X])^{an}, \delta).$$

where the differential on $\Omega^i_X \otimes M$ is $\delta^i = d \otimes 1 + (-1)^i 1 \wedge \nabla$.

Another useful functor is the solution functor

$$\mathrm{Sol} : D^b(\mathcal{D}_X) \to D^b(X^{an}; \mathbb{C}) \tag{4.2.1}$$

$$M \mapsto \underline{\mathrm{Hom}}_{\mathcal{D}_{X^{an}}}(M^{an}, \mathcal{O}_{X^{an}}). \tag{4.2.2}$$

It is easy to show that when restricted to coherent left $\mathcal{D}_X$-modules

$$\mathrm{Sol}[\dim X] = \mathrm{dR} \circ \mathbb{D}_X.$$

Remark. In the definition of Sol, it is important to first analytify and then take Hom, otherwise there will not be enough "solutions".

4.3. **Theorem** (R-H correspondence)**.**

(1) *The functor* $\mathrm{dR}_{\mathrm{rh}}$ *induces an exact functor*

$$\mathrm{dR}_{\mathrm{hol}} : D^b_{\mathrm{hol}}(\mathcal{D}_X^{op}) \to D^b_{\mathrm{con}}(X^{an}; \mathbb{C})$$

which is t-exact with respect to the natural t-structure on the LHS and the perverse t-structure on the RHS;

(2) *The functor* dR *induces an equivalence*

$$\mathrm{dR}_{\mathrm{rh}} : D^b_{\mathrm{rh}}(\mathcal{D}_X^{op}) \cong D^b_{\mathrm{con}}(X^{an}; \mathbb{C}).$$

which is compatible with Verdier dualities and six functors.

Remark. By the first definition of section 3.2, the above theorem also holds for singular schemes.

4.3.1. **Corollary.**

(1) *The functor* dR *induces an equivalence of abelian categories:*

$$\mathrm{dR}_{ab} : \mathrm{rh}(\mathcal{D}_X) \cong \mathrm{Perv}(X^{an}; \mathbb{C})$$

which further specializes to the well known equivalence:

$$\{\text{Vector bundles with flat regular connection on } X\} \leftrightarrow \{\text{Local systems on } X^{an}\}$$

(2) *(A.Beilinson* [1]*)The functor* $\mathrm{dR}_{\mathrm{rh}} \circ D^b(\mathrm{dR}_{ab}^{-1})$ *gives a realization functor which is an equivalence*

$$D^b(\mathrm{Perv}(X^{an}, \mathbb{C})) \cong D^b(\mathrm{rh}(\mathcal{D}_X)) \overset{\text{Beilinson}}{\cong} D^b_{\mathrm{rh}}(\mathcal{D}_X) \cong D^b_{\mathrm{con}}(X^{an}, \mathbb{C}).$$

Remark. The de-Rham functor behaves well for holonomic $\mathcal{D}_X$-modules, but it is not an equivalence. The reason is when we pass to analytic $\mathcal{D}_{X^{an}}$-modules, we already lose information. Consider the case $X = \mathbb{A}^1$ and the right $\mathcal{D}_X$-modules M generated by e^x. We have $M^{an} \cong \mathcal{O}_{X^{an}}$ but $M \ncong \mathcal{O}_X$.

The same example shows that $\mathrm{dR}_{\mathrm{hol}}$ does not commute with f_*.

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