Notes on Equivariant Derived Categories

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January 19, 2006

This is a brief review of the construction and properties of equivariant derived categories following [1]. By a sheaf we always mean a constructible sheaf of vector spaces over a fixed coefficient field. Standard operations of sheaves are understood to be derived.

1 Definition of equivariant sheaves

Let G be a topological group(usually assumed to be a Lie group) acting on a topological space X. Let $\alpha : G \times X \to X$ and $p : G \times X \to X$ be action and projection.

Definition 1. *A sheaf $\mathcal{F}$ on X is called G-*equivariant *if there is an isomorphism $\phi : \alpha^*\mathcal{F} \cong p^*\mathcal{F}$ satisfying the natural cocycle condition.*

Denote by $Sh_G(X)$ the abelian category of G-equivariant sheaves on X, and by $Sh(X)$ the abelian category of sheaves on X.

There is an alternative definition using simplicial spaces and sheaves. Let $[G\backslash X]$. be the simplicial space defined in [2]. Then a G-equivariant sheaf on X is, by definiton, a Cartesian sheaf on $[G\backslash X]$., i.e., a sheaf $\mathcal{F}_n$ on each $[G\backslash X]_n$ and an isomorphism $\phi_\theta : \theta^*\mathcal{F}_n \to \mathcal{F}_m$ for each structure map $\theta : [G\backslash X]_m \to [G\backslash X]_n$ satisfying natural compatibility conditions.

Proposition 1. *The two definitions of equivariant sheaves are equivalent.*

This is a combinatorial exercise.

Proposition 2. *If the action of G on X is free and let $\bar{X}$ be the quotient, then the pull-back functor defines an equivalence of categories $Sh(\bar{X}) \to Sh_G(X)$.*

2 Acyclic resolutions

Let n be a natural number or infinity.

Definition 2. *A map $f : X \to Y$ is called (universal) n-*acyclic*, if for any base change $Y' \to Y$ and the resulting map $f' : X' \to Y'$, and any sheaf $\mathcal{F}$ on Y', the natural map*

$$\mathcal{F} \to \tau_{\leq n} f'_* f'^* \mathcal{F}$$

is an isomorphism in $D^b(Y')$.

For example, if Z is a space with $H^i(Z) = 0$ for $1 \leq i \leq n$, then the projection $X \times Z \to X$ is n-acyclic.

Proposition 3. *If $f : X \to Y$ is n-acyclic and I is an interval of length $\leq n$, then*

$$f^* : D^I(Y) \to D^I(X)$$

is an equivalence of categories.

Definition 3. *A G-map $p : P \to X$ from a free G-space to X is called a* G-resolution *(or simply a* resolution*) of X.*

We denote by $Res_G(X)$ the category of G-resolution of X and G-maps between them. Finite product exists in this category (fibered product over X with diagonal action). There is a distinguished object in this category, namely the action map

$$\alpha : G \times X \to X$$

with G action on $G \times X$ by left multiplication on G.

We can talk about n-acyclic resolutions, smooth resolutions etc. We always assume X has ∞-acyclic resulotions, which is the case when pt admits ∞-acyclic resolutions.

3 Equivariant derived categories

We shall give two equivalent definitions. The first one is "categorical" or "stacky". Let D^b (resp. D^+)be the fibered category over the category Top of topological spaces with fiber over $X \in Top$ the bounded (resp. bounded below) derived category of sheaves on X. Let $\Phi : Res_G(X) \to Top$ be the functor sending P to the G-quotient $\bar{P}$. We define $D^b_G(X)$ (resp. $D^+_G(X)$)to be the "fiber" $D^b(\Phi)$ (resp. $D^+(\Phi)$)over the functor Φ. In down to earth language, to give an object of $D^b(X)$ amounts to give an object $\mathcal{F}(P) \in D^b(\bar{P})$ for each $P \in Res_G(X)$ and for each G-map $f : P \to R$ over X inducing $\bar{f} : \bar{P} \to \bar{R}$ an isomorphism $f^*\mathcal{F}(R) \cong \mathcal{F}(P)$ compatible with compositions.

This definition can be rephrased as follows. Let $p : P \to X$ be a G-resolution of X and $\pi : P \to \bar{P}$ be the quotient map. Let $D^b_G(X, P)$ be the category with objects $(\mathcal{F}_X, \mathcal{F}(P), \phi)$ where $\mathcal{F}_X \in D^b(X)$,$\mathcal{F}(P) \in D^b(\bar{P})$ and ϕ an isomorphism

$$\phi : p^*\mathcal{F}_X \cong \pi^*\mathcal{F}(P).$$

Morphisms are defined in a natural way. For a G-map $f : P \to R$ of resolutions over X, there is a natural pull-back functor:

$$f^* : D^b_G(X, R) \to D^b_G(X, P).$$

We define $D^+_G(X, P)$ in the same way. Moreover, for an interval I of integers, we denote by $D^I_G(X, P)$ the full subcategory of $D^b_G(X, P)$ consisting of objects $(\mathcal{F}_X, \mathcal{F}(P), \phi)$ with $\mathcal{F}_X \in D^I(X)$.

$D^b_G(X)$ has a natural structure of a triangulate category. A diagram

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to \mathcal{F}'[1]$$

is an exact triangle if for all $P \in Res_G(X)$,

$$\mathcal{F}'(P) \to \mathcal{F}(P) \to \mathcal{F}''(P) \to \mathcal{F}'(P)[1]$$

is an exact triangle, or equivalently,

$$\mathcal{F}'_X \to \mathcal{F}_X \to \mathcal{F}''_X \to \mathcal{F}'_X[1]$$

is an exact triangle.

Proposition 4. *Let $n \geq$ the length of the interval I. Let P be an n-acyclic resolution of X, then the natural functor:*

$$D^I_G(X) \to D^I_G(X, P)$$

is an equivalence of categories. If $n = \infty$, we have the similar statement for D^b.

In particular,

Proposition 5. *If the action of G on X is free and let $\bar{X}$ be the quotient, then $D^b_G(X) \cong D^(\bar{X})$.*

In practice, to give an object of $D^b_G(X)$, we only need to assign $\mathcal{F}(P) \in D^b(\bar{P})$ for P in a sufficiently rich subcategory of $Res_G(X)$. Basically, we require this subcategory to have products, contain the distinguished resolution and contain sufficiently acyclic resolutions. We will use this remark in defining standard operations of equivariant sheaves.

The second definition is simplicial. Let $D^b_{Car}([G\backslash X].)$ be the derived category of simplicial sheaves on the simplicial space $[G\backslash X].$ with Cartesian cohomology sheaves.

Theorem 1. *There is a natual functor*

$$\alpha : D^b_G(X) \to D^b_{Car}([G\backslash X].)$$

which is an equivalence of triangulated categories.

We sketch a proof. First we construct the functor α. Each $G^{\Delta_n} \times X \to X$ is a G-resolution of X with G-quotient $[G\backslash X]_n$. Therefore an object $\mathcal{F} \in D^b_G(X)$ determines $\mathcal{F}^n \in D^b([G\backslash X]_n)$. The collection $(\mathcal{F}^n)_{n\geq 0}$ satisfies compatibility and Cartesian condition, hence defines an object $\alpha(\mathcal{F}) \in D^b_{Car}([G\backslash X].)$.

Let $P \to X$ be an ∞-acyclic resolution of X with G-quotient $\bar{P}$. Consider the following commutative diagram

$$\begin{tikzcd}
D^b_G(X,P) \arrow[r, "\cong"] \arrow[d, "\beta"] & D^b_G(X) \arrow[r, "\alpha_X"] \arrow[d, "\beta"] & D^b_{Car}([G\backslash X].) \arrow[d, "\gamma"] \\
D^b(\bar{P}) \arrow[r, "\cong"] & D^b_G(P) \arrow[r, "\alpha_P"] & D^b_{Car}([G\backslash P].)
\end{tikzcd}$$

By the properties of ∞-acyclic resolutions, we see β and γ are fully faithful. We claim α_P is an equivalence. In fact, the simplicial space $[G\backslash P].$ is isomorphic to the (augmented) simplicial space $\mathrm{cosq}(P \to \bar{P})$(notation is the same as in [2]). And the claim follows from the following general result.

Proposition 6. *If $X \to S$ locally admits a section, the the pull-back by the augmentation map:*

$$D^b(S) \cong D^b_{Car}(\mathrm{cosq}(X \to S)).$$

is an equivalence of triangulated categories.

Now the diagram shows α_X is fully faithful. But an object $(\mathcal{F}^n) \in D^b_{Car}([G\backslash P].)$ comes from $D^b_{Car}([G\backslash X].)$ if and only if each $\mathcal{F}^n$ comes from $D^b([G\backslash X]_n)$. In particular, $\mathcal{F}^0 \in D^b(P)$ comes from $D^b X$, and therefore $(\mathcal{F}^n)$ is the image ofsome object $\mathcal{F} \in D^b_G(X,P)$ via α_X. This shows the essential surjectivity.

Note that the distinguished resolution is sent to X by Φ, therefore for each $\mathcal{F} \in D^b_G(X)$ we get an object $\mathcal{F}_X \in D^b(X)$. This defines the forgetfull functor:

$$\mathrm{For} : D^b_G(X) \to D^b(X).$$

(Similar for D^+.) In particular, we can pull-back both the natural t-structure and the perverse t-structure from $D^b(X)$ to $D^b_G(X)$.

Proposition 7. *The pull-back of the natural t-structure via* For *is a t-structure on $D^b_G(X)$. The heart of this t-structure is naturally equivalent to $Sh_G(X)$.*

This is a consequence of the above theorem and the combinatorial exercise in section 1.

In fact, $\mathcal{F} \in D^{\leq a}_G(X)$ (resp. $D^{\geq a}_G(X)$) if and only if for any resolution P, $\mathcal{F}(P) \in D^{\leq a}(\bar{P})$ (resp. $D^{\geq a}(\bar{P})$).

However, it is not in general true that $D^b(Sh_G(X))$ is equivalent to $D^b_G(X)$.

Proposition 8. *Suppose we are in the complex algebraic setting. The pull-back of the perverse t-structure via* For *is a t-structure on $D^b_G(X)$. The heart of this t-structure is, by definition, the abelian category of G-equivariant perverse sheaves, denoted $Perv_G(X)$.*

This easily follows from the simplicial definition. A Cartesian complex $(\mathcal{F}^n) \in D^b_{Car}([G\backslash X].)$ (where $\mathcal{F}^n \in D^b([G\backslash X]_n)$) is in ${}^pD^{\geq 0}$ of the perverse t-structure if and only if $\mathcal{F}^n \in^p D^{\geq n\dim(G)}([G\backslash X]_n)$. This criterion enables us to check

$$Hom({}^pD^{\leq 0}, {}^pD^{<0}) = 0.$$

If X is equipped with a G-invariant stratification $\mathcal{S}$, we can define $D^b_{G,\mathcal{S}}(X)$ to be the full subcategory of complexes with $\mathcal{S}$-constructible cohomology sheaves (viewed as a complex on X via For). Similarly, we can define $Perv_{G,\mathcal{S}}(X)$.

All the above constructions carry over to the D^+ case.

4 Change of groups: restriction, induction and quotient functors

Let H be a subgroup of G. We shall define a restriction functor

$$\mathrm{Res}^G_H : D^b_G(X) \to D^b_H(X)$$

in the following way. Let $\mathcal{F} \in D^b_G(X)$. For any H-resolution $p : P \to X$ consider the induced G-resolution $p' : P' = G \times^H P \to X$ where $P \times^H X$ means twisted product with respect to the right H-multiplication on P and left H-action on X. We have $\bar{P} = \bar{P}'$ where the first bar is the H-quotient and second is the G-quotient. We define:

$$\mathrm{Res}^G_H(\mathcal{F})(P) = \mathcal{F}(P').$$

If K is a subgroup of H, we have an obvious transitivity relation:

$$\mathrm{Res}^H_K \circ \mathrm{Res}^G_H = \mathrm{Res}^G_K.$$

In particular, if H is the trivial group, Res^G_H becomes the forgetful functor.

Let X be an H-space and $X' = G \times^H X$ the induced G-space and $\iota : X \to X'$ the inclusion sending x to $(1, x)$. We shall define a functor

$$\iota^* : D^b_G(X') \to D^b_H(X)$$

in the following way. Let $\mathcal{F} \in D^b_G(X')$ and P an H-resolution of X. Then $P' = G \times^H X$ is a G-resolution of X'. We also have $\bar{P} = \bar{P}'$ with bars understood as above. We define:

$$(\iota^*\mathcal{F})(P) = \mathcal{F}(P').$$

Proposition 9. *The functor constructed above is an equivalence of categories, called the* induction equivalence*.*

The restriction functor has a left adjoint functor $\mathrm{Ind}^G_{H!}$ and a right adjoint functor Ind^G_{H*}. The definitions logically depend on the six operation formalism of the next subsection, nevertheless we shall give them here for coherence. Let H be a subgroup of G and X a G-space. Let $\alpha : X' = G \times^H X \to X$ be the action map and $\iota : X \to X'$ as above. Now the restriction can be viewed as a composition:

$$\mathrm{Res}^G_H = \iota^* \circ \alpha^* : D^b_G(X) \to D^b_G(X') \to D^b_H(X).$$

Therefore the right adjoint is easily seen to be

$$\mathrm{Ind}^G_{H*} = \alpha_* \circ (\iota^*)^{-1} : D^b_H(X) \to D^b_G(X') \to D^b_G(X).$$

To define the left adjoint, observe that

$$\alpha^* = D_\alpha^{-1} \otimes \alpha^!$$

where D_α is the relative dualizing complex of α. Hence

$$\operatorname{Res}_H^G = \iota^* \circ (D_\alpha^{-1} \otimes \alpha^!).$$

Therefore the left adjoint is easily seen to be

$$\operatorname{Ind}_{H!}^G = \alpha_! \circ (D_\alpha \otimes (\iota^*)^{-1}) : D_H^b(X) \to D_G^b(X') \to D_G^b(X).$$

Summarizing:

Proposition 10. *For any $\mathcal{F}_1 \in D_G^b(X)$ and $\mathcal{F}_2 \in D_H^b(X)$, we have*

$$Hom_{D_H^b(X)}(\operatorname{Res}_H^G \mathcal{F}_1, \mathcal{F}_2) = Hom_{D_G^b(X)}(\mathcal{F}_1, \operatorname{Ind}_{H*}^G \mathcal{F}_2)$$

and

$$Hom_{D_H^b(X)}(\mathcal{F}_2, \operatorname{Res}_H^G \mathcal{F}_1) = Hom_{D_G^b(X)}(\operatorname{Ind}_{H!}^G \mathcal{F}_2, \mathcal{F}_1).$$

Let's describe the induction functor $\operatorname{Ind}_{H*}^G$ in down-to-earth language. Consider the commutative diagram:

$$\begin{tikzcd} G \times X \arrow[r, "\pi"] \arrow[d, "pr"] & G \times^H X \arrow[d, "\alpha"] \\ X & X \end{tikzcd}$$

where pr is the projection onto the second factor. For $\mathcal{F} \in D_H^b(X)$, $pr^*\mathcal{F}$ is H-equivariant with respect to the anti-diagonal action of H on $G \times X$, hence descends to a sheaf $\mathcal{F}' \in D^d(G \times^H X)$. Finally, $\operatorname{Ind}_{H*}^G \mathcal{F} = \alpha_* \mathcal{F} \in D_G^b(X)$.

Let K be a normal subgroup of H with quotient G. Consider X as an K-space, the quotient is denoted by $\pi : X \to K\backslash X$. It inherits a G-action. We shall define a quotient functor

$$\operatorname{Quo} : D_G^b(K\backslash X) \to D_H^b(X)$$

in the following way. Let $\mathcal{F} \in D_G^b(K\backslash X)$. For any H-resolution $p : P \to X$, let $K\backslash P$ be the K-quotient. Then $K\backslash P$ is a K-resolution of $K\backslash X$. And we have $\bar{P} = \overline{K\backslash P}$ where the bars are understood in an obvious way. We define:

$$\operatorname{Quo}(\mathcal{F})(P) = \mathcal{F}(K\backslash P).$$

In particular, if $K = H$, we get the usual quotient functor:

$$\operatorname{Quo} : D^b(\bar{X}) \to D_H^b(X).$$

Proposition 11. *If the K-action on X is free, then the functor* Quo *defined above is an equivalence of categories, called the* quotient equivalence*.*

5 The six operation formalism

Let X and Y be G-spaces and $f : X \to Y$ be a G-map. We first define the functor

$$f_! : D_G^b(X) \to D_G^b(Y)$$

in the following way. Let $\mathcal{F} \in D_G^b(X)$ and P be a resolution of Y. Then $P_X = P \times_Y X$ is a resolution of X. We denote by $\bar{f}$ the natural projection map of quotient spaces $\bar{P_X} \to \bar{P}$. We define:

$$(f_!\mathcal{F})(P) = \bar{f}_!\mathcal{F}(P_X).$$

For a morphism $h : P \to R$ of resolutions, the proper base change theorem ensures that

$$(f_! \mathcal{F})(P) = \bar{h}^*(f_! \mathcal{F})(R).$$

The next is the pull-back functor:

$$f^* : D^b_G(Y) \to D^b_G(X).$$

Let $\mathcal{F} \in D^b_G(Y)$. To define an object in $D^b_G(X)$, we may only consider resolutions of the form $P_X = X \times_Y P$ where P is a resolution of Y since they are rich enough. We define:

$$(f^* \mathcal{F})(P_X) = \bar{f}^* \mathcal{F}(P)$$

where $\bar{f}$ is the same as above.

To define the functors f_* and $f^!$, we could proceed in the same way. However, in order for the sheaves defined to be Cartesian with respect to morphisms of resolutions, we need these morphisms to be smooth so that we can apply smooth base change theorem. Therefore we consider the subcategory $SRes_G(X)$ consisting of resolutions $p : P \to X$ where p is smooth and smooth G-maps between them. For a nice Lie group G, this subcategory is rich enough, so that we can only assign sheaves to these smooth resolutions. With $Res_G(X)$ replaced by $SRes_G(X)$, we can construct f_* and $f^!$ in exactly the same way as above.

The tensor functor

$$\otimes : D^b_G(X) \times D^b_G(X) \to D^b_G(X)$$

is defined by

$$(\mathcal{F}_1 \otimes \mathcal{F}_2)(P) = \mathcal{F}_1(P) \otimes \mathcal{F}_2(P)$$

for any smooth resolution P. The inner Hom functor

$$\mathcal{H}om : D^b_G(X)^\circ \times D^b_G(X) \to D^b_G(X)$$

is defined by

$$\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)(P) = \mathcal{H}om(\mathcal{F}_1(P), \mathcal{F}_2(P))$$

for any smooth resolution P.

We define the constant sheaf $C_X \in Sh_G(X)$ to be the one assigning a constant sheaf to each resolution. The relative dualing sheaf of a G-map $f : X \to Y$to be $D_f = f^! C_Y$. The (absolute) dualizing sheaf D_X of X is D_π where $\pi : X \to pt$. The Verdier duality functor

$$\mathcal{D} : D^b_G(X) \to D^b_G(X)$$

is defined by

$$\mathcal{D}\mathcal{F} = \mathcal{H}om(\mathcal{F}, D_X).$$

Proposition 12. *All these functors satisfy the usual adjointness,duality and base change properties.*

Let H be a subgroup of G.

Proposition 13. *All these functors commute with* Res^G_H*, the induction equivalence and the quotient equivalence.* Ind^G_{H*} *commutes with* f_* *and* $f^!$ *and* $\mathrm{Ind}^G_{H!}$ *commutes with* $f_!$ *and* f^*.

All the above constructions extends trivially to D^+_G.

Let Z be a G-stable locally closed subset of X. Then it makes sense to take the intermediate extension of the constant sheaf on Z, therefore the intersection cohomology complex $IC_{\bar{Z}} \in Perv_G(X)$ is defined.

6 The general pull-back and push-forward functors

In this section, we consider the following situation. Let $\phi : H \to G$ be a homomorphism of Lie groups and $f : X \to Y$ be a ϕ-map, i.e.,X is an H-space, Y is a G-space and the map f is H-G-equivariant. We want to define pull-back functor:

$$f_H^{G*} : D_G^b(Y) \to D_H^b(X).$$

Let $\mathcal{F} \in D_G^b(Y)$, $P \in Res_H(X)$ and $R \in Res_G(Y)$. We assume there is a ϕ-morphism $h : P \to R$ such that the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "h"] & X \arrow[d, "f"] \\ R \arrow[r] & Y \end{tikzcd}$$

is commutative. Such a map h is called *compatible*. We define:

$$(f_H^{G*}\mathcal{F})(P) = \bar{h}^*\mathcal{F}(R).$$

This definition is independent of R:for another choice $h' : P \to R'$, we consider $h'' : P \to R'' = R \times_Y R'$ and use Cartesian property of $\mathcal{F}$ to deduce canonical isomorphisms

$$\bar{h}^*\mathcal{F}(R) \cong \bar{h''}^*\mathcal{F}(R'') \cong \bar{h'}^*\mathcal{F}(R').$$

Since such P's (the ones with compatible maps to some $R \in Res_G(Y)$) form a rich enough subcategory of $Res_H(X)$, we have actually defined $f^*\mathcal{F}$ as an object of $D_H^b(X)$.

Special cases:

- If $\phi : H \hookrightarrow G$ is the inclusion of a subgroup and $f = id : X \to X$, then $f_H^{G*} = \mathrm{Res}_H^G$.
- If ϕ is injective and $Y = G \times^H X$ and f the natural inclusion of X into Y, then f_H^{G*} is the induction equivalence.
- If $\phi : H \to G$ is surjective with kernel K, K acts freely on X and $f : X \to Y = K\backslash X$ the quotient map, then f_H^{G*} is the quotient equivalence.

Next we define push-forward functor

$$f_{H*}^G : D_H^+(X) \to D_G^+(Y).$$

For $\mathcal{F} \in D_H^+(X)$, R a G-resolution of Y and and P an ∞-acyclic resolution of X. Consider the compatible map $h : P' = P \times_Y R \to R$. We define:

$$(f_{H*}^G\mathcal{F})(R) = \bar{h}_*\mathcal{F}(P').$$

It is easy to see this definition is independent of the choice of P as long as P is ∞-acyclic. In order for the Cartesian condition to hold, we have to use certain base change theorem. Therefore we have to restrict ourselves to those "good" resolutions(maybe infinite dimensional spaces) any "good" morphisms between them which enable us to apply base change. Fortunately for nice Lie groups, this subcategory is rich enough.

Special cases:

- If $\phi : H \hookrightarrow G$ is the inclusion of a subgroup and $f = id : X \to X$, then $f_{H*}^G = \mathrm{Ind}_{H*}^G$.
- If ϕ is the injective and $f : X \to Y = G \times^H X$ the natural inclusion, then f_{H*}^G is the inverse of the induction equivalence.
- If $\phi : H \to G$ is surjective with kernel K, K acts freely on X and $f : X \to Y = K\backslash X$ the quotient map, then f_{H*}^G is the inverse of the quotient equivalence.

Sometimes the push-forward functor is not as intuitive as it looks like. For example, let $G =< 1 >$ be the trivial group and $Y = pt$. Then the push-forward:

$$f_{H*}^{<1>} : D_H^+(X) \to D^+(pt)$$

is NOT first restricting to $D^+(X)$ then taking global sections. It is the equivariant cohomology functor that we will define in the next section.

Proposition 14. *f_H^{G*} is left adjoint to f_{H*}^G; they satisfy the obvious transitivity relations. Moreover, f_H^{G*} preserves the natural t-structure.*

Note that in general f_{H*}^G does not send $D_H^b(X)$ to $D_G^b(Y)$. For example, when G is trivial and $Y = pt$, this is the equivariant cohomology functor, which is usually not bounded.

7 Classifying space and equivariant cohomology

Let G be a connected Lie group throughout this section. Consider the category $D_G^+(pt)$. Let EG be a contractible free G-space and $BG = G\backslash EG$ the classifying space of G. By definition,

$$D_G^+(pt) = D_G^+(pt, EG)$$

consisting of triples $(\mathcal{F}_{pt}, \mathcal{F}(EG), \phi)$ where $\mathcal{F}_{pt} \in D^+(pt)$ is a complex of vector spaces, $\mathcal{F}(EG) \in D^+(BG)$ and ϕ an isomorphism of their pull-backs on EG. Therefore, we have an equivalence

$$D_G^+(pt) = D_{const}^+(BG)$$

where *const* denotes cohomologically constant(which is the same as locally constancy since BG is now simply connected) of finite rank.

Let $\mathcal{A}_G$ be the DG-algebra isomorphic to the cohomology ring $A_G = H^*(BG)$ with zero differentials. Note that the cohomology group of any complex in $D^+(BG)$ is in a natural way a graded module over A_G, i.e., we have the global section functor

$$H : D^+(BG) \to A_G - mod.$$

For a G space X with $\pi : X \to pt$,we define the *equivariant cohomology* and *equivariant cohomology with compact support* functors by:

$$H_G(X, -) = H \circ \pi_* : D_G^+(X) \to A_G - mod$$

and

$$H_{G,c} = H \circ \pi_! : D_G^+(X) \to A_G - mod.$$

We define equivariant (intersection) cohomology (with compact support) of X to be:

$$H_G(X, C_X), H_{G,c}(X, C_X), IH_G(X) = H_G(X, IC_X), IH_{G,c}(X) = H_{G,c}(X, IC_X).$$

Theorem 2. *Let G be a connected Lie group. Then we have a canonical equivalence of triangulated categories*

$$D^+(\mathcal{A}_G - mod) \cong D_G^+(pt)$$

$$D^f(\mathcal{A}_G - mod) \cong D_G^b(pt)$$

where the LHS are the derived category of bounded below (resp. finitely generated) DG-modules over $\mathcal{A}_G$.

We indicate the main ingredients of the proof. First we recall the general construction of a canonical sheaf of DG-algebras on a smooth manifold. Let X be an inductive limit of smooth manifolds by smooth embeddings(such as a smooth model for BG). Let $\Omega^{\cdot}_X$ be the de Rham complex of X. It is a soft resolution of the constant sheaf C_X and it is a sheaf of DG-algebras in a natural way. Let $\mathcal{A}_X$ be the complex of global sections of $\Omega^{\cdot}_X$:

$$\mathcal{A}^i_X = \Gamma(X, \Omega^i_X).$$

This is a DG-algebra in a natural way. We have a pair of functors:

$$D^+(\mathcal{A}_X - mod) \underset{\Gamma}{\overset{L}{\rightleftarrows}} D^+(X)$$

defined as follows. For $M \in D^+(\mathcal{A}_X - mod)$, the localization functor L takes M to $\Omega^{\cdot}_X \otimes^L_{\mathcal{A}_X} M$ where $\otimes^L$ means the the left derived functor of the tensor product, i.e., we first resolve M then take tensor product. For $\mathcal{F} \in D^+(X)$, the global sections functor Γ takes $\mathcal{F}$ to $\Gamma(X, \Omega^{\cdot}_X \otimes \mathcal{F})$. The key lemma is:

Lemma 1. *The localization and global sections functors induce equivalence of subcategories:*

$$D^+(\oplus\mathcal{A}_X) \cong D^+(\oplus C_X)$$

$$D^f(\oplus\mathcal{A}_X) \cong D^f(\oplus C_X)$$

where the LHS means the full triangulated subcategory of $D^+(\mathcal{A}_X - mod)$ generated by bounded-below (resp. finite) direct sums of shifts of $\mathcal{A}_X$ and the RHS means the similar subcategory of $D^+(X)$.

Now the proof of the theorem for the D^b part becomes a diagram chasing:

$$\begin{tikzcd}
D^f(\mathcal{A}_G - mod) \arrow[r, "\psi"] & D^f(\mathcal{A}_{BG} - mod) \arrow[r, shift left] & D^b_{const}(BG) \arrow[l, shift left] \\
D^f(\oplus\mathcal{A}_G) \arrow[u, "\iota_0"] \arrow[r, "\psi"] & D^f(\oplus\mathcal{A}_{BG}) \arrow[u, "\iota_1"] \arrow[r, shift left, "L"] & D^f(\oplus C_{BG}) \arrow[u, "\iota_{BG}"] \arrow[l, shift left, "\Gamma"]
\end{tikzcd}$$

We first define $\psi : \mathcal{A}_G \to \mathcal{A}_{BG}$. We know $\mathcal{A}_G = H(\mathcal{A}_{BG})$. Now since for any connected Lie group G, $\mathcal{A}_G$ is always a polynomial ring. Therefore we can define ψ by simply assigning values for each generator and make ψ a quasi-isomorphism of DG-algebras. This shows the two ψ's in the diagram are equivalences. ι_0 is also an equivalence(cf.[1]). L and Γ are inverses to each other by the lemma. Finally, ι_{BG} is an equivalence since for $\mathcal{F} \in D^b_{const}(BG)$ we can truncate it so it has a finite filtration with constant graded pieces.

The assertion for D^+ does not follow directly from the similar diagram because it is not apriorily obvious that $\iota_{BG} : D^+(\oplus C_X) \to D^+_{const}(X)$ should be an equivalence of categories. For details, see [1].

This canonical equivalence sends the hypercohomology functor to the global sections functor:

$$\begin{tikzcd}
D^+(\mathcal{A}_G - mod) \arrow[rr] \arrow[dr, "H"] & & D^+_{const}(BG) \arrow[dl, "R\Gamma"] \\
& \mathcal{A}_G - mod &
\end{tikzcd}$$

It enjoys functoriality with respect to group homomorphisms. To be more precise, let $\phi : H \to G$ be a homomorphism of connected Lie groups. Then we have

$$B\phi : BH \to BG$$

and

$$\Phi : \mathcal{A}_G \to \mathcal{A}_H.$$

This Φ further induces the *extension of scalars* functor

$$\Phi^* : D^+(\mathcal{A}_G - mod) \to D^+(\mathcal{A}_H - mod)$$

and the *restriction of scalars* funtor

$$\Phi_* : D^+(\mathcal{A}_H - mod) \to D^+(\mathcal{A}_G - mod).$$

We have the following commutative diagram:

$$\begin{tikzcd}
D^+(\mathcal{A}_H - mod) \arrow[r] \arrow[d, shift left, "\Phi^*"] & D^+_H(pt) \arrow[r] \arrow[d, shift left, "1^{G*}_H"] & D^+_{const}(BH) \arrow[d, shift left, "B\phi^*"] \\
D^+(\mathcal{A}_G - mod) \arrow[r] \arrow[u, shift left, "\Phi_*"] & D^+_G(pt) \arrow[r] \arrow[u, shift left, "1^G_{H*}"] & D^+_{const}(BG) \arrow[u, shift left, "B\phi_*"]
\end{tikzcd}$$

where $1 : pt \to pt$ is the identity map.

8 Discrete group actions

In this section, all groups are endowed with discrete topology. In this case, a lot of constructions can be carried in the categories $Sh_G(X)$.

Let $\phi : H \to G$ be a surjection with kernel K and $f : X \to Y$ be a ϕ-map. Then

Proposition 15. *The functor*

$$f^* : Sh_G(Y) \to Sh_H(X)$$

has a right adjoint

$$f^K_* : Sh_H(X) \to Sh_G(Y)$$

defined by first taking direct image then taking K-invariants. Furthermore, if K acts freely on X and $f : X \to Y = K\backslash X$ is the quotient map, then the above functors are equivalences inverse to each other.

Let H be a subgroup of G. Then restriction functor

$$\mathrm{Res}^G_H : Sh_G(X) \to Sh_H(X)$$

has a left adjoint

$$\mathrm{Ind}^G_H : Sh_H(X) \to Sh_G(X)$$

In face, given $\mathcal{F} \in Sh_H(X)$, pull it back to $G \times X$, it becomes an H-equivariant sheaf on $G \times X$, where H acts on G by right multiplication and acts on X by left action. This action is free. By the above proposition, this descends to a sheaf $\tilde{\mathcal{F}}$ on $G \times^H X$. Let $\alpha : G \times^H X \to X$ be the action, then we define

$$\mathrm{Ind}^G_H \mathcal{F} = \alpha_! \tilde{\mathcal{F}}.$$

The main result is:

Theorem 3. *Let $* = b, +$. The natural functor*

$$D^*(Sh_G(X)) \to D^*_G(X)$$

is an equivalence of triangulated categories.

We sketch a proof. The statement for D^+ follows from the statement for D^b by truncating and taking limit. For D^b, it reduces to showing that, for $\mathcal{F}_1, \mathcal{F}_2 \in D^b(Sh_G(X))$,

$$Hom_{D^b(Sh_G(X))}(\mathcal{F}_1, \mathcal{F}_2) = Hom_{D^b_G(X)}(\mathcal{F}_1, \mathcal{F}_2).$$

Let $p : P \to X$ be an ∞-acyclic resolution with quotient $\bar{P}$. Then

$$Hom_{D^b_G(X)}(\mathcal{F}_1, \mathcal{F}_2) = Hom_{D^b(\bar{P})}(\mathcal{F}_1(P), \mathcal{F}_2(P)) = Hom_{D^b(Sh_G(P))}(p^*\mathcal{F}_1, p^*\mathcal{F}_2).$$

Therefore it reduces to show

$$Hom_{D^b(Sh_G(X))}(\mathcal{F}_1, \mathcal{F}_2) = Hom_{D^b(Sh_G(P))}(p^*\mathcal{F}_1, p^*\mathcal{F}_2).$$

A devissage reduces to the case $\mathcal{F}_1 = Ind^G_{<1>}(C_U)$ for some open U in X. Then we apply the adjoint property of Ind and Res.

Now we return to the situation of the beginning of the section where $f : X \to Y = K\backslash X$. Assume K acts properly discontinuously on X and the coeffient field of all sheaves has characteristic 0. Then we have

Proposition 16. *We have:*

- f^K_* *is exact. Both* f^* *and* f^K_* *extending to derived categories;*
- $f^* = f^{G*}_H, f^K_* = f^G_{H*}$;
- $f^K_* \circ f^* = id_{D^+_G(Y)}$;
- f^* *induces an equivalence of categories between* $Sh_G(Y)$ *and a full subcategory of* $Sh_H(X)$ *consisting sheaves with trivial action of stabilizers on the stalks.*

There is an algebraic analogue. Suppose everything is complex algebraic and H, G, K are affine algebraic groups. Assume K acts on X with finite stabilizeers and assume the GIT quotient $f : X \to Y = K\backslash X$ exists and is affine.

Proposition 17. *We have:*

- $f^G_{H*} : D^+_H(X) \to D^+_G(Y)$ *preserves the natural* t*-structure;*
- $f^G_{H*} \circ f^{G*}_H = id_{D^+_G(Y)}$;
- f^G_{H*} *preserves the perverse* t*-structure up to shift:*

$$f^G_{H*} : Perv_H(X) \to Perv_G(Y)[\dim K];$$

- $f^G_{H*} IC_X = IC_Y[\dim K]$.

References

[1] J.Bernstein, V.Lunts, Equivariant sheaves and functors, Springer Lecture Notes 1578 (1994)

[2] P.Deligne, Théorie de Hodge III, Publ. IHES 44 (1974)