Galois theory for differential equations
(Picard–Vessiot theory) Lecture 1.

Consider the differential equation

$\frac{dF}{dz} = AF$, where $A \in \mathfrak{gl}_n(C(z))$.

Let $U$ be a simply connected open set in $C$ avoiding the poles of $A(z)$. Then we have a basis $\vec{f}_1, \ldots, \vec{f}_n$ of the space of solutions of (\*) on $U$, $\vec{f}_i = (f_{ij})$, $j = 1, \ldots, n$, where $f_{ij}(z)$ are holomorphic functions on $U$. Consider the field $L = C(z)(\{f_{ij}\})$, a subfield of the field of meromorphic functions on $U$. It is called the solution field of (\*). It is easy to see that $L$ does not depend on the choice of $U$.

Definition. The differential Galois group of (\*) is the group $G$ of automorphisms of $L$ which act trivially on $C(z)$ and commute with $\frac{d}{dz} : L \to L$.

Remark. Note that $\frac{d}{dz}$ preserves $L$ since

$\frac{d}{dz} f_{ij} = \sum a_{ij}(z) f_{kj}(z)$. In other words, $L$ is a differential field.
Proposition. The group $G$ admits an embedding into $GL_n(\mathbb{C})$ which is defined canonically up to conjugation in $GL_n(\mathbb{C})$. Moreover, $G \subseteq GL_n(\mathbb{C})$ is a Zariski closed subgroup, so $G$ is an affine algebraic group.

Proof. Let $F = (f_{ij})$ be the matrix solution of (x). Then $\forall g \in G$, since $g(A) = A$ and $g$ commutes with $\frac{d}{dz}$, we have that $g(F)$ is also a solution of (x), hence $g(F) = F \rho(g)$, where $\rho : G \to GL_n(\mathbb{C})$ is a map. Moreover, $g(h)(F) = g(F \rho(h)) = F \rho(g) \rho(h)$, so $\rho$ is a homomorphism. Finally, $\rho(g) = 1$ implies $g(F) = F$, hence $g$ acts trivially on $L$, so $g = 1$. Thus $\rho$ is injective, i.e., an embedding $G \to GL_n(\mathbb{C})$. Moreover, if we change the basis $\vec{F}_1, \ldots, \vec{F}_n$ by a matrix $S$, then $\rho$ gets conjugated by $S$. Hence $\rho$ is defined canonically up to conjugation.

Now let us show that $G \subseteq GL_n(\mathbb{C})$ is Zariski closed. To this end, note that
we have a natural homomorphism
\[ \varphi : C(\mathbb{Z})[[x_{ij} \mid i, j \in [1, n]]] \rightarrow L, \] sending \( x_{ij} \) to \( f_{ij} \).
Let \( I \subset C(\mathbb{Z})[[x_{ij}]] \) be the kernel of \( \varphi \).
Then \( \varphi : R = C(\mathbb{Z})[[x_{ij}]] / I \rightarrow L \), and \( \varphi \) identifies \( L \) with \( \text{Frac}(R) \).
Now, given \( g \in G \subset GL_n(\mathbb{C}) \),
g maps \( F \) to \( Fp(g) \), hence defines an automorphism of \( \text{Rat}(F) \) by \( X \mapsto Xp(g) \),
\( X = (x_{ij}) \). Thus, \( G \subset GL_n(\mathbb{C}) \) is the subgroup of all \( g \in GL_n(\mathbb{C}) \) such that the map
\( X \mapsto Xp \) preserves the ideal \( I \), i.e., the subvariety \( \text{Spec}_R \mathcal{R} \subset \text{Mat}_n(\mathbb{C}) \), where for any \( M \in \text{Spec}_R \mathcal{R} \) we must have
\[ u(Mg) = 0 \quad \forall u \in I. \] This is a system of polynomial equations on \( g \) which give rise to a closed subset in \( GL_n(\mathbb{C}) \) in Zariski topology.

Examples 1. Suppose all \( f_{ij} \) are algebraic functions, i.e., \( f_{ij} \in C(\mathbb{Z}) \). Then \( L \) is a finite Galois extension of \( C(\mathbb{Z}) \), since \( \text{Aut}(C(\mathbb{Z}) / \mathcal{R}(\mathbb{Z})) \)
we have that \( g(F) \) satisfies (*), so
\[ g(F) : y \in L, \] hence \( g : L \to L. \) Thus in this case \( G = \text{Gal}(L/\mathbb{Q}(z)) \).

2. \( n = 1 \), \( \frac{dF}{dz} = \frac{\lambda}{z} F. \)

Then \( F = z^\lambda. \) If \( \lambda \in \mathbb{Z} \), \( L = \mathbb{C}(z) \) and \( G = 1. \)
If \( \lambda \in \mathbb{Q} \), \( \lambda = \frac{p}{q} \) (in lowest terms), \( L = \mathbb{C}(z)(z^{1/q}) \)
and \( G = \mathbb{Z}/q\mathbb{Z}. \) If \( \lambda \notin \mathbb{Q} \) then \( L = \mathbb{C}(z)(z^\lambda) \)
and \( G = \mathbb{C}^* = GL_1(\mathbb{C}) = \mathbb{C}_m. \)

3. \( n = 2 \), \( \frac{dF}{dz} = \begin{pmatrix} 0 & \frac{1}{z} \\ 0 & 0 \end{pmatrix} F. \)

So \( \frac{df_2}{dz} = 0, \frac{df_1}{dz} = \frac{1}{z} f_2. \)

So we get \( F = \begin{pmatrix} 1 & \log z \\ 0 & 1 \end{pmatrix} \) and \( L = \mathbb{C}(z)(\log z). \)

Under differential automorphisms \( \log z \mapsto \log z \),
so \( \begin{pmatrix} 1 & \log z \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & \log z + c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \log z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}. \)
Thus \( G = \{ (1, c)^n \in GL_2 \mid c \in \mathbb{C} \} = \mathbb{C} = \mathbb{C}_a. \)
Fundamental theorem of differential Galois theory:

The assignment $H \mapsto L^H$ is a bijection between closed subgroups $H \leq G$ and subfields $K \subset \mathbb{C}$ invariant under $\frac{d}{dz}$. The inverse is given by $K \mapsto \text{Aut}_{\mathbb{C}}(L/K, d/dz)$.

We will discuss the proof later.

In particular, we have $L^G = \mathbb{C}(z)$.

Theorem. Assume that the differential equation (*) has regular singularities. Then $G$ is the Zariski closure $\overline{\Gamma}$ of the monodromy group $\Gamma \subset GL_n(\mathbb{C})$.

Proof. First of all, it is clear that $\Gamma \subset G$, hence $\Gamma \subset G$. To prove the opposite inclusion, consider $f \in L^\Gamma$. This function is single-valued, and has polynomial growth at poles of $A(z)$ since (*) has regular singularities. Hence, by removable singularity theorem $f$ is meromorphic, hence $f \in \mathbb{C}(z)$. Thus $\overline{\Gamma} = G$. 
Example. Consider the equation
\[ \frac{dF}{dt} = \left( \frac{A}{t^2} + \frac{B}{t^2 - 1} \right) F, \quad A, B \in \mathbb{C}. \]
This is a regular equation, so \( G = \Gamma \).
We claim that for generic \( A, B \) we have \( \Gamma = \text{SL}_2(\mathbb{C}) \). First of all, \( \Gamma \subseteq \text{SL}_2(\mathbb{C}) \) since \( A, B \) have trace 0. On the other hand, \( \Gamma \) contains matrices \( PQ \) conjugate to \( e^{2\pi i A} \), \( e^{2\pi i B} \), and \( PQ \) is conjugate to \( e^{2\pi i (A+B)} \).
Looking at eigenvalues, we see that such matrices should define an irreducible representation on \( \mathbb{C}^2 \). So \( \Gamma = \text{SL}_2(\mathbb{C}) \) or \( \Gamma = \text{SL}_2(\mathbb{C}) \), but the last possibility can't happen since \( \Gamma \) has no abelian subgroup of index 2 (basically, \( \Gamma \) is a free group).

Theorem. (Inverse problem of differential Galois theory). Any affine algebraic group over \( \mathbb{C} \) can be the differential Galois group of an equation \( (\ast) \).
Proof. It is known that \( \forall \Gamma \subseteq \text{GL}_n(\mathbb{C}) \) with \( n \) generators there exists
a regular differential equation(*) with
moles in G and regular singularities
for which the monodromy group is \( \Gamma \).
So it remains to show that for any
algebraic group \( G \leq GL_n(\mathbb{C}) \) there exists
a finitely generated subgroup \( \Gamma \leq G \)
such that \( \overline{\Gamma} = G \). To construct such \( \Gamma \),
construct a sequence of elements
\( \delta_1, \ldots, \delta_n \in G \) such that
\[ \dim \langle \delta_1, \ldots, \delta_n \rangle > \dim \langle \delta_1, \ldots, \delta_{n-1} \rangle \]
until we reach \( \dim G \).
To this end, we proceed by induction.

If \( \dim \langle \delta_1, \ldots, \delta_{n-1} \rangle < \dim G \),
let \( x \in \text{Lie} G \), \( x \notin \text{Lie} \langle \delta_1, \ldots, \delta_{n-1} \rangle \),
and consider the 1-parameter subgroup
\( e^{tx} \) (in the sense of complex Lie groups).
Set \( \delta_n \) to be a generic element of this
subgroup. Then \( \text{Lie} \langle \delta_1, \ldots, \delta_n \rangle \ni x \), so
\[ \dim \langle \delta_1, \ldots, \delta_n \rangle = \dim \langle \delta_1, \ldots, \delta_{n-1} \rangle. \]
Thus we have \( \delta_1, \ldots, \delta_n \) such that
\( \langle \delta_1, \ldots, \delta_n \rangle \) and we can add fin. many elements to
generate the whole $G$.

**Liouville extensions.**

Let $K \subset \text{Mer}(U)$ be a differential subfield of Mer$(U)$.

**Additive extension:** $L = K\left( u \right)$ where
\[
\frac{du}{dz} = \alpha(z), \quad \alpha \in \text{Hol}(U).
\]
Thus,
\[
L = K\left( \int \alpha(z) \, dz \right).
\]
Note that if $\int \alpha(z) \, dz \notin K$ then $G = G_a$.

**Multiplicative extension:** $L = K\left( u \right)$ where
\[
\frac{du}{dz} = \alpha(z)u, \quad \alpha \in \text{Hol}(U), \quad \text{i.e.,}
\]
\[
L = K\left( \int \alpha(z)u \, dz \right).
\]
If $u \notin K$ and $u$ is not algebraic over $K$ then
\[
G = G_a.
\]

$\text{Gal}(L:K)$

**Def. A Liouville extension** $K \subset L \subset \text{Mer}(U)$ is a differential field extension which
obtained by a succession of additive, multiplicative and algebraic extension. So if $K \subset L$ is a Liouville extension, then all $f \in L$ express "in quadrature".

**Theorem.** $K \subset \mathbb{C}(z)$ is a Liouville extension if and only if $G^0$ is a solvable group (composition factors $G_a$ and $G_m$).

**Proof.** Suppose $L \supset \mathbb{C}(z)$ is Liouville. Then we have a chain

$$L = L_n \supset L_{n-1} \supset \cdots \supset L_0 = \mathbb{C}(z)$$

where $L_{i+1} = L_i(u)$ is an additive or multiplicative extension or $L_{i+1}$ is an algebraic Galois extension of $L_i$ let $G_i = \text{Aut}(L_n/L_i, \frac{d}{dx})$.

Then $1 = G_n \subset G_{n-1} \subset \cdots \subset G_0 = G$.

Moreover, $G_i$ are normal subgroups in $G$ and $G/G_i = \text{DGal} (L_i/L_0, \frac{d}{dx})$.

Finally, $G_i^{-1}/G_i = \text{DGal} (L_i/L_{i-1}, \frac{d}{dz}) = \bigcup G_m$.
This implies \( G_0 \) is solvable. Concretely, if \( G_0 \) is solvable, form a sequence

\[
G = G_n \supset G_{n-1} \supset \cdots \supset G_0 = 1
\]

with \( G_i \) normal in \( G \) and

\[
G_i/G_i = \{
\{ G_{a_i}, \quad G_{m_i}
\}
\text{ finite.}
\]

Take \( L_i = L_i G_i \). Then \( L_i \) form a sequence such that \( L_{i+1} \) is an additive, multiplicative or finite Galois extension of \( L_i \).

**Corollary.** The equation \( \frac{dF}{dz} = \frac{A}{z} + \frac{B}{z-1} \) does not admit solutions in quadratures in the above sense.

**Remark.** In fact, solutions of this equation express via hypergeometric functions.
Lecture 2.

Finite zone potentials.

Consider the Schrödinger operator

\[ L = \Delta^2 + u(x) \]

where \( u \) is a smooth or holomorphic function of one variable called a potential.

**Def.** \( u(x) \) and \( L \) are called **finite zone** if there exists a differential operator of odd order, \( M = a_0 \Delta^{2n+1} + \cdots + a_{2n+1} \)

which commutes with \( L \).

**Remark** It is clear that if \( [L, M] = 0 \) then \( a_0 = \text{const} \), so we may assume that \( a_0 = 1 \). Also it's easy to check that \( a_1 = \text{const} \), so by subtracting \( a_1 \), \( L^n \)

we can assume that \( a_1 = 0 \).

**Example.** 1. \( L = \Delta^2 \) is finite zone (\( M = 0 \)).

2. Consider \( L = \Delta^2 - \frac{\alpha}{x^2}, x \in \mathbb{R} \), and look for \( M = \Delta^3 + a \Delta + b \). We have

\[ [\Delta^2 - \frac{\alpha}{x^2}, \Delta^3 + a \Delta + b] = 0. \]

This gives

\[ (\alpha a' + a' \alpha) \Delta + (\alpha b' + b' \alpha) - \left[ \frac{\alpha}{x^2}, \Delta^3 \right] \]

\[ = 2 \alpha \Delta = 0. \]
Simplifying, we get
\[ 2a' \theta^2 + a'' \theta + b'' + 2b' \theta + \left( \frac{6}{x^3} \theta^2 + \frac{18}{x^4} \theta - \frac{24}{x^5} \right) \theta \]
\[ - \frac{2\alpha}{x^3} = 0 \]

This yields:
\[ 2a' = \frac{6\alpha}{x^3} \]
\[ a'' + 2b' = -\frac{18\alpha}{x^4} \]
\[ b'' - \frac{2\alpha a}{x^3} = + \frac{24\alpha}{x^5} \]

This gives:
\[ a = -\frac{3\alpha}{2x^2} + C_1 \]
\[ -\frac{9\alpha}{x^4} + 2b' = -\frac{18\alpha}{x^4} \quad \Rightarrow \]
\[ b' = -\frac{9\alpha}{2x^4} = \quad \begin{array}{c}
\text{Boxed: } b = +\frac{3\alpha}{2x^3} + C_2 \\
\text{(can assume } C_2 = 0)\end{array} \]
\[ + \frac{18\alpha}{x^5} - \frac{2\alpha}{x^3} \left( \frac{3\alpha}{2x^2} + C_1 \right) = + \frac{24\alpha}{x^5} \]

This implies that \( C_1 = 0 \) and
\[ \theta^2 = 2\alpha. \]

So \( \theta = 0 \) (solution we already know) and
\[ \alpha = 2. \]
Thus, we get: the operator $L = \frac{-3}{x^2} \frac{\partial^3}{\partial x^3} - \frac{2}{x^2}$ is finite zone, with $M = \frac{3}{x^2} \frac{\partial}{\partial x} + \frac{3}{x^3}$.

More generally, it is easy to show that $L = \frac{-\alpha}{x^2}$ is finite zone if $\alpha = m(m+1)$ for $m \in \mathbb{Z}_{\geq 0}$, and $M = \partial^{2m+1} + \cdots$.

Exercise. Compute $M$.

Proposition. If $L$ is finite zone and $[L,M] = 0$ with $M = \partial^{2m+1} + a_2 \partial^{2m-1} + \cdots$ (of minimal odd order), $M^* = -M$ then there exists a polynomial $P$ of degree $2m+1$ such that $M^2 = P(L)$.

Proof. Consider the operator $M^2$. Then $A = M^2 - L^{2m+1}$ has order $\leq 4m+1$, and it commutes with $L$, so has constant leading coefficient. If $A$ has even order, subtract a multiple of $L^{s-m}$ to obtain an operator of lower order. If $A$ has odd order, by assumption this order $2s+1$ is $\geq 2m+1$, so subtract $cML$ to lower the order.

Continuing this procedure, we will eventually get $0$, so $M^2 = Q(L)M + P(L)$. But $M^* = -M$, which implies that $Q = 0$ and $M^2 = P(L)$.
Remark: We can always assume $M^* = -M$ by replacing $M$ with $\frac{M - M^*}{2}$.

The equation $\mu^2 = P(x)$ defines a hyperelliptic algebraic curve, called the spectral curve. The meaning of this curve is the following:

Consider the space $V_2$ of solutions of the eigenvalue equation

$$L\psi = \lambda^2\psi.$$ 

Then $M$ acts on this space, and the eigenvalues of $M$ are $\mu$ such that $\mu^2 = P(x)$, i.e., $\mu = \pm \sqrt{P(x)}$.

For generic $\lambda$, these eigenvalues are distinct, so the joint eigenvalue problem

$$L\psi = \lambda^2\psi \quad M\psi = \mu\psi$$

has a 1-dimensional solution space, hence solutions express in quadratures.

Note that the coefficients of $M$ are rational, so if $u(x)$ is rational, it is easy to deduce...
Indeed, \( M = \sqrt{P(x)} \), where RHS is understood in terms of pseudodifferential operators. This means that the Galois group \( G \) of the equation \( L\psi = x^2 \psi \) (or, rather, the Galois group of the associated 2x2 system) is \( \mathbb{C}^* \leq \text{SL}_2(\mathbb{C}) \).

We can also compute the Galois group over \( \mathbb{C}(x^2) \). Then it's a nonsplit torus, and all 1-dimensional tori over a field \( F \) are classified by quadratic extensions \( E/F \) (namely, \( T = \{ z \in E \mid N(z) = 1 \} \}). In our case, \( E \) is the field of functions on the spectral curve \( X \).

Assume that \( u(x) \to 0 \) as \( x \to \infty \) (and \( u \) is rational).

**Proposition.** The eigenfunction \( L\psi = x^2 \psi \), where we take the branch \( M\psi = \sqrt{P(x^2)} \psi \), where we take the branch.

\[ \psi(x, \lambda) = e^{x\lambda} Q(x, \lambda) \] where \( Q \) is a rational function, \( Q \to 1 \) as \( x \to \infty \).

**Proof:** This follows since the system reduces to a first order equation with rational coefficients.
Remark. This means that the spectral curve has genus 0 in this case (and is singular).

Def. The function \( y \) is called the Baker-Abhiezer function.

Example. For \( L = 3^2 - \frac{2}{x^2} \),
\[ y = e^{\lambda x} \left( 1 - \frac{1}{\lambda x} \right). \]

Exercise. Compute \( P \) in this case.

Corollary. If \( L \) is finite zone then it has no monodromy at the poles. Ex. \( 3^2 - m(m+1) \) is finite zone \( \leftrightarrow m \in \mathbb{Z} \).

Prop. The converse also holds.

(we will skip the proof).

Now consider \( L = 3^2 - 2 \sum_{i=1}^{n} \frac{1}{(x-x_i)^2} \).

The "no monodromy" condition at \( x \) results in the system of equations
\[ \sum_{j \neq i} \frac{1}{(x_i - x_j)^3} = 0 \quad (**). \]

i.e. \( x \) is a critical point of the Calogero-Moser Hamiltonian.
\[ H = \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}. \]

**Theorem.** This system has no solutions unless \( N = \frac{m(m+1)}{2} \) (i.e., \( L \) can be finite only in this case).

**Proof.** Let us assume that \( L \) is finite. Then \( L_t = \theta^2 - 2 \sum_{j=1}^{N} \frac{1}{(x - t \varepsilon_j)^2} \) is finite for all \( t \) (by scaling). Taking limit \( t \to 0 \), we get

\[ L_0 = \theta^2 - \frac{2N}{\varepsilon^2} \]

is finite zone. Hence \( N = \frac{m(m+1)}{2} \). \( \Box \)