1. Pronipotent completions of groups.

Let $G$ be a group, and $k$ a field of characteristic $0$. Let $kG$ be the group algebra.

Let $I$ be the kernel of the augmentation map $\varepsilon: kG \to k$. Let $H = \varprojlim \frac{kG}{I^n}$. This is a topological Hopf algebra (since $I$ is a Hopf ideal).

**Definition.** The protonipotent completion of $G$ over $k$ is the group $\hat{G}$ of grouplike elements of $H$, i.e., elements $g \in H$ such that $\Delta g = g \otimes g$ and $\varepsilon(g) = 1$.

(Note that in this case $g$ is invertible and $g^{-1} = S(g)$, where $S$ is the antipode).

**Example.** 1. If $G$ is finite then $I^2 = I$, $I^N = I$ for $N \geq 1$ and $H = k$. Hence $\hat{G} = 1$.

2. Let $G = \mathbb{Z}$. Then $kG = k[x, x^{-1}]$, and $I = (x-1)$. Thus $H = k[[t]]$, where $t = x-1$.

A more convenient representation is $H = k[[u]]$ where $u = \log(x) = \log(1+t) = \sum (-1)^{n-1} \frac{t^n}{n}$. (Indeed, $\Delta t = t \otimes 1 + 1 \otimes t + t \otimes t$)
but \( \Delta u = u \otimes 1 + 1 \otimes u \). It is easy to check that group-like elements of \( k[[u]] \) are of the form \( \exp(xu) \), \( x \in k \), so \( \hat{G} = k \) (the additive group).

Assume for simplicity that \( G \) is finitely generated. The reason \( \hat{G} \) is called the pro-nilpotent completion is that it has a natural structure of pro-nilpotent algebraic group over \( k \) (i.e., inverse limit of nilpotent groups). Indeed, it is clear that \( g \in H \) is group-like \( \iff x = \log(g) = \log(1 + (g-1)) = \sum (-1)^{n-1}(g-1)^n \) is primitive: \( \Delta x = x \otimes 1 + 1 \otimes x \).

Thus we can consider the Lie algebra \( \hat{g} = \{ x \in H \mid \Delta x = x \otimes 1 + 1 \otimes x^2 \} \) and \( \hat{G} = \exp(\hat{g}) \).

Moreover, \( g \) is pro-nilpotent (inverse limit of nilpotent finite dimensional Lie algebras), namely, \( g = \lim_{n \to \infty} g_n \), where \( g_n \) is the image of \( g \) in \( kG/n \) and \( g_n \) is nilpotent of index \( n-1 \).

Then \( \hat{G} = \lim_{n \to \infty} G_n \), where \( G_n = \exp(g_n) \).
Example: Let $G$ be the free group in $n$ generators $g_1, \ldots, g_n$. Then $G$ is the completed free Lie algebra $\hat{\mathbb{F}}_n$ in generators $x_i = \log g_i$, so $\hat{G} = \text{exp}(\mathbb{F}_n)$.

Finally, consider the notion of pro-nilpotent completion with respect to a subgroup. Let $K \triangleleft G$ be a normal subgroup of finite index, and let $J \subset kK$ be the augmentation ideal. Let $I = kG \cdot J \subset kG$, and let $H = \lim_{N \to \infty} kG/I^N$.

Definition. The pro-nilpotent completion of $G$ relative to $K$ is the group $\hat{G} = \hat{G}_K = \left\{ g \in H \mid \Delta g = g \circ g, \varepsilon(g) = 1 \right\}$.

One can show that one has a short exact sequence

$$1 \to \hat{K} \to \hat{G}_K \to G_K \to 1 \tag{*}$$

and $\hat{G}_K$ is a pro-algebraic group with $(\hat{G}_K)_0 = \hat{K}$. Note that the exact sequence $(*)$ is automatically split since $H^2(\hat{G}, U) = 0$ for any finite group $\hat{G}$ and for any cohomologically trivial $U$. 
2. Prunipotent completions of braid groups.

Recall that the braid group $B_n$ is generated by $b_i$, $i=1, \ldots, n-1$, with relations $b_i b_j = b_j b_i$ if $|i-j| \geq 2$ and $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$.

Geometrically, $b_i$ corresponds to the braid

\[ \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
i \ \ \ \ \ \ \ \ \ \ \ \ i+1
\end{array} \]

We have a surjective homomorphism $\varphi: B_n \rightarrow S_n$, $\varphi(b_i) = s_i = (i, i+1)$.

The kernel of $\varphi$ is called the pure braid group, denoted $PB_n$.

The pure braid group is generated by the elements $\tau_{ij} = \prod_{k=1}^{j-i} b_{i+1}$, $i < j \leq n$.

($\tau_{ij} = \prod_{k=i+1}^{j} b_k$).
Consider the pro-\(n\)potent completion \(\widehat{PB}_n\) of \(PB_n\) over \(k\). It is generated by elements \(t_{ij} = \text{log} E_{ij}\). One can show that the defining relations for \(t_{ij}\) look like:

\[
[t_{ij}, t_{pq}] = 0 \quad \text{if} \quad i, j, p, q \text{ are distinct};
\]

\[
[t_{ij}, t_{ip} + t_{jp}] = 0
\]

where \(\cdots\) denotes cubic and higher degree expressions (in fact, it involves all degrees). Denote by \(T_n\) the Lie algebra generated by \(t_{ij}\) with the graded version of these relations,

\[
[t_{ij}, t_{pq}] = 0, \quad i, j, p, q \text{ distinct}; \\
[t_{ij}, t_{ip} + t_{jp}] = 0.
\]

**Theorem.** One has an isomorphism \(\mu : \exp (\widehat{T}_n) \leftrightarrow \widehat{PB}_n\).
Moreover, it extends to an isomorphism
\[ \tilde{\mu}: S_n \times \exp(\hat{T}_n) \leftrightarrow \hat{B}_n \]
where \( \hat{B}_n \) is the pro-unipotent completion of \( B_n \) with respect to \( PB_n \), \( \hat{B}_n = \hat{B}_n \mid_{PB_n} \).

3. We will prove this theorem for \( k = 0 \) using complex analysis. Namely recall that \( PB_n = \pi_1(X_n) \), where \( X_n = C^n \setminus \{ (z_1, \ldots, z_n) \in C^n : z_1^2 + \cdots + z_n^2 = 0 \} \). Thus to construct a representation of \( B_n \), it suffices to define a flat connection on \( X_n \); then the monodromy of this connection will provide the desired representation.

Consider the trivial bundle over \( X_n \) with fiber \( U(T_n) \).

Definition. The KZ (Knizhnik-Zamolodchikov) connection is the connection
on this bundle defined by the 1-form
\[ \Omega = \sum_{i < j} \frac{1}{2\pi i t_{ij}} \frac{d(z_i - z_j)}{z_i - z_j} \]
(i.e., \( \nabla = d + \frac{1}{2} \)).

In other words, flat sections of \( \nabla \) are solutions of the differential equations
\[ \frac{\partial F}{\partial z_i} = \sum_{j \neq i} \frac{t_{ij}}{z_i - z_j} F \quad (F \in U(\mathbb{T}_n)). \]

Lemma. The KZ connection is flat.

Proof. It is easy to show that \( d\Omega = [\Omega, \Omega] = 0 \), so \( \Omega \) satisfies MC equations.

Now fix a base point \( P \in X_n \), and consider the monodromy representation \( \mu : PB_n \to \exp(\hat{\mathfrak{t}}_n) \) with this base point. It is easy to see that \( \mu(t_{ij}) = 1 + t_{ij} + \text{higher order terms} \), which implies that \( \mu \) is an isomorphism.
A categorical generalization.

We'd like to prove a similar statement in the categorical setting.

Recall that a monoidal category is a category \( \mathcal{C} \) with a functor \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) and an isomorphism of functors \( \alpha = (\alpha_{xyz} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)) \) satisfying the pentagon relation

\[
\begin{array}{ccc}
(w \otimes (x \otimes y)) \otimes z & \xrightarrow{\alpha} & (w \otimes x) \otimes (y \otimes z) \\
\downarrow & & \downarrow \\
(w \otimes (x \otimes y)) \otimes z & \xleftarrow{\alpha} & (w \otimes x) \otimes (y \otimes z)
\end{array}
\]

which contains a unit object \((1, i)\), \(i : 1 \otimes 1 \cong 1\), such that the functors \(1 \otimes -\) and \(- \otimes 1\) are auto-equivalences of \(\mathcal{C}\).

We'll consider monoidal categories which are also bimodule linear.
over \( k \), so that \( \otimes \) is bilinear on morphisms.

**Def.** A braiding on \( \mathcal{C} \) is an isomorphism of functors \( \gamma = (\gamma_{xy} : X \otimes Y \to Y \otimes X) \) satisfying the hexagon relations:

\[
\begin{align*}
X \otimes (Y \otimes Z) & \to (Y \otimes Z) \otimes X \\
(X \otimes Y) \otimes Z & \to Y \otimes (X \otimes Z) \\
Y \otimes (Z \otimes X) & \\
(Y \otimes X) \otimes Z & \to Y \otimes (X \otimes Z) \\
Z \otimes (X \otimes Y) & \\
(X \otimes Y) \otimes Z & \to Z \otimes (X \otimes Y) \\
(Z \otimes X) \otimes Y & \\
X \otimes (Z \otimes Y) & \to (X \otimes Z) \otimes Y
\end{align*}
\]

A category equipped with a braiding is called a braided monoidal category.

A braiding is called symmetric if \( \gamma_{xy} \circ \gamma_{yx} = \text{id} \). A category equipped
with a symmetric braiding is called a symmetric monoidal category.

If \((\mathcal{C}, \otimes)\) is a braided category and \(X_1, \ldots, X_n \in \mathcal{C}\), then \(X_1 \otimes \cdots \otimes X_n\) carries a natural action of \(PB_n\), and \(\bigoplus_{\sigma \in S_n} X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(n)}\) carries an action of \(B_n\). It is given by

\[ b_i = c_{i, i+1} \] (the braiding acting in the \(i\)th and \(i+1\)th factors). Also for \(X \in \mathcal{C}\), \(X^{\otimes n}\) carries an action of \(B_n\) (defined in the same way). If \(c\) is symmetric, this action factors through the symmetric group \(S_n\).

Now let \((\mathcal{C}, c^0)\) be a symmetric category (e.g. the representation category of a group or a Lie algebra).

**Def.** An infinitesimal braiding on \(\mathcal{C}\) is a morphism \(t = (t_{xy} : X \otimes Y \to X \otimes Y)\)
such that $t_{x_1 \otimes x_2, y} = 1_{x_1} \otimes t_{x_2, y} + \left(1_{x_2} \otimes t_{x_1, y} \right) \left(C_{x_1 x_2} \otimes 1_y \right)$.

(ignoring associativity isomorphisms).

Shorthand: $t_{x_1 \otimes x_2, y} = t_{x_1, y} + t_{x_2, y}$.

**Example.** Let $g$ be a Lie algebra over $k$, and $\mathcal{C} = g$-module. Let $t \in (S^2 g)^F$.

Then $t_{x y} = t \mid_{x \otimes y}$ defines an infinitesimal braiding on $\mathcal{C}$.

**Motivation:** If $(\mathcal{C}, c)$ is braided over $k[[t]]$ and $c = c^0 (1 + t t^2 + O(t^2))$ then $t$ is an infinitesimal braiding.

**Def:** $t$ is symmetric if it commutes with $c_0$.

**Lemma.** If $(\mathcal{C}, c^0, t)$ is a symmetric category with infinitesimal braiding then there is a natural action of the Lie algebra $T_n$ on $\mathcal{C}$.

Namely, $t_{ij} = t \mid_{x_i \otimes x_j}$.
Theorem (Drinfeld). Let \((\mathcal{C}, c^0, t)\) be a symmetric category with infinitesimal braiding, and let \(\mathcal{C}[[t]]\) be the category with the same objects and \(\text{Hom}_{\mathcal{C}[[t]]}(X,Y) = \text{Hom}_{\mathcal{C}}(X,Y)[[t]]\) a new associativity \(a = a^0(1+O(t))\). Then \(\mathcal{C}\) carries an infinitesimal braiding \(c\) such that \(c = c^0(1+t+O(t^2))\). In other words, any infinitesimal braiding can be "quantized".

We will prove this theorem for \(k = \mathbb{C}\) using the KZ equations. But plain monodromy will not be sufficient now: we will need to extract from KZ equations a new object called the Drinfeld associator.