On the smooth transfer conjecture of Jacquet–Rallis for $n = 3$

Wei Zhang*†

Contents

1 Introduction 1

2 Germ expansion of relative orbital integrals 5
2.1 Nilpotent orbital integral 5
2.2 Theorem on relative germ expansion 15
2.3 Application to the density principle 17

3 Proof of the germ expansion 18

4 Existence of refined smooth transfer 27
4.1 An extended Fourier transform 27
4.2 Refined smooth transfer. 30

1 Introduction

Let F be a field and let H be a reductive group over F. Consider a linear representation of H on a finite dimensional F-vector space $\mathcal{V}$ (considered as an affine space)

$$\rho : H \to GL(\mathcal{V}).$$

*Address: Department of Mathematics, Columbia University, 2990 Broadway New York, NY 10027 USA.
†Email: wzhang@math.columbia.edu

Let $(\mathcal{V}_H, \pi)$ be the categorical quotient (cf. [1]) where π is a morphism $\mathcal{V} \to \mathcal{V}_H$. One may construct the categorical quotient by taking

$$\mathcal{V}_H := \mathrm{Spec}(\mathcal{O}(\mathcal{V})^H), \quad \pi : \mathcal{V} \to \mathrm{Spec}(\mathcal{O}(\mathcal{V})^H),$$

where $\mathcal{O}(\mathcal{V})^H$ is the ring of invariant regular functions on the affine space $\mathcal{V}$ and π is the natural morphism induced by the inclusion $\mathcal{O}(\mathcal{V})^H \hookrightarrow \mathcal{O}(\mathcal{V})$.

In this paper we will consider two (series of) representations ρ that arise from the relative trace formulae of Jacquet–Rallis's approach to the Gan–Gross–Prasad conjecture for unitary groups (cf. [4], [9]):

- The general linear case: $H = GL_{n-1}$ acts on the Lie algebra $\mathcal{V} = \mathfrak{gl}_n$ of GL_n.
- The unitary case: $H = U_{n-1}$ (a unitary group in $n-1$ variables) acts on the Lie algebra of U_n.

We explain in more details. In the first case, we embed GL_{n-1} into GL_n by taking g to the block-diagonal matrix $diag[g, 1]$. Then the first representation is the restriction of the adjoint representation of GL_n on its Lie algebra $\mathfrak{gl}_n$ to the subgroup $H = GL_{n-1}$. To describe the second case, let E/F be a quadratic extension. Let W be a (non-degenerate) Hermtian space of dimension $n-1$. Let $V = W \oplus Eu$ be the orthogonal sum of W and a one-dimensional Hermitian space Eu with a generator u of norm one. We restrict the adjoint representation of $U(V)$ on $\mathfrak{u}(V)$ to the subgroup $H = U(W)$.

Now we let F be a non-archimedean local field of characteristic zero (though the results below should hold when the characteristic is large enough). By abuse of notations, we will use $\mathcal{V}$ to denote the the set of F-points of $\mathcal{V}$ and similarly for $\mathcal{V}/H$ etc., and π the continuous map from $\mathcal{V}$ to $\mathcal{V}/H$. Then $\mathcal{V}/H$ has a distinguished point, also denoted by 0, being the image of $0 \in \mathcal{V}$. By the *H-nilpotent cone*, denoted by $\mathcal{N}$, we mean the pre-image $\pi^{-1}(0)$ of 0. An element in $\mathcal{N}$ is called *H-nilpotent*. An element $X \in \mathcal{V}$ is called *H-regular* (*H-semisimple*, resp.) if its stabilizer has minimal dimension (if its H-orbit is closed, resp.). When there is no confusion, we will drop the word H and simply speak of nilpotent,regular and semisimple. The terminology is clearly borrowed from the example where ρ is the adjoint representation of a reductive group H on its Lie algebra $\mathfrak{h}$. In that case, all the notions coincide with the usual ones.

Let $\mathcal{C}_c^\infty(\mathcal{V})$ be the space of locally constant and compactly supported functions on $\mathcal{V}$. Then for $f \in \mathcal{C}_c^\infty(\mathcal{V})$ and a semisimple element $x \in \mathcal{V}$ with stabilizer H_x, the orbit $H \cdot x \simeq H/H_x$ is a closed subset of $\mathcal{V}$ so that the orbital integral (relative to H) for a suitable Haar measure

$$O(x, f) = \int_{H/H_x} f(h \cdot x) dx$$

is absolutely convergent. We now focus on the two cases above, where the stabilizer of a regular semisimple element is trivial. We consider the orbital integrals, first introduced by Jacquet–Rallis in [4], for regular semisimple elements twisted by a character η of H (cf. §2.1). The character η is trivial in the unitary case; in the general linear case, η is the quadratic character associated to the quadratic extension E/F (considered as a character of $GL_{n-1}(F)$ by composing with the determinant). Then our first result is a *density principle* in the sense of [6, p. 100]:

Theorem 1.1. *Assume that $n = 3$. Consider the regular orbital integrals as (H, η)-invariant distributions on $\mathcal{V}$. Then they span a weakly dense subspace in the space of all (H, η)-invariant distributions (namely, if all of regular semisimple orbital integrals of $f \in \mathcal{C}_c^\infty(\mathcal{V})$ vanish, then so does $T(f)$ for every (H, η)-invariant distribution T).*

As demonstrated by [6], the density principle does not hold in general for symmetric pairs (not even the the rank one case).

In the two cases above, their categorical quotients are isomorphic (cf. [9]; the case $n = 3$ is given explicitly below in terms of invariants). This isomorphism induces a bijection between the set of regular semisimple orbits in $\mathfrak{gl}_n$ and the disjoint union of regular semisimple orbits in $\mathfrak{u}(V)$ where $V = W \oplus Eu$ and W runs over all isomorphism classes of Hermitian spaces over E. In terms of the matching of orbits, Jacquet–Rallis conjectured ([4]) the existence of smooth matching of test functions. Denote by W_1, W_2 the two isomorphism classes of Hermirian spaces of dimension $n-1$, and $V_i = W \oplus Eu$ where u has norm one. Our second result is to establish a refined version of their conjecture when $n = 3$:

Theorem 1.2. *Assume that $n = 3$. Given $f \in \mathcal{C}_c^\infty(\mathcal{V})$, there exists a pair $f_i \in \mathcal{C}_c^\infty(\mathfrak{u}(V_i))$, $i = 1, 2$, that matches f. Conversely, given a pair $f_i \in \mathcal{C}_c^\infty(\mathfrak{u}(V_i))$, $i = 1, 2$, there exists $f \in \mathcal{C}_c^\infty(\mathcal{V})$ that matches $f_i \in \mathcal{C}_c^\infty(\mathfrak{u}(V_i))$, $i = 1, 2$. Moreover, their nilpotent orbital integrals (defined in §2.1) also match.*

This is proved in §4.2. We refer to Prop. 4.4 in §4.2 for the precise meaning of matching nilpotent orbital integrals. The existence part is now proved in [9] for all n. However, the current proof is different and yields a refined version of smooth matching that gives the relation between nilpotent orbital integrals. Some further application requires this refinement. In this aspect, the work of Jacquet–Chen ([2]) is a prototype which requires the existence of a refined version of smooth matching. In a sequel of this paper, we will pursue similar application to the refined Gan–Gross–Prasad conjecture.

To prove the two theorems, we study the "singularity" of the regular semisimple orbital integral near $0 \in \mathcal{V}/H$. We want to have a germ expansion of the orbital integral $O(\cdot, f)$ in a neighborhood of $0 \in \mathcal{V}/H$ and one hopes this expansion only involves the nilpotent orbital integrals (whose definition is not clear in general). When ρ is the adjoint representation of a reductive group H and F is non-archimedean, this is known as the Shalika germ expansion (on Lie algebra). However, the explicit form the singularity, namely, the so-called Shalika germ, is hard to obtain. The question makes sense in the above general setting and when the germ expansion exists we may call it a *relative Shalika germ* expansion. In the study of various relative trace formulae by Jacquet and many others, similar notion was already introduced (for example, in a series of work by Jacquet–Ye [5], which does not exactly fit this framework since the group H appeared there is not reductive). The main technical part of this paper is to establish an explicit germ expansion (Theorem 2.7 and 2.8). The case $n = 2$ was known already in [3]. One main difficulty in the case $n \geq 3$ is that, contrary to the classical Shalika germ expansion, there are infnitely many nilpotent orbits. To the arthur's knowledge, our result is the first of such examples. It is not surprising that the germ expansion turns out to be a continuos "sum" of nilpotent orbital integrals. The question for $n > 3$ looks formidable for the moment. One of the purpose of this paper is to draw the reader's attention to this type of question.

Note that a large part of the treatment of the unitary case uses technique similar to the general linear case. The reader may skip the unitary case without losing the idea of the proof. The paper is organized as follows. We will only consider the case $n = 3$. In section 2, we classify nilpotent orbits in both representations, and state the germ expansion. Then we deduce the application to the density principle. In section 3, we show the germ expansion. In section 4, we prove the refined version of the existence of smooth matching.

Some notation: F is a non-archimedean local field of characteristic zero and $\mathcal{O}_F$ its ring of integers. Let ϖ be a uniformizer and $k = \mathcal{O}_F/(\varpi)$ the residue field of cardinality q. Let E/F be a quadratic extension and let η be the quadratic character of $F^\times$ associated to E/F by class field theory. We denote by η_1 the character $\eta|\cdot|^{-1}$.

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2 Germ expansion of relative orbital integrals

We first define the relevant orbital integrals for nilpotent orbits. Then we present the main results of relative germs.

2.1 Nilpotent orbital integral

We first classify nilpotent orbits then study their orbital integrals.

Nilpotent orbits in the general linear case. An element in $\mathfrak{gl}_3$ can be written as

$$X = \begin{bmatrix} A & u \\ v & d \end{bmatrix}, \quad A \in \mathfrak{gl}_2.$$

We define an invariant subspace $\mathcal{V}$ of $\mathfrak{gl}_3$ of codimension two consisting of all X such that

$$tr(A) = 0, d = 0. \tag{2.1}$$

It is easy to see that, as an H-module, $\mathfrak{gl}_3$ is a direct sum of $\mathcal{V}$ and a two dimensional vector space with trivial H-action. We thus consider only the H-module $\mathcal{V}$. Then the ring of H-invariants are generated by (cf.[9])

$$-det(A), \quad vu, \quad vAu \in \mathcal{O}[\mathcal{V}]^H.$$

They define an H-invariant morphism π from $\mathcal{V}$ to $\mathbb{A}^3$, the affine 3-space (with the trivial H-action). This allows us to identify the categorical quotient $\mathcal{V}//H$ with $(\mathbb{A}^3, \pi)$ (cf. [9]). We also define a *discriminant*

$$\Delta(X) = -det(A)(vu)^2 - (vAu)^2.$$

Then $X \in \mathcal{V}$ is H-regular-semisimple if and only if $\Delta(X) \neq 0$. The condition $\Delta = 0$ defines a hypersurface of $\mathbb{A}^3$. We fix a section of $\pi : \mathcal{V} \to \mathbb{A}^3$:

$$\begin{aligned} \mathfrak{s} : \mathbb{A}^3 &\longrightarrow \mathcal{V} \\ (\lambda, a, b) &\mapsto \begin{bmatrix} 0 & \lambda & 1 \\ 1 & 0 & 0 \\ a & b & 0 \end{bmatrix}. \end{aligned} \tag{2.2}$$

It is obviously an adaption of the classical *companion matrix*. It has the property of the usual companion matrix does, namely: the image of $\mathfrak{s}$ lies in the H-regular locus. Recall that the H-regularity of X means that the stabilizer of X has minimal dimension among all elements in $\mathcal{V}$. In this case, the minimality condition amounts the fact that the stubblier of $\mathfrak{s}(\lambda, a, b)$ is always trivial for any $(\lambda, a, b) \in \mathbb{A}^3$. Moreover, it is self-evident how to generalize this section to $\mathfrak{gl}_n$.

Let η be the quadratic character associated to a quadratic extension E/F. We will be interested in the (η-twisted) orbital integral of regular semisimple X

$$O(X, f) = \int_H f(h \cdot X)\eta(H)dh,$$

where we for short write $\eta(h) = \eta(det(h))$. Our goal is to calculate the germ of the function $O(X, f)$ around $X = 0$. We see that the germ depends only the nilpotent orbital integrals (defined below) in a way that resembles the classical Shalika germ expansion. The difference has been indicated in the introduction part.

We now classify the H-orbits in the nilpotent cone $\mathcal{N} := \pi^{-1}(0)$ of $\mathcal{V}$ together with their stabilizers. More precisely, $X \in \mathcal{V}$ is nilpotent if and only if

$$A^2 = 0, \quad vA^i u = 0, i = 0, 1.$$

Then it is not hard to analyze them case by case. We omit the intermediate steps which are elementary and we only enumerate a complete set of representatives as follows. it consists of a continuous family and eight discrete orbits. The continuous family is

$$n(c) := \begin{bmatrix} 0 & c & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad c \in F$$

with stabilizer N. The others are

(1)

$$n_{0,+} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, n_{1,+} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, n_{2,+} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

their stabilizers are respectively: $\{1\}$, N, $B_+ = A_+N$ with A_+ consisting of matrices of the form $diag[1,x]$. The stabilizer of $n_{i,+}$ is of dimension i for $i = 0, 1, 2$.

(2) their transposes denoted by $n_{i,-}, i = 0, 1, 2$ respectively,

(3)

$$n_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad 0,$$

their stabilizers are respectively: ZN (where Z is the center of H), H.

Note that we list them in the order such that the later ones are limits of the earlier ones. Moreover, obviously three of them: $n_{2,\pm}$, 0, cannot support non-zero (H, η)-invariant distribution as the character η has non-trivial restriction on their stabilizers. We will see that they won't enter into the germ expansion.

Nilpotent orbits in the unitary case. Let $E = F[\sqrt{\tau}]$, $\tau \in F^\times \setminus (F^\times)^2$. In fact, as in [9], we need to consider $\sqrt{\tau} \cdot \mathfrak{u}(V)$. There are two isomorphic classes of hermitian space of a fixed dimension. We will use $J_i \in Her_{n-1}(E), i = 0, 1$ to denote the two isomorphism classes of hermitian matrix and let $H = U(J_i)$ be the unitary group:

$${}^t\overline{g} J_i g = J_i.$$

Let $Q_i = diag[J_i, 1]$ and $\mathfrak{u}_i$ be the Lie algebra of the corresponding unitary group $U(Q_i)$. Then $\sqrt{\tau}\mathfrak{u}(Q_i)$ consists of $Y \in M_n(E)$ satisfying

$${}^t\overline{Y} = Q_i^{-1} Y Q_i.$$

And H acts on $\sqrt{\tau}\mathfrak{u}_i$ induced by the restriction of the adjoint action of $U(Q_i)$ on its Lie algebra.

We now specify to case $n = 3$. We first consider the case $-det(J_0) \in \mathrm{N}E^\times$. Recall that $E = F[\sqrt{\tau}]$. We may set $J_0 = \begin{bmatrix} & \sqrt{\tau} \\ -\sqrt{\tau} & \end{bmatrix}$. It is easy to see that we then may identify $SU(J_0)$ with SL_2. Let $J = diag[J_0, 1]$ and consider the action of $H = U(J_0)$ on a subspace $\mathcal{W}_0$ of $\sqrt{\tau}\mathfrak{u}_0$ consisting of Y such that

$$Y = \begin{bmatrix} A & w \\ w' & d \end{bmatrix}, \quad tr(A) = 0, d = 0.$$

Such Y can be written as

$$Y = \begin{bmatrix} \alpha & \lambda' & z_1 \\ \lambda & -\alpha & z_2 \\ -\bar{z}_2\sqrt{\tau} & \bar{z}_1\sqrt{\tau} & 0 \end{bmatrix}, \alpha, \lambda, \lambda' \in F\sqrt{\tau}, z_1, z_2 \in E.$$

Modulo the $U(J_0)$-action, a regular seimisimple element can be written as

$$\begin{bmatrix} 0 & \lambda_2 & 1 \\ \lambda_1 & 0 & z_2 \\ -\bar{z}_2\sqrt{\tau} & \sqrt{\tau} & 0 \end{bmatrix}, \lambda_1, \lambda_2 \in \sqrt{\tau}F, z_2 \in E.$$

The ring of H-invariants is generated by

$$(\lambda_1\lambda_2, (z_2 - \bar{z}_2)\sqrt{\tau}, (\lambda_1 - \lambda_2 z_2 \bar{z}_2)\sqrt{\tau}).$$

This defines a morphism $\pi : \mathcal{W} \to \mathbb{A}^3$ and $(\mathbb{A}^3, \pi)$ defines a categorical quotient of $\mathcal{W}$ by H ([9]).

The nilpotent orbits are classified as follows

$$n(c) := \begin{bmatrix} 0 & \beta\sqrt{\tau} & 1 \\ 0 & 0 & 0 \\ 0 & \sqrt{\tau} & 0 \end{bmatrix}, \beta \in F; n_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -\sqrt{\tau} & \sqrt{\tau} & 0 \end{bmatrix},$$

$$n_{0,\pm} = \begin{bmatrix} 0 & \beta_\pm\sqrt{\tau} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \beta_\pm \in F^\times/\mathrm{N}E^\times, \eta(\beta_\pm) = \pm 1,$$

and finally 0. We will show that n_1 does not appear in the germ expansion.

When $-det(J_1) \in F^\times \setminus \mathrm{N}E^\times$, the associated unitary group $U(J_1)$ is anisotropic. In this case the only nilpotent element in an analogous subspace $\mathcal{W}_1 \subset \sqrt{\tau}\mathfrak{u}_1$ is zero itself.

Nilpotent orbital integrals

Before we proceed, we need to define the orbital integrals for nilpotent elements. In our setting, the nilpotent orbital integral usually diverges. This is different from the classical case when ρ is the adjoint representation, where by Deligne and Ranga Rao's theorem ([7]), the nilpotent orbital integral is absolutely convergent. In our situation, we have to use analytic continuation to define the nilpotent orbital integrals.

The general linear case. For $X \in \mathcal{N}$ whose stabilizer H_X is contained in SL_2, we denote

$$O(X, f, s) = \int_{H/H_X} f(h \cdot X)\eta(h)|h|^s d\bar{h}, \tag{2.3}$$

where the quotient measure on $d\bar{h}$ will be self-evident in view of the explicit list above and will be discussed in more details below. As we will show, this integral is absolutely convergent when $\mathrm{Re}(s)$ is large enough and meromorphic extends to all $s \in \mathbb{C}$ as a rational function of q^{-s}. When it has no pole at $s = 0$, we will denote for simplicity $O(X, f) = O(X, f, 0)$.

For $f \in \mathcal{C}_c^\infty(\mathcal{V})$, we say that f has *period radius* (at least) r if f is invariant under translation by elements $X \in \mathcal{V}$ in the ball in $\mathcal{V}$ of radius r. We say that f has *support radius* (at most) D if the support of f is contained in the ball in $\mathcal{V}$ of radius D. We will frequently use the Iwasawa decomposition $H = KB = KAN = KNA$, where $K = SL_2(\mathcal{O})$. We define f_K to be the function in $\mathcal{C}_c^\infty(\mathcal{V})$ by

$$f_K(X) = \int_K f(k \cdot X) dk.$$

We now also make convention for the measures. We fix a measure on F and a measure on K such that $\mathrm{vol}(K) = 1$. We will then choose the Haar measure on H by

$$dh = |y|^{-2} dk dx dy du, \quad h = k \begin{bmatrix} x & \\ & y \end{bmatrix} \begin{bmatrix} 1 & u \\ & 1 \end{bmatrix}.$$

As the stabilizer H_X for nilpotent X is trivial or N, ZN, or their transpose, we may choose a measure e du on N and $|x|^{-1}dx$ on Z, and similarly for the transpose of N.

Lemma 2.1. • *The integral*

$$O(n_{0,\pm}, f, s) = \int_H f(h \cdot n_{0,\pm})\eta(h)|h|^s dh \tag{2.4}$$

can be analytically extended to all $s \in \mathbb{C}$ with only a simple pole at $s = 1/2$. They are all rational functions of q^{-s}. For each of them, the value at $s = 0$ defines an (H, η)-invariant distributions.

- *The integral*

$$O(n_0, f) = \int_{H/ZN} f(h \cdot n_0)\eta(h)d\bar{h} \tag{2.5}$$

is absolutely convergent and defines an (H, η)-invariant distributions.

Proof. The assertion for n_0 is trivial. We now prove the one for $n_{0,+}$. By Iwasawa decomposition we have

$$O(n_{0,+}, f, s) = \int_{u\in F}\int_{x,y\in F^\times} f_K \begin{bmatrix} 0 & xy^{-1} & ux \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \eta(xy)|xy|^s \frac{dudxdy}{|y|^2}. \tag{2.6}$$

Substitute $x \to xy$ and then $u \to u(xy)^{-1}$:

$$O(n_{0,+}, f, s) = \int_{u\in F}\int_{x,y\in F^\times} f_K \begin{bmatrix} 0 & x & u \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \eta(x)|xy^2|^s \frac{dudxdy}{|x||y|^2}. \tag{2.7}$$

When $\mathrm{Re}(s) > 1/2$, it is easy to see the absolute convergence from the formula above. The rest follows from the Tate's thesis. □

Lemma 2.2. *The integral*

$$O(n_{1,\pm}, f, s) = \int_{H/N_-} f(h \cdot n_{1,\pm})\eta(h)|h|^s d\bar{h} \tag{2.8}$$

can be analytically extended to all $s \in \mathbb{C}$ with a simple pole at $s = 0$. The residue for $n_{1,+}, n_{1,-}$ are given respectively by

$$\frac{1}{2\zeta(1)\log q}O(n_0, f), \quad \frac{1}{2\zeta(1)\log q}\eta(-1)O(n_0, f).$$

Let

$$O'(n_0, f) := \int_{x\in F^\times} \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{\eta(x)\log|x|dx}{\log q}.$$

Then the constant term of the sum

$$O(n_{1,+}, f, s) + \eta(-1)O(n_{1,-}, f, s) - \frac{1}{\zeta(1)\log q}O'(n_0, f),$$

denoted by $O(n_{1,+,-}, f)$, defines an (H, η)-invariant distribution. Moreover, on the orbits of each of $n_{1,\pm}$, there is an (H, η)-invariant distribution; but none of them extend to an invariant distribution on $\mathcal{V}$.

Proof. By definition we have

$$O(n_{1,+}, f, s) = \int_{x,y\in F^\times} f_K \begin{bmatrix} 0 & xy^{-1} & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \eta(xy)|xy|^s \frac{dxdy}{|y|^2}.$$

Substitute $y \to y^{-1}x$:

$$O(n_{1,+}, f, s) = \int_{x,y\in F^\times} f_K \begin{bmatrix} 0 & y & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \eta(y)|x^2y^{-1}|^s \frac{dxdy}{|x|}.$$

Then it follows from the Tate's thesis that this has a simple pole at $s = 0$ with residue

$$\frac{1}{2\zeta(1)\log q}\int_{y\in F^\times} f_K \begin{bmatrix} 0 & y & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \eta(y)dy = \frac{1}{2\zeta(1)\log q}O(n_0, f). \tag{2.9}$$

Similarly for $O(n_{1,-}, f, s)$. For the (H, η)-invariance, one may prove it directly. But in any way it follows from the formula in the next lemma.

Finally, it is easy to see the existence of (H, η)-invariant distribution on the orbit of $n_{1,+}$ ($n_{1,-}$, resp.), but this distribution does not extend (H, η)-invariantly to $\mathcal{V}$. □

Finally we have for the one-parameter family of nilpotent orbits

$$O(n(\mu), f) = \int_{H/N} f(h \cdot n(\mu))\eta(h)d\bar{h}.$$

The integral converges absolutely from the following equivalent expression:

$$\int_{x,y\in F^\times} f_K \begin{bmatrix} 0 & \mu xy & x \\ 0 & 0 & 0 \\ 0 & y & 0 \end{bmatrix} \eta(xy) dx dy. \tag{2.10}$$

They all define (H, η)-invariant distributions.

Lemma 2.3. *When $|\mu|$ is large enough, $O(n(\mu), f)$ is equal to*

$$\frac{\eta(\mu)}{\zeta(1)|\mu|} O(n_{1,+,-}, f) + \frac{\eta(\mu) \log |\mu|}{\zeta(1)|\mu| \log q} O(n_0, f).$$

Proof. We break the integral (2.10) into two pieces according to $|x| \leq 1$ or not. Without loss of generality, we may assume that f_K has period radius at least 1 and support radius at most D. The part from $|x| \leq 1$ can be written as the sum of two terms:

$$\int_{x,y\in F^\times} f_K \begin{bmatrix} 0 & \mu xy & 0 \\ 0 & 0 & 0 \\ 0 & y & 0 \end{bmatrix} \eta(xy)|y|^s dx dy|_{s=0},$$

$$-\int_{|x|>1, y\in F^\times} f_K \begin{bmatrix} 0 & \mu xy & 0 \\ 0 & 0 & 0 \\ 0 & y & 0 \end{bmatrix} \eta(xy)|y|^s dx dy|_{s=0}.$$

Both have a simple pole with opposite residues. When $|\mu| > D$, in the second term we must have $|y| \leq 1$. So we rewrite the two terms respectively as

$$\frac{\eta(\mu)}{\zeta(1)|\mu|} \int_{x,y\in F^\times} f_K \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & y & 0 \end{bmatrix} \eta(x)|y|^s dx d^\times y|_{s=0}$$

and

$$\begin{aligned} &- \frac{\eta(\mu)}{\zeta(1)|\mu|} \int_{y\in F^\times} f_K \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \eta(x) \int_{|y|<|x/\mu|} |y|^s d^\times y dx|_{s=0} \\ &= - \frac{\eta(\mu)}{\zeta(1)|\mu|} \int_{y\in F^\times} f_K \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \eta(x) \left(|\frac{x}{\mu}|^s \int_{|y|<1} |y|^s d^\times y \right) dx|_{s=0}. \end{aligned}$$

Similarly the part from $|x| > 1$ can be written as the sum of two terms when $|\mu| > D$:

$$\frac{\eta(\mu)}{\zeta(1)|\mu|}\int_{x,y\in F^\times} f_K\begin{bmatrix} 0 & y & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\eta(y)|x|^s d^\times x dy|_{s=0}$$

and

$$-\frac{\eta(\mu)}{\zeta(1)|\mu|}\int_{|x|\leq 1, y\in F^\times} f_K\begin{bmatrix} 0 & y & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\eta(y)|x|^s d^\times x dy|_{s=0}.$$

Finally we note

$$|\frac{x}{\mu}|^s\int_{|y|<1}|y|^s d^\times y dx + \int_{|x|\leq 1}|x|^s d^\times x = \frac{|\varpi x/\mu|^s + 1}{1-q^{-s}}.$$

Its constant term is given by

$$\frac{\log|x/\mu|}{\log q}.$$

Together we obtain when $|\mu| > D$:

$$\frac{\eta(\mu)}{\zeta(1)|\mu|}O(n_{1,+,-}, f) + \frac{\eta(\mu)\log|\mu|}{\zeta(1)|\mu|\log q}O(n_0, f).$$

□

We now call all the nilpotent orbital integrals in above lemmas *relevant* if they define (H,η)-invariant distribution. We will see that the relevant ones are precisely those “visible” by the germ expansion.

The unitary case. Now we discuss the nilpotent orbital integrals in the unitary case.

Let T be the maximal torus of $U(J_0)$ consisting of elements of the form $diag(z, \bar{z}^{-1})$, $z \in E^\times$. As $SU(J_0) = SL_2$ and $U(J_0) = T\cdot SU(J_0) = SU(J_0)\cdot T$, we have a variant of the Iwasawa decomposition

$$H := U(J_0) = KTN,$$

where $K = SL_2(\mathcal{O})$ and $N \subset SL_2$ as before (by abuse of notations, from now on, $T, SU(J_0)$ etc. mean theire F-points). We will fix a Haar measure dz on E. Then we choose a Haar measure on $H = KTN$ by

$$dh = dkdzdu, \quad h = k \begin{bmatrix} z & \\ & \bar{z}^{-1} \end{bmatrix} \begin{bmatrix} 1 & u \\ & 1 \end{bmatrix}.$$

where the Haar measure on K is normalized by $\mathrm{vol}(K) = 1$. In this way we define the nilpotent orbital integral

$$O(n, f) = \int_{H/N} f_K(h \cdot n) d\bar{h}, \quad n = n(c), n_{1,\pm},$$

and

$$O(0, f) = q^{-1}\mathrm{vol}(\mathcal{O}_E)L(1, \eta)f(0).$$

It is easy to see that all of them converges absolutely. We have an asymptotic expansion analogous to Lemma 2.3.

Lemma 2.4. *When $|\mu|$ is large enough, $O(n(\mu), f)$ is of the form*

$$A\frac{\eta(\mu)}{|\mu|} + B\frac{1}{|\mu|}.$$

Moreover, we may express the constants A and B explicitly in terms of $O(n_{1\pm}, f)$.

Lemma 2.5. *There is no H-invariant distribution on the orbit of n_1 that extend to an H-invariant distribution on $\mathcal{W}$.*

The proofs of the two lemmas are simple and we omit them.

For the other case J_1, H is compact and we normalize the Haar measure such that $\mathrm{vol}(H) = 1$. There is only one nilpotent orbit: $O(n, f) = f(0)$ if $n = 0$.

Based on the second lemma, we call the orbit of n_1 *irrelevant* and the others are *relevant*. Again we will see that the relevant ones are precisely those "visible" by the germ expansion.

2.2 Theorem on relative germ expansion

We now state our results on relative germ expansion.

Definition 2.6. *The space $\mathcal{C}_1(F)$ and $\mathcal{C}_2(F)$, resp., consists of locally constant functions ϕ on F with the property: there are constants c_1, c_2 such that when $|x|$ is large enough, we have for $\mathcal{C}_1(F)$*

$$\phi(x) = c_1\eta(x)|x|^{-1} + c_2\eta(x)\log|x||x|^{-1},$$

and for $\mathcal{C}_2(F)$ respectively:

$$\phi(x) = c_1|x|^{-1} + c_2\eta(x)|x|^{-1}.$$

Let $f \in \mathcal{C}_c^\infty(\mathcal{V})$ and a pair $\{f_i \in \mathcal{C}_c^\infty(\mathcal{W}_i)_{i=0,1}\}$. We will denote

$$\phi_f(x) = O(n(x), f), \quad \phi_{f_0}(x) = O(n(x), f_0). \tag{2.11}$$

Then $\phi_f \in \mathcal{C}_1(F)$ and $\phi_{f_0} \in \mathcal{C}_2(F)$ by Lemma 2.3 and 2.4.

The first theorem is for the germ expansion in the general linear case. Recall that

$$X = \begin{bmatrix} 0 & 1 & 1 \\ \lambda & 0 & 0 \\ a & b & 0 \end{bmatrix}, \tag{2.12}$$

whose invariants are

$$(\lambda, a, b), \quad \Delta = \lambda a^2 - b^2.$$

Theorem 2.7. *There is a neighborhood of $0 \in \mathbb{A}^3$ on which the orbital integral $O(X, f)$ at $X = \mathfrak{s}(\lambda, a, b)$ is the sum of the following two terms*

(i)

$$\int_F \phi_f(\mu)\Gamma_\mu(\lambda, a, b)d\mu,$$

where the germ $\Gamma_\mu(\lambda, a, b) = 0$ if $(a^2\mu - 2b)^2 + 4\Delta$ is a non-square, and

$$\Gamma_\mu(\lambda, a, b) = \frac{2\eta(-u)|a|}{|(a^2\mu - 2b)^2 + 4\Delta|^{1/2}},$$

if $(a^2\mu - 2b)^2 + 4\Delta$ is a square and u denotes one of the two roots of $a^2\mu = u - \frac{\Delta}{u} + 2b$.

(ii) $\eta(-1)(O(n_{0+}, f) + \eta(\Delta)O(n_{0-}, f))$.

Conversely, let $\Psi(\lambda, a, b)$ be a function in a neighborhood of $0 \in \mathbb{A}^3$ such that for some $\phi \in \mathcal{C}_1(F)$ and two constant A_1, A_2, we have

$$\Psi(\lambda, a, b) = \int_F \phi(\mu)\Gamma_\mu(\lambda, a, b)d\mu + A_1 + A_2\eta(\Delta).$$

Then there exists a function $f \in \mathcal{C}_c^\infty(\mathcal{W})$ such that in a neighborhood (possibly smaller) of 0, we have

$$\Psi(\lambda, a, b) = O(X(\lambda, a, b), f_1),$$

when $\Delta \neq 0$.

We also have a germ expansion in the unitary case.

Theorem 2.8. *Let $f_i \in \mathcal{C}_c^\infty(\mathcal{W}_i), i \in \{0, 1\}$. Let $Y(\lambda, a, b) \in \mathfrak{u}_i, i \in \{0, 1\}$ be any element with invariants $(\lambda, a, b) \in \mathbb{A}^3$.*

(1) *There is a neighborhood of $0 \in \mathbb{A}^3$ on which the orbital integral $O(Y, f_0)$ at $Y = Y(\lambda, a, b)$ with $\eta(-\Delta) = 1$ is the sum of the following two terms:*

  (i)
$$\int_F \phi_{f_0}(\mu)\widetilde{\Gamma}_\mu(\lambda, a, b)d\mu,$$
  where
$$\widetilde{\Gamma}_\mu(\lambda, a, b) = \frac{2|a|}{|\tau((a^2\mu - 2b)^2 + 4\Delta)|^{1/2}}$$
  if $\tau((a^2\mu - 2b)^2 + 4\Delta)$ is a square, and zero otherwise.

  (ii) $-O(0, f)$.

(2) *There is a neighborhood of $0 \in \mathbb{A}^3$ on which the orbital integral $O(Y, f_1)$ at $Y = Y(\lambda, a, b)$ with $\eta(-\Delta) = -1$ is given by*

$$O(Y, f_1) = O(0, f_1).$$

Conversely, let $\Phi(\lambda, a, b)$ be a function in a neighborhood of $0 \in \mathbb{A}^3$ such that for some $\phi \in \mathcal{C}_2(F)$ and two constant A_1, A_2,we have

$$\Phi(\lambda, a, b) = \int_F \phi(\mu)\widetilde{\Gamma}_\mu(\lambda, a, b)d\mu + A_1 + A_2\eta(\Delta).$$

Then there exists a pair of function $f_i \in \mathcal{C}_c^\infty(\mathcal{W}_i), i \in \{0, 1\}$ such that in a neighborhood (possibly smaller) of 0, we have

$$\Phi(\lambda, a, b) = \begin{cases} O(Y(\lambda, a, b), f_0), & \eta(-\Delta) = 1; \\ O(Y(\lambda, a, b), f_1), & \eta(-\Delta) = -1. \end{cases}$$

2.3 Application to the density principle

One immediate corollary of Theorem 2.7 and Theorem 2.8 is

Theorem 2.9. *Let $f \in \mathcal{C}_c^\infty(\mathcal{V})$ be such that all its regular orbital integrals vanish. Then its relevant nilpotent orbital integrals vanish too.*

From this we may deduce the *density principle*, Theorem 1.1 of Introduction.

Proof of Theorem 1.1. We show this in the general linear case. We let $\mathcal{D}(\mathcal{V})$ denote the space of distributions on $\mathcal{V}$. The unitary case is similar and easier. Consider the invariants $\pi : \mathcal{V} \to \mathbb{A}^3$. Over each point in the complement of $\Delta = 0$, the fiber consists of precisely one regular semisimple orbit. By Bernstein's localization principle (cf.[6, p. 99]), it is enough to show that for any point $x \in \mathbb{A}^3$ lying on the hyperspace defined by $\Delta = 0$, and $T \in \mathcal{D}(\pi^{-1}(x))^{H,\eta}$, we have $T(f) = 0$ for $f \in \mathcal{C}_c^\infty(\mathcal{V})$ with vanishing regular semisimple orbital integrals. We show this for the extreme case $x = 0$ (so we are in the case $\mathcal{N} = \pi^{-1}(0)$). For the other point x, it is easier and essentially use similar expansion for the case $n = 2$.

To treat the case $x = 0$, we note that by Theorem 2.9 above, it suffices to show that the relevant unipotent orbital integrals are weakly dense in $\mathcal{D}(\mathcal{N})^{H,\eta}$. Since the orbit $H \cdot n_{0,+}$ is characterized inside $\mathcal{N}$ by the condition that u, Au are linearly independent, it is an open orbit. Consider the closed subset $\mathcal{N} \setminus \mathcal{N}_\pm, \mathcal{N}_\pm = H \cdot n_{0,+} \coprod H \cdot n_{0,-}$. By the exact sequence

$$0 \to \mathcal{D}(\mathcal{N} \setminus \mathcal{N}_\pm)^{H,\eta} \to \mathcal{D}(\mathcal{N})^{H,\eta} \to \mathcal{D}(\mathcal{N}_\pm)^{H,\eta},$$

we are reduced to show that weak density of relevant nilpotent orbital integrals in $\mathcal{D}(\mathcal{N} \setminus \mathcal{N}_{\pm})^{H,\eta}$. Define an open subsubvariety of $\mathcal{N}$

$$\mathcal{N}_0 := \{X \in \mathcal{N} | A \neq 0, u \neq 0, v \neq 0\}$$

Then it defines a smooth subvariety of $\mathcal{V}$ (by checking the dimension of the tangent space of every point) and the F-points of it (by abuse of notation still denoted by $\mathcal{N}_0$) inherit an F-manifold structure. Then the closed subset $\mathcal{N} \setminus (\mathcal{N}_{\pm} \coprod \mathcal{N}_0)$ consists of finitely many H-orbits. By similar argument above and Lemma 2.2 on the irrelevant orbit, we may show the relevant nilpotent orbital integrals span $\mathcal{D}(\mathcal{N} \setminus (\mathcal{N}_{\pm} \coprod \mathcal{N}_0))^{H,\eta}$. It remains to show that the restriction of nilpotent orbital integrals $O(n(\mu), \cdot)$ to $\mathcal{N}_0$ are weakly dense in $\mathcal{D}(\mathcal{N}_0)^{H,\eta}$. We consider the continuous map

$$\begin{aligned} H/N \times F^{\times} &\to \mathcal{N}_0 \\ (h, \mu) &\mapsto hn(\mu)h^{-1}. \end{aligned}$$

It is clearly bijective. One can also verify that it is subversive (namely, the induced map at every point on tangent spaces is surjective), hence open. In particular it is an homeomorphism and its inverse composed with the projection to the second factor defines a continuous map $\mathcal{N}_0 \to F^{\times}$. This is H-equivariant for the trivial action of H on the target and each fiber consists of precisely one H-orbit. Now the desired weak density follows again from the localization principle of Bernstein.

□

3 Proof of the germ expansion

We will only prove the germ expansion in the general linear case. The unitary case is similar and indeed simpler.

We use the Iwasawa decomposition $H = KAN$ for $K = SL_2(\mathcal{O})$:

$$h = k \begin{bmatrix} y & \\ & x^{-1}y \end{bmatrix} \begin{bmatrix} 1 & u \\ & 1 \end{bmatrix}, \quad dg = dk \frac{dxdydu}{|y|}.$$

Noting that

$$\begin{bmatrix} x & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & u \\ & 1 \end{bmatrix} \cdot \begin{bmatrix} & \lambda \\ 1 & \end{bmatrix} = \begin{bmatrix} u & (\lambda - u^2\epsilon)x \\ 1/x & -u \end{bmatrix},$$

we may write $O(X,f)$ for $X = \mathfrak{s}(\lambda, a, b)$:

$$\text{(3.1)} \qquad \int_{x,y\in F^\times, u\in F} f_K \begin{bmatrix} u & x(\lambda-u^2) & y \\ 1/x & -u & 0 \\ y^{-1}a & (b-au)xy^{-1} & 0 \end{bmatrix} \eta(x)\frac{dxdydu}{|y|}.$$

Here as before we have a K-invariant function $f_K(X) := \int_K f(k^{-1}Xk)dk$. Without loss of generality, we may assume that f and hence f_K has period radius 1 and support radius $D > 1$.

We now split the integral over $(x,y) \in (F^\times)^2$ into four cases. We summarize the total contribution of each case at the end of each subsection. We make the following convention:

- All integrals are understood as the value obtained by analytic continuation. As all analytic continuation appeared in this paper is elementary, we often do not mention it.
- For simplicity, we only write the integrand involving f_K and the domain of integration. For example the first equation below means

$$\int_{|y|>1,|x|>D,u\in F} f_K \begin{bmatrix} 0 & (\lambda-u^2)x & y \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix} \eta(x)\frac{dxdydu}{|y|}.$$

Case (1)**:** $|y| > 1, |x| > D^2$**.**

Then $|u| \le 1$ is automatically true:

$$f_K \begin{bmatrix} 0 & (\lambda-u^2)x & y \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix}; \{|y| > 1, |x| > D^2, u\}.$$

This is the sum of three terms

(3.2)

$$-f_K \begin{bmatrix} 0 & (\lambda-u^2)x & y \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix} = -f_K \begin{bmatrix} 0 & (\lambda-u^2)x & 0 \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix}; \{x, |y| \le 1, u\},$$

(3.3)

$$-f_K \begin{bmatrix} 0 & (\lambda-u^2)x & y \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix} = -f_K \begin{bmatrix} 0 & -u^2x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \{|x| \le D^2, |y| > 1, u\}.$$

(note: $\gamma \to 0$ and $|(\lambda - u^2)x| \le D$ implies that $|ux| \le (|u^2x||x|)^{1/2} < D^2$) and

$$(3.4) \qquad f_K \begin{bmatrix} 0 & (\lambda - u^2)x & y \\ 0 & 0 & 0 \\ 0 & (b - au)xy^{-1} & 0 \end{bmatrix}; \{x, y, u\}.$$

To summarize, Case (1) *contributes: (3.2),(3.3),(3.4).*

Case (2)**:** $|y| > 1, |x| \le D^2$.

Similarly $|(\lambda - u^2)x| \le D$ (and $\lambda \to 0$) implies that $|ux| = (|u^2x||x|)^{1/2} < D^2$. We have (note: $\lambda, a, b \to 0$)

$$f_K \begin{bmatrix} u & -xu^2 & y \\ 1/x & -u & 0 \\ 0 & 0 & 0 \end{bmatrix}; \{|x| \le D^2, |y| > 1, u \in F\}.$$

which is the sum of three terms:

(3.5)

$$-f_K \begin{bmatrix} u & -xu^2 & y \\ 1/x & -u & 0 \\ 0 & 0 & 0 \end{bmatrix} = -f_K \begin{bmatrix} 0 & -xu^2 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \{|x| > D^2, |y| > 1, u \in F\}.$$

(3.6)

$$-f_K \begin{bmatrix} u & -xu^2 & y \\ 1/x & -u & 0 \\ 0 & 0 & 0 \end{bmatrix} = -f_K \begin{bmatrix} u & -xu^2 & 0 \\ 1/x & -u & 0 \\ 0 & 0 & 0 \end{bmatrix}; \{x, |y| \le 1, u \in F\},$$

and

$$(3.7) \qquad f_K \begin{bmatrix} u & -xu^2 & y \\ 1/x & -u & 0 \\ 0 & 0 & 0 \end{bmatrix}; \{x, y, u \in F\}.$$

(Note that all of them need to be regularized.)

To summarize, the Case (2) *contributes: (3.5), (3.6), (3.7).*

Case (3)**:** $|y| \leq 1, |x| > D^2$

As $|x| > D$ (note $\lambda \to 0$), we have $|u| \leq 1$:

$$f_K \begin{bmatrix} 0 & (\lambda - u^2)x & 0 \\ 0 & 0 & 0 \\ y^{-1}a & (b-au)xy^{-1} & 0 \end{bmatrix}; \{|y| \leq 1, |x| > D^2\}.$$

This is the sum of two terms:

$$f_K \begin{bmatrix} 0 & (\lambda - u^2)x & 0 \\ 0 & 0 & 0 \\ y^{-1}a & (b-au)xy^{-1} & 0 \end{bmatrix}; \{|y| \leq |a|, |x| > D^2\},$$

and

(3.8)

$$f_K \begin{bmatrix} 0 & (\lambda - u^2)x & 0 \\ 0 & 0 & 0 \\ y^{-1}a & (b-au)xy^{-1} & 0 \end{bmatrix} = f_K \begin{bmatrix} 0 & (\lambda - u^2)x & 0 \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix}; \{|a| < |y| \leq 1, |x| > D^2\}.$$

In the first term, it is easy to see that $|(b/a-u)x| \leq D|y||a|^{-1} \leq D$ and $|b/a-u| \leq D/|x| \leq 1/D$, $|u| < 1/D$. Note that

$$(b/a)^2 - u^2 = (b/a-u)^2 + 2(b/a-u)u.$$

We conclude that $|((b/a)^2 - u^2)x| \leq 1$ (this possibly fails only when $p = 2$; but the equality below holds if we assume in the beginning that the period radius of the test function f is $1/|2|_p$):

$$f_K \begin{bmatrix} 0 & (\lambda - u^2)x & 0 \\ 0 & 0 & 0 \\ y^{-1}a & (b-au)xy^{-1} & 0 \end{bmatrix} = f_K \begin{bmatrix} 0 & (\lambda - (b/a)^2)x & 0 \\ 0 & 0 & 0 \\ y^{-1}a & (b-au)xy^{-1} & 0 \end{bmatrix}; \{|y| \leq |a|, |x| > D^2\}$$

which is the sum of

$$(3.9) \qquad f_K \begin{bmatrix} 0 & (\lambda - (b/a)^2)x & 0 \\ 0 & 0 & 0 \\ y^{-1}a & (b-au)xy^{-1} & 0 \end{bmatrix}; \{x, y, u\}$$

$$(3.10) \qquad -f_K \begin{bmatrix} 0 & (\lambda - (b/a)^2)x & 0 \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix}; \{x, |y| > |a|, u\}$$

and

$$-f_K \begin{bmatrix} 0 & -(b/a)^2x & 0 \\ 0 & 0 & 0 \\ y^{-1}a & (b-au)xy^{-1} & 0 \end{bmatrix}; \{|x| \le D^2, |y| \le |a|, u\}. \tag{3.11}$$

To summarize, Case (3) *contributes:*(3.8), (3.9), (3.10), (3.11).

Case (4)**:** $|y| \le 1, |x| \le D^2$**.**

$$f_K \begin{bmatrix} u & -u^2x & 0 \\ 1/x & -u & 0 \\ y^{-1}a & (b-au)xy^{-1} & 0 \end{bmatrix}; \{|x| \le D^2, |y| \le 1, u\}.$$

We may write this as a sum of three terms:

$$f_K \begin{bmatrix} u & -u^2x & 0 \\ 1/x & -u & 0 \\ y^{-1}a & (b-au)xy^{-1} & 0 \end{bmatrix}; \{x, y, u\}, \tag{3.12}$$

(3.13)

$$-f_K \begin{bmatrix} u & -u^2x & 0 \\ 1/x & -u & 0 \\ y^{-1}a & (b-au)xy^{-1} & 0 \end{bmatrix} = -f_K \begin{bmatrix} u & -u^2x & 0 \\ 1/x & -u & 0 \\ 0 & 0 & 0 \end{bmatrix}; \{|x| \le D^2, |y| > 1, u\},$$

and

$$-f_K \begin{bmatrix} u & -u^2x & 0 \\ 1/x & -u & 0 \\ y^{-1}a & (b-au)xy^{-1} & 0 \end{bmatrix}; \{|x| > D^2, y, u\}. \tag{3.14}$$

Subcase (4i)**: the term (3.13).** It is the sum of two terms

$$-f_K \begin{bmatrix} u & -u^2x & 0 \\ 1/x & -u & 0 \\ 0 & 0 & 0 \end{bmatrix}; \{x, |y| > 1, u\} \tag{3.15}$$

and

(3.16)

$$f_K \begin{bmatrix} u & -u^2x & 0 \\ 1/x & -u & 0 \\ 0 & 0 & 0 \end{bmatrix} = f_K \begin{bmatrix} 0 & -u^2x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \{|x| > D^2, |y| > 1, u\}.$$

Subcase $(4ii)$ **: the term (3.14).** It is the sum of two terms:

$$
\begin{aligned}
&-f_K \begin{bmatrix} u & -u^2x & 0 \\ 1/x & -u & 0 \\ y^{-1}a & (b-au)xy^{-1} & 0 \end{bmatrix}; |x| > D^2, |y| > |a|, \\
&= -f_K \begin{bmatrix} 0 & -u^2x & 0 \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix}; \{|x| > D^2, |y| > |a|, u\}
\end{aligned} \tag{3.17}
$$

(note: $|u| \leq 1$ is automatic in this term) and

$$
-f_K \begin{bmatrix} u & -u^2x & 0 \\ 1/x & -u & 0 \\ y^{-1}a & (b-au)xy^{-1} & 0 \end{bmatrix} = -f_K \begin{bmatrix} 0 & -u^2x & 0 \\ 0 & 0 & 0 \\ y^{-1}a & (b-au)xy^{-1} & 0 \end{bmatrix}; \{|x| > D^2, |y| \leq |a|, u\}
$$

In the second term, the same argument as in Case (3) allows us to replace u^2 by $(b/a)^2$:

$$
-f_K \begin{bmatrix} 0 & -(b/a)^2x & 0 \\ 0 & 0 & 0 \\ y^{-1}a & (b-au)xy^{-1} & 0 \end{bmatrix}; \{|x| > D^2, |y| \leq |a|, u\}. \tag{3.18}
$$

Combine (3.11) and (3.18):

$$
-f_K \begin{bmatrix} 0 & -(b/a)^2x & 0 \\ 0 & 0 & 0 \\ y^{-1}a & (b-au)xy^{-1} & 0 \end{bmatrix}; \{x, y, u\}, \tag{3.19}
$$

plus

$$
f_K \begin{bmatrix} 0 & -(b/a)^2x & 0 \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix}; \{x, |y| > |a|, u\}. \tag{3.20}
$$

To summarize, Case (4) *contributes:(3.12) (3.15),(3.16),(3.17),(3.19),(3.20), minus (3.11).*

Finish of the proof

We now put together all four cases.

Lemma 3.1. *The term (3.4) is equal to*

$$\int_F O(n(\mu), f)\Gamma_\mu(\lambda, a, b)d\mu, \tag{3.21}$$

where the germ

$$\Gamma_\mu(\lambda, a, b) = \frac{2\eta(-u)|a|}{|(a^2\mu - 2b)^2 + 4\Delta|^{1/2}},$$

if $(a^2\mu - 2b)^2 + 4\Delta$ *is a square and* $\eta(-\Delta) = 1$ *(then* u *denotes one of the two roots of* $u - \frac{\Delta}{u} = a^2\mu - 2b$*), and* $\Gamma_\mu(\lambda, a, b) = 0$ *otherwise.*

Proof. A suitable substitution in (3.4) yields

$$\eta(-1)|a|^{-1}\int f_K \begin{bmatrix} 0 & \frac{(u+b)^2 - a^2\lambda}{a^2u}xy & y \\ 0 & 0 & 0 \\ 0 & x & 0 \end{bmatrix} \eta(xyu)\frac{dxdydu}{|u|}; x, y \in F^\times, u \in F.$$

Now substitute $\mu = \frac{(u+b)^2 - a^2\lambda}{a^2u}$. Then we have

$$a^2\mu = u - \frac{\Delta}{u} + 2b,$$

$$|a|^2 d\mu = \frac{du}{|u|}|u + \frac{\Delta}{u}|.$$

Notice the symmetry $u \to -\frac{\Delta}{u}$ and we conclude that the integral is zero if $\eta(-\Delta) = -1$. Now assume that $\eta(-\Delta) = 1$. Note that

$$(u + \frac{\Delta}{u})^2 = (a^2\mu - 2b)^2 + 4\Delta.$$

The map $u \to \mu$ is 2-to-1. The lemma then follows easily. $\square$

Lemma 3.2. *The sum of (3.3) and (3.5) is zero. The sum of (3.6) and (3.15) is zero.*

Proof. Recall that their values are understood in the sense of analytic continuation. We prove the second assertion. The first one is simpler. We substitute $u \to u/x$ in both (3.6) and (3.15). We first consider the inner integral of (3.6) and (3.15):

$$\int f_K \begin{bmatrix} u & -xu^2 & 0 \\ 1/x & u & 0 \\ 0 & 0 & 0 \end{bmatrix} \eta(x)|x|^{-s}dxdu; \{x, u\}.$$

Note that

$$\begin{bmatrix} 1 & u \\ & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -u^2 \\ 1 & -u \end{bmatrix},$$
$$\begin{bmatrix} 1 & u \\ & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1/u \end{bmatrix} \begin{bmatrix} 1 & \\ u & 1 \end{bmatrix} \begin{bmatrix} 1/u & \\ & u \end{bmatrix}.$$

Since f_K is invariant under $\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 1/u \end{bmatrix}$ ($\begin{bmatrix} 1 & u \\ & 1 \end{bmatrix}$, resp.) when $|u| > 1$ ($|u| \leq 1$, resp.), we may write the integral as a sum according to $u > 1$ or not:

$$\int f_K \begin{bmatrix} 0 & 0 & 0 \\ 1/x & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \eta(x)|x|^{-s-1} dx du; \{x, |u| \leq 1\},$$

and

$$\int f_K \begin{bmatrix} 0 & 0 & 0 \\ u^2/x & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \eta(x)|x|^{-s-1} dx du; \{x, |u| > 1\}.$$

The sum can be simplified as

$$\left(\int_{|u| \leq 1} du + \int_{|u|>1} |u|^{-2s} du \right) \int_{x \in F} f_K \begin{bmatrix} 0 & 0 & 0 \\ 1/x & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \eta(x)|x|^{-s-1} dx du.$$

It is easy to see that the first factor $\left(\int_{|u| \leq 1} du + \int_{|u|>1} |u|^{-2s-2} du \right)$ has a zero at $s = 0$ and the second one is homomorphic at $s = 0$. But the integral $\int_{|y| \leq 1} |y|^{2s-1} dy$ and $\int_{|y|>1} |y|^{2s-1} dy$ each has a simple pole at $s = 0$. Moreover the residue of those two sum up to zero. Therefore (3.6) and (3.15) sum up to zero. This completes the proof.

□

Lemma 3.3. *The sum of the following terms is zero: (3.2), (3.8) (3.10) (3.16) (3.17) (3.20).*

Proof. The sum of (3.2) and (3.8) is equal to the sum of the same integral over the domain

(I) $|x| > D^2, |y| \leq |a|$;

(II) $|y| \le 1, |x| \le D^2$.

The term for (I) is then reduced to

$$-f_K \begin{bmatrix} 0 & (\lambda - (b/a)^2)x & 0 \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix}; \{|x| > D^2, |y| \le |a|\}.$$

This term and the term (3.10) sum up to

$$f_K \begin{bmatrix} 0 & (\lambda - (b/a)^2)x & 0 \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix} = f_K \begin{bmatrix} 0 & -(b/a)^2 x & 0 \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix}; \{|x| \le D^2, |y| \le |a|\}.$$

This last one and (3.20) together sum up to

$$-f_K \begin{bmatrix} 0 & -(b/a)^2 x & 0 \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix}; \{|x| > D^2, |y| \le |a|\}. \tag{3.22}$$

The term for (II) is equal to a sum of two terms

$$-f_K \begin{bmatrix} 0 & -u^2 x & 0 \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix}; \{|x| \le D^2, y\}, \tag{3.23}$$

and

$$f_K \begin{bmatrix} 0 & -u^2 x & 0 \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix} = f_K \begin{bmatrix} 0 & -u^2 x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \{|x| \le D^2, |y| > 1\}.$$

The last term cancels the term (3.16).

The term (3.17) is the sum of two terms

$$-f_K \begin{bmatrix} 0 & -u^2 x & 0 \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix}; \{|x| > D^2, y\},$$

and

$$f_K \begin{bmatrix} 0 & -u^2 x & 0 \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix} = f_K \begin{bmatrix} 0 & -(b/a)^2 x & 0 \\ 0 & 0 & 0 \\ 0 & (b-au)xy^{-1} & 0 \end{bmatrix}; \{|x| > D^2, |y| \le |a|\}.$$

The first one cancels (3.23) and the second one cancels (3.22). This completes the proof. $\square$

To finish the proof of Theorem 2.7, we note that (3.19) cancels (3.12). The term (3.7) is the same as $\eta(-1)O(n_{0+}, f)$ and (3.9) is the same as $\eta(-\Delta)O(n_{0-}, f)$. Finally, to show the converse, it suffices to show that all $\phi \in \mathcal{C}_1(F)$ can be one of ϕ_f. This is easy to prove and we omit it.

4 Existence of refined smooth transfer

We will restrict ourselves to the case the residue characteristic $p \neq 2$ for some technical reason. We choose the additive measure on F such that $\mathrm{vol}(\mathcal{O}_F) = 1$. And we choose τ such that $E = F[\sqrt{\tau}]$ and its valuation $v(\tau) \in \{0, 1\}$.

4.1 An extended Fourier transform

Recall we have defined earlier the space $\mathcal{C}_1(F)$ and $\mathcal{C}_2(F)$.

Definition 4.1. *Let $\mathcal{C}(F)$ ($\mathcal{C}_0(F)$, resp.) be the span (intersection, resp.) of $\mathcal{C}_1(F)$ and $\mathcal{C}_2(F)$.*

We now define *an extended Fourier transform* which is an automorphism of $\mathcal{C}(F)$.

Definition 4.2. *For $\phi \in \mathcal{C}(F)$, we define*

$$\widetilde{\phi}(v) := \int_{F^\times} \phi(v+x)\eta(x)\frac{dx}{|x|}. \tag{4.1}$$

We explain the meaning of this integral. Let C be greater than $|v|$ and such that when $|x| > C$, $\phi(x)$ is the linear combination of $|x|^{-1}$, $\eta(x)|x|^{-1}$ and $\eta(x)\log|x||x|^{-1}$. Then the integral is understood as a sum $\int_{|x|\leq C} \cdots$ and $\int_{|x|>C} \cdots$ where the first term is the value at $s = 0$ of the (analytic extension of) Tate integral

$$\zeta(1)^{-1}\int_{F^\times} 1_{|x|\leq C}\phi(v+x)\eta(x)|x|^s d^\times x$$

and the second one is absolutely convergent.

We choose a non-trivial additive character ψ such that the self-dual measure dx gives $\mathcal{O}_F$ volume one. Recall the gamma factor is given by

$$\gamma(s,\chi) = \frac{Z(1-s, \widehat{f}, \chi^{-1})}{Z(s, f, \chi)}, \quad f \in \mathcal{C}_c^\infty(F)$$

where the Fourier transform is defined as $\widehat{f}(x) = \int_F f(y)\psi(-xy)dy$ and

$$Z(s, f, \chi) = \int_F f(x)\chi(x)|x|^s d^\times x.$$

Proposition 4.3. *1. The transform $\phi \mapsto \widetilde{\phi}$ defines an automorphism of $\mathcal{C}(F)$ and $C_0(F)$, respectively. And it takes $\mathcal{C}_1(F)$ ($\mathcal{C}_2(F)$, resp.) to $\mathcal{C}_2(F)$ ($\mathcal{C}_1(F)$, resp.).*

2. We have

$$\widetilde{\widetilde{\phi}}(v) = \gamma(1, \eta)^2 \phi(v).$$

And the square of the gamma factor is

$$\gamma(1, \eta)^2 = \begin{cases} \left(\frac{L(0,\eta)}{L(1,\eta)}\right)^2 = \frac{(1+q^{-1})^2}{4}, & \eta \text{ unramified}, \\ \eta(-1)q^{-1}, & \eta \text{ ramified}. \end{cases}$$

Proof. Consider the codimension four subspace W of $\mathcal{C}(F)$ consisting of function $\phi \in \mathcal{C}_c^\infty(F)$ such that $\widehat{\phi}(0) = 0$. We claim that W is stable under the transform and for $\phi \in W$, we have the desired identity for $\widetilde{\widetilde{\phi}}$. Let us denote by ϕ_v the function $\phi_v(x) := \phi(v + x)$. Then we have

$$\widetilde{\phi}(v) = \zeta(1)^{-1} Z(0, \phi_v, \eta).$$

By the local functional equation we have

$$\widetilde{\phi}(v) = \gamma(1, \eta)\zeta(1)^{-1} Z(1, \widehat{\phi_v}, \eta) = \gamma(1, \eta) \int_F \psi(vx)\widehat{\phi}(x)\eta(x)dx.$$

From $\widehat{\phi}(0) = 0$ it follows that $\widehat{\phi} \cdot \eta \in \mathcal{C}_c^\infty(F)$ and hence

$$\widetilde{\phi}(v) = \gamma(1, \eta)\widehat{\widehat{\phi} \cdot \eta}(-v).$$

We then immediately see that $\widehat{\widetilde{\phi}}(0) = 0$ and $\widetilde{\widetilde{\phi}}(v) = \gamma(1, \eta)^2 \phi(v)$ (note that $\eta^2 = 1$).

Now we consider the following functions:

$$\phi_0 = 1_{\mathcal{O}_F}, \quad \phi_1(x) = \begin{cases} \frac{\eta(x)}{|x|}, & |x| > 1, \\ 0, & |x| \le 1, \end{cases}$$

$$\phi_2(x) = \begin{cases} \frac{1}{|x|}, & |x| > 1, \\ 0, & |x| \leq 1, \end{cases} \quad \phi_3(x) = \begin{cases} \frac{\eta(x)\log|x|}{|x|}, & |x| > 1, \\ 0, & |x| \leq 1. \end{cases}$$

Note that the ϕ_i's and W together span $\mathcal{C}(F)$ (and ϕ_0, ϕ_1 together span $\mathcal{C}_0(F)$, etc.).

All of them are invariant under translation by $\mathcal{O}_F$. So when $|v| \leq 1$, we have in all four cases

$$\widetilde{\phi}(v) = \int_F \phi(x)\eta(x)\frac{dx}{|x|}.$$

To calculate $\widetilde{\phi}_i(v)$ when $|v| > 1$, we will sometimes use when $v \neq 0$

$$\widetilde{\phi}(v) = \eta(v)\int_{F^\times} \phi(vx)\frac{\eta(x-1)}{|x-1|}dx.$$

First we assume that η is unramified:

$$\widetilde{\phi}_0(v) = \begin{cases} \frac{L(0,\eta)}{\zeta(1)}, & |v| \leq 1; \\ \frac{\eta(-v)}{|v|}, & |v| > 1. \end{cases} \quad \widetilde{\phi}_1(v) = \begin{cases} q^{-1}, & |v| \leq 1; \\ -\frac{L(0,\eta)}{\zeta(1)}\frac{\eta(v)}{|v|}, & |v| > 1. \end{cases}$$

$$\widetilde{\phi}_2(v) = \begin{cases} -\frac{L(1,\eta)}{q\zeta(1)}, & |v| \leq 1; \\ -\frac{1}{\zeta(1)\log q}\frac{\eta(v)\log|v|}{|v|} + \frac{1}{2}L(1,\eta)(q^{-2} - 4q^{-1} - 1)\frac{\eta(v)}{|v|}, & |v| > 1. \end{cases}$$

$$\widetilde{\phi}_3(v) = \begin{cases} -\frac{\zeta'(1)}{\zeta(1)}, & |v| \leq 1; \\ \log q\frac{\zeta(1)L(0,\eta)^2}{L(1,\eta)^2}\frac{1}{|v|} - \log q\frac{L(0,\eta)^2}{\zeta(1)}\frac{\eta(v)}{|v|}, & |v| > 1. \end{cases}$$

Now we assume that η is ramified. Then we have

$$\widetilde{\phi}_0(v) = \begin{cases} 0, & |v| \leq 1; \\ \frac{\eta(-v)}{|v|}, & |v| > 1. \end{cases} \quad \widetilde{\phi}_1(v) = \begin{cases} q^{-1}, & |v| \leq 1; \\ 0, & |v| > 1. \end{cases}$$

$$\widetilde{\phi}_2(v) = \begin{cases} 0, & |v| \leq 1; \\ \frac{\eta(-1)}{\log q\zeta(1)}\frac{\eta(v)\log|v|}{|v|} - \frac{\eta(-v)}{|v|}, & |v| > 1. \end{cases} \quad \widetilde{\phi}_3(v) = \begin{cases} -\frac{\zeta'(1)}{\zeta(1)}, & |v| \leq 1; \\ -\frac{\zeta'(1)}{\zeta(1)}\frac{1}{|v|}, & |v| > 1. \end{cases}$$

From them we immediately prove the first part. Note that when η is ramified, the gamma factor $\gamma(1,\eta)$ is the Gauss sum $G(\eta,\psi)$ for the quadratic residue. Then we see that $\gamma(1,\eta)^2 = \eta(-1)|\gamma(1,\eta)|^2 = \eta(-1)q^{-1}$. Then we can verify the second part for ϕ_i's. $\square$

Remark 1. It seems that the transform actually extends to an automorphism of the space of all locally constant functions f on F such that when $|x| \to \infty$, $f(x)$ is a linear combination of functions of the forms $\eta(x)^m(\log|x|)^n/|x|^k$ for $m = 0, 1, n \geq 0, k \geq 1$. But we don't need this in this paper.

4.2 Refined smooth transfer.

Let $f \in \mathcal{C}_c^\infty(\mathcal{V})$ and a pair $\{f_i \in \mathcal{C}_c^\infty(\mathcal{W}_i)_{i=0,1}\}$. Recall that $\phi_f \in \mathcal{C}_1(F)$ and $\phi_{f_0} \in \mathcal{C}_2(F)$. Let

$$\kappa_{E/F} = e_{E/F} L(1,\eta)^{-1} = \begin{cases} (1+q^{-1}), & E/F \text{ unramified };\\ 2, & E/F \text{ ramified }. \end{cases}$$

where $e_{E/F}$ is the ramification index of E/F.

Proposition 4.4. *If the functions $f, \{f_i\}_{i=0,1}$ match each other, then we have*

(i) $\phi_{f_0} = 2\eta(-1)\kappa_{E/F}^{-1}\widetilde{\phi}_f$.

(ii)
$$\begin{cases} -O(0, f_0) & = \eta(-1)O(n_{0+}, f) + O(n_{0-}, f);\\ O(0, f_1) & = \eta(-1)O(n_{0+}, f) - O(n_{0-}, f). \end{cases}$$

Proof. The second assertion is clear in view of Theorem 2.7 and 2.8. We now prove the first one.

Let
$$\lambda' = \lambda/a^2, b' = b/a^2, \Delta' = \Delta/a^4 = \lambda' - b'^2.$$

So we have $\eta(-\Delta') = \eta(-\Delta) = 1$. Substitute u by a^2u in (3.21):

$$\eta(-1)|a|^{-1}\int_F \phi_f(u - \Delta'/u + 2b')\eta(u)du/|u|.$$

Fix b'. When we vary (λ, a, b) in a neighborhood of 0, b', λ' can take any value in F. We let λ' be close to b'^2 so that Δ' is close to zero.

We first simplify the germ on the linear side. By the local constancy of ϕ_f at b', when Δ' is small enough, either u or Δ'/u is small so that $\phi_f(u - \Delta'/u + 2b')$ can be replaced by $\phi_f(-\Delta'/u + 2b')$ or $\phi_f(u + 2b')$. We may choose an appropriate $\epsilon > 0$ (of size $|\Delta'|^{1/2}$) so that we may write the integral above as a sum of two terms:

$$\int_{|u|\geq\epsilon} \phi_f(u+2b')\eta(u)du/|u|$$

and (note: $\eta(-\Delta')=1$)

$$\int_{|u|<\epsilon} \phi_f(-\Delta'/u+2b')\eta(u)du/|u| = \int_{|u|>|\Delta'|/\epsilon} \phi_f(u+2b')\eta(u)du/|u|.$$

The sum differs from $2\int_F \phi_f(u+2b')\eta(u)du/|u|$ by an error term

$$\phi(b')\left(\int_{|x|<\epsilon} \eta(u)du/|u| + \int_{|x|\leq|\Delta'|/\epsilon} \eta(u)du/|u|\right).$$

It is easy to verify that the error gives zero (indeed, this is obviously true if η is ramified; when η is unramified, note that $\eta(-\Delta')=1$). In summary, in this case, the germ for $\mathcal{V}$ is given by

$$2\eta(-1)|a|^{-1}\int_F \phi_f(u+2b')\eta(u)du/|u|.$$

Now we may also simplify the germ for $\mathcal{W}_0$ under the same choice of λ'. Note that the germ can be written as

$$2|a|^{-1}\int_F \phi_{f_0}(u+2b')\Gamma'_u(\lambda',b')du$$

where $\Gamma'_u(\lambda',b') = |\tau(u^2+\Delta)|^{-1/2}$ (0, resp.) if $\tau(u^2+\Delta)$ is a square (otherwise, resp.). Note that when $|u|^2 > |\Delta'|$, it is obvious that $\tau(u^2+\Delta')$ is not a square. We may choose Δ' to be a non-square. Then we have by the local constancy of ϕ_{f_0}

$$2|a|^{-1}\phi_{f_0}(2b')\int_{|u|^2\leq|\Delta'|} \Gamma'_u(\lambda',b')du.$$

By Lemma 4.5 below, this is equal to

$$2|a|^{-1}\phi_{f_0}(2b')\frac{1}{2}\kappa_{E/F}.$$

Comparing with the germ for $\mathcal{V}$, we have shown that

$$\phi_{f_0}(2b') = 2\eta(-1)\kappa_{E/F}^{-1}\widetilde{\phi}_f(2b').$$

□

Lemma 4.5. *Let $\theta : F \to \mathbb{C}$ be the function mapping x to $|x|^{1/2}$ if $x \in (F)^2$, and zero otherwise. Let $\Delta \in F^\times$ be such that $\eta(-\Delta) = 1$. Then the integral*

$$\int_F \frac{du}{\theta(\tau(u^2+\Delta))} = \frac{1}{2}\kappa_{E/F}.$$

Proof. Obviously the integral depends only on the coset $\Delta(F^\times)^2$. We first assume that E is unramified (so that τ is a non-square unit). It follows from $\eta(-\Delta) = 1$ that $v(\Delta)$ is even. It suffices to consider $v(\Delta) = 0$. If Δ is a square, then the integrand vanishes unless u is a unit. Let

$$N_\Delta := \#\{\bar{u} \in k | \bar{u}^2 + \bar{\Delta} \in (k)^2\} = \begin{cases} \frac{q-1}{2} + 1, & (-1) \in (k^\times)^2; \\ \frac{q-1}{2}, & (-1) \notin (k^\times)^2. \end{cases}$$

When $(-1) \in (k^\times)^2$, there is an extra contribution from $u \in \pm\sqrt{-\Delta} + \varpi\mathcal{O}_F$ given by

$$2\int_{\varpi\mathcal{O}_F} \frac{du}{\theta(\tau u)} = 2\frac{1}{2}q^{-1} = q^{-1}.$$

Together we show that the integral is equal to

$$q^{-1}(q - N_\Delta) + \begin{cases} q^{-1}, & (-1) \in (k^\times)^2; \\ 0, & (-1) \notin (k^\times)^2. \end{cases}$$

which is always given by $\frac{1}{2}(1+q^{-1})$. If Δ is a non-square, we have

$$N_\Delta := \#\{\bar{u} \in k | \bar{u}^2 + \bar{\Delta} \in (k^\times)^2\} = \begin{cases} \frac{q-1}{2}, & (-1) \in (k^\times)^2; \\ \frac{q-1}{2} + 1, & (-1) \notin (k^\times)^2. \end{cases}$$

Similarly, in this case when $(-1) \notin (k^\times)^2$, the contribution from $u \in \pm\sqrt{-\Delta} + \varpi\mathcal{O}_F$ is given by q^{-1}. Together we still get $\frac{1}{2}(1+q^{-1})$.

We now assume that E is ramified. It follows from $\eta(-\Delta) = 1$ that $v(\Delta)$ is even and $-\Delta \in (F^\times)^2$or $v(\Delta)$ is odd and $-\Delta \in -\tau(F^\times)^2$. In the former case, it suffices to consider the case $\Delta = -1$. Then the integrand vanishes unless $u \in \pm 1 + \varpi\mathcal{O}_F$. It is then easy to see that the integral is equal to

$$2\int_{\varpi\mathcal{O}_F} \frac{du}{\theta(\tau u)} = 2 \cdot \frac{1}{2} = 1.$$

In the latter case it suffices to assume that $\Delta = \tau$. Then the integrand vanishes unless $u \in \varpi\mathcal{O}_F$. It follows that the integral is equal to (note: by our normalization, $v(\tau) = 1$)

$$\int_{\varpi\mathcal{O}_F} \frac{du}{|\tau|} = 1.$$

$\square$

Lemma 4.6. *We have*

$$\int_F \eta_1(u^2+\xi)du = \begin{cases} \kappa_{E/F}|\xi/\tau|^{-1/2}, & \xi \in -\tau(F^\times)^2; \\ 0, & \textit{otherwise.} \end{cases}$$

Here $\eta_1(x) := \eta(x)|x|^{-1}$.

Proof. Substitution $u \to \xi/u$ yields a multiple $\eta(\xi)$. Hence the integral vanishes if $\eta(\xi) = -1$. Now we assume that

$$\eta(\xi) = 1.$$

Note that if the result holds for ξ, then it also holds for $\xi(F^\times)^2$. We may thus assume that $v(\xi)$ is either 0 or -1. If $v(\xi) = -1$ which is only possible when η is ramified (and then $v(\tau) = 1$), the integral is equal to

$$(1+\eta(\xi))\int_{|u|\le 1} |\xi|^{-1}du = 2q^{-1}.$$

Hence we now assume that $\eta(\xi) = 1$ and $|\xi| = 1$. By the symmetry $u \to \xi/u$, the integral is the sum of

$$2\int_{|u|^2<|\xi|} |\xi|^{-1}du = 2q^{-1}, \tag{4.2}$$

and

$$\int_{|u|^2=|\xi|} \eta_1(u^2+\xi)du. \tag{4.3}$$

First we consider the case that η is unramified. There are two cases according to $-\tau\xi\,(mod\,(F^\times)^2)$. If $\xi \in -\tau(F^\times)^2$, then $u^2+\xi \in \mathrm{N}E^\times$, $\eta(u^2+\xi) = 1$ and $|u^2+\xi| = max\{|u|^2, |\xi|\} = 1$. In this case the integral is equal to

$$2q^{-1} + (1-q^{-1}) = 1 + q^{-1} = L(1,\eta)^{-1}.$$

Then $-\tau\xi$ is not a square and hence $-\xi = t^2$ is a square. From the expression:

$$\int_{|u|^2=1} \eta_1(u^2+\xi)du = \int_{|u|^2=1} \eta_1(u+t)\eta_1(u-t)du,$$

we break this as a some of three terms: $u \equiv t, u \equiv -t \,(mod\, \varpi\mathcal{O}_F)$ and the other $u \,(mod\, \varpi\mathcal{O}_F)$. The first two yields:

$$(4.4) \qquad 2\int_{|u|^2=1, u\equiv t} \eta_1(u-t)du = 2\int_{\varpi\mathcal{O}_F} \eta(u)\frac{du}{|u|} = -(1-q^{-1}).$$

The third one yields

$$(q-1-2)q^{-1} = 1-3q^{-1}.$$

Together with (4.2), we have showed that the integral is equal to zero.

Now we assume that η is ramified and we need to show the integral vanishes. Since $\eta(\xi) = 1$ and $|\xi| = 1$, ξ is then a square. Let k denote the residue field $\mathcal{O}_F/(\varpi)$. We first assume that $-\xi$ is not a square. Then it is not hard to show that

$$N_\xi := \#\{\bar{u} \in k^\times | \bar{u}^2 + \bar{\xi} \in (k^\times)^2\} = \frac{q-3}{2}.$$

Then we see that (4.3) is equal to

$$N_\xi q^{-1} - ((q-1) - N_\xi q^{-1} = -2q^{-1}.$$

Finally we assume that $-\xi = t^2$ is a square. And the term (4.3) is a sum of three terms: $u \equiv t, u \equiv -t \,(mod\, \varpi\mathcal{O}_F)$ and the other $u \,(mod\, \varpi\mathcal{O}_F)$. The first two contributes zero as η is ramified (cf. (4.4)). To calculate the third one, we note that

$$N_\xi := \#\{\bar{u} \in k^\times | \bar{u}^2 + \bar{\xi} \in (k^\times)^2\} = \frac{q-5}{2}.$$

Hence the third one contributes

$$N_\xi q^{-1} - (q-3-N_\xi)q^{-1} = -2q^{-1}.$$

This completes the proof. □

Now we recall from [9] that we say that the functions $f, \{f_i\}_{i=0,1}$ are *local transfer* of each other around zero, if there is a neighborhood of zero in $\mathbb{A}^3$ such that $O(\mathfrak{s}(\lambda, a, b), f)$ is equal to $O(Y(\lambda, a, b), f_i)$ whenever $\mathfrak{s}(\lambda, a, b)$ is in this neighborhood.

Proposition 4.7. *If $f, \{f_i\}_{i=0,1}$ satisfy the conditions in Prop. 4.4, then $f, \{f_i\}_{i=0,1}$ are local transfer of each other around zero. In particular, by Prop. 4.3, local transfer around zero always exists.*

Proof. To show that $f, \{f_i\}_{i=0,1}$ are local transfer, we can use the germ expansion. It is easy to see that the discrete terms are equal. It suffices to treat the continuous part. Denote by $\phi = \phi_{f_0}$ and by the inverse extended Fourier transform, ϕ_f is a constant times $\widetilde{\phi}$. We may ignore the constant since the proof of Prop 4.4 already shows that for some (λ, a, b), the orbital integrals match.

To compute the continuous part of the germ expansion of the orbital integral of f, we substitute u by a^2u in (3.21), use the definition of $\widetilde{\phi}$, and up to some constants independent of (λ, a, b):

$$\begin{aligned}
&\int_F \left(\int_F \phi(u - \Delta/u + 2b + x)\eta_1(x)dx \right) \eta_1(u)du \\
=&\int_F \left(\int_F \phi(u + 2b + x)\eta_1(x + \Delta/u)dx \right) \eta_1(u)du \\
=&\int_F \left(\int_F \phi(u + 2b + x)\eta_1(xu + \Delta)dx \right) du \\
=&\int_F \left(\int_F \phi(x)\eta_1((x - u - 2b)u + \Delta)dx \right) du.
\end{aligned}$$

Now we interchange the order of integration to obtain (note that the integrals are regularized):

$$\begin{aligned}
&\int_F \phi(x) \left(\int_F \eta_1((x - u - 2b)u + \Delta)du \right) dx \\
=&\eta(-1) \int_F \phi(x) \left(\int_F \eta_1((u + b - x/2)^2 - ((b - x/2)^2 + \Delta))du \right) dx \\
=&\eta(-1) \int_F \phi(x) \left(\int_F \eta_1(u^2 - ((b - x/2)^2 + \Delta))du \right) dx.
\end{aligned}$$

By Lemma 4.6, the inner integral is equal to

$$\begin{cases} \kappa_{E/F}|((b-x/2)^2+\Delta)/\tau| & ((b-x/2)^2+\Delta) \in \tau(F^\times)^2 \\ 0, & \text{otherwise.} \end{cases}$$

This shows that up to a constant the continuous part of the germ expansion of f does match that of f_0. This completes the proof. $\square$

Finally we conclude with a new proof of existence of transfer for the case $n = 3$.

Theorem 4.8. *Given $f \in \mathcal{C}_c^\infty(\mathcal{V})$, there exists $\{f_i\}_{i=0,1}$ such that they are transfer of each other. And given $\{f_i\}_{i=0,1}$, there exists $f \in \mathcal{C}_c^\infty(\mathcal{V})$ such that they are transfer of each other.*

Proof. By [9], it suffices to show the existence of local transfer around zero for $n \leq 3$. This follows from Prop. 4.7 and that the local transfer around zero for $n = 2$ is proved in [3]. $\square$

Remark 2. By Cayley transform as in [9], the theorem implies the existence of smooth transfer on groups. It should also be evident how to translate Prop. 4.4 to its analogue on groups.

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