SHTUKAS AND THE TAYLOR EXPANSION OF L-FUNCTIONS OVER A FUNCTION FIELD

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(JOINT WITH ZHIWEI YUN)

In this talk, we explained a joint work with Zhiwei Yun [3].

Let $k = \mathbb{F}_q$ be a finite field of characteristic $p > 0$. Let X be a geometrically connected smooth proper curve over k. Let $\nu : X' \to X$ is a finite étale cover of degree 2 such that X' is also geometrically connected. Let $F = k(X)$ and $F' = k(X')$ be their function fields.

Let $G = \mathrm{PGL}_2$ and $T = (\mathrm{Res}_{F'/F}\,\mathbb{G}_m)/\mathbb{G}_m$ the non-split torus associated to the double cover X' of X. Let Bun_2 be the stack of rank two vector bundles on X. The Picard stack Pic_X acts on Bun_2 by tensoring a line bundle. Then $\mathrm{Bun}_G = \mathrm{Bun}_2/\mathrm{Pic}_X$ is the moduli stack of G-torsors over X.

0.1. **The Heegner–Drinfeld cycle.** Let r be an *even* integer. Let $\mu \in \{\pm\}^r$ be an r-tuple of signs such that exactly half of them are equal to $+$. The Hecke stack Hk_2^μ is the stack whose S-points is the groupoid of the data $(\mathcal{E}_0, ..., \mathcal{E}_r, x_1, ..., x_r, f_1, ..., f_r)$ where $\mathcal{E}_i$'s are vector bundles of rank two over $X \times S$, x_i's are S-points of X, and each f_i is a minimal upper (i.e., increasing) modification if $\mu_i = +$, and minimal lower (i.e., decreasing) modification if $\mu_i = -$, and the modification takes place along the graph of x_i

$$\mathcal{E}_0 \overset{f_1}{\dashrightarrow} \mathcal{E}_1 \overset{f_2}{\dashrightarrow} \cdots \overset{f_r}{\dashrightarrow} \mathcal{E}_r \ .$$

The Picard stack Pic_X acts on Hk_2^μ by simultaneously tensoring a line bundle. Define $\mathrm{Hk}_G^\mu = \mathrm{Hk}_2^\mu/\mathrm{Pic}_X$. Assigning $\mathcal{E}_i$ to the data above descends to a morphism $p_i : \mathrm{Hk}_G^\mu \to \mathrm{Bun}_G$.

The moduli stack Sht_G^μ of Drinfeld G-Shtukas with r-modifications of type μ for the group G is defined by the following cartesian diagram

$$\begin{tikzcd} \mathrm{Sht}_G^\mu \arrow[r] \arrow[d] & \mathrm{Hk}_G^\mu \arrow[d, "(p_0,p_r)"] \\ \mathrm{Bun}_G \arrow[r, "(\mathrm{id},\mathrm{Fr})"] & \mathrm{Bun}_G \times \mathrm{Bun}_G \end{tikzcd} \tag{0.1}$$

The stack Sht_G^μ is a Deligne-Mumford stack over X^r and the natural morphism

$$\pi_G^\mu : \mathrm{Sht}_G^\mu \longrightarrow X^r$$

is smooth of relative dimension r, and locally of finite type. We remark that Sht_G^μ as a stack over X^r is canonically independent of the choice of μ. The stack Sht_T^μ of T-Shtukas is defined analogously, with the $\mathcal{E}_i$ replaced by line bundles on X', and the points x_i on X'. Then we have a map

$$\pi_T^\mu : \mathrm{Sht}_T^\mu \longrightarrow X'^r$$

which is a torsor under the finite Picard stack $\mathrm{Pic}_{X'}(k)/\mathrm{Pic}_X(k)$. In particular, Sht_T^μ is a proper smooth Deligne-Mumford stack over $\mathrm{Spec}\, k$.

There is a natural finite morphism of stacks over X^r

$$\mathrm{Sht}_T^\mu \longrightarrow \mathrm{Sht}_G^\mu \ .$$

It induces a finite morphism

$$\theta^\mu : \mathrm{Sht}_T^\mu \longrightarrow \mathrm{Sht}_G'^\mu := \mathrm{Sht}_G^\mu \times_{X^r} X'^r \ .$$

This defines a class in the Chow group of proper cycles of dimension r with $\mathbb{Q}$-coefficient

$$\theta_*^\mu[\mathrm{Sht}_T^\mu] \in \mathrm{Ch}_{c,r}(\mathrm{Sht}_G'^\mu)_{\mathbb{Q}}.$$

In analogy to the classical Heegner cycles [1] in the number field case, we will call $\theta^\mu_*[\mathrm{Sht}^\mu_T]$ the *Heegner–Drinfeld cycle* in our setting.

0.2. **The spectral decomposition of the cycle space.** We denote the set of closed points (places) of X by $|X|$. For $x \in |X|$, let $\mathcal{O}_x$ be the completed local ring of X at x and let F_x be its fraction field. Let $\mathbb{A} = \prod'_{x\in|X|} F_x$ be the ring of adeles, and $\mathbb{O} = \prod_{x\in|X|}\mathcal{O}_x$ the ring of integers inside $\mathbb{A}$. Let $K = \prod_{x\in|X|} K_x$ where $K_x = G(\mathcal{O}_x)$. The (spherical) Hecke algebra $\mathscr{H}$ is the $\mathbb{Q}$-algebra of bi-K-invaraint functions $C_c^\infty(G(\mathbb{A})/\!\!/K, \mathbb{Q})$ with the product given by convolution.

Let $\mathcal{A} = C_c^\infty(G(F)\backslash G(\mathbb{A})/K, \mathbb{Q})$ be the space of everywhere unramified $\mathbb{Q}$-valued automorphic functions for G. Then $\mathcal{A}$ is an $\mathscr{H}$-module. By an everywhere unramified cuspidal automorphic representation π of $G(\mathbb{A})$ we mean an $\mathscr{H}$-submodule $\mathcal{A}_\pi \subset \mathcal{A}$ that is irreducible over $\mathbb{Q}$. For every such π, $\mathrm{End}_{\mathscr{H}}(\mathcal{A}_\pi)$ is a number field E_π, which we call the *coefficient field* of π. Then by the commutativity of $\mathscr{H}$, $\mathcal{A}_\pi$ is a one-dimensional E_π-vector space.

The Hecke algebra $\mathscr{H}$ acts on the Chow group $\mathrm{Ch}_{c,r}(\mathrm{Sht}'^\mu_G)_\mathbb{Q}$ via Hecke correspondences. Let $\widetilde{W} \subset \mathrm{Ch}_{c,r}(\mathrm{Sht}'^\mu_G)_\mathbb{Q}$ be the sub $\mathscr{H}$-module generated by the Heegner–Drinfeld cycle $\theta^\mu_*[\mathrm{Sht}^\mu_T]$. There is a bilinear and symmetric intersection pairing

$$\langle\cdot,\cdot\rangle_{\mathrm{Sht}'^\mu_G} : \widetilde{W} \times \widetilde{W} \longrightarrow \mathbb{Q}. \tag{0.2}$$

Let $\widetilde{W}_0$ be the kernel of the pairing. The quotient $W := \widetilde{W}/\widetilde{W}_0$ is then equipped with a *non-degenerate* pairing induced from $\langle\cdot,\cdot\rangle_{\mathrm{Sht}'^\mu_G}$

$$(\cdot,\cdot) : W \times W \longrightarrow \mathbb{Q}\ .$$

The Hecke algebra $\mathscr{H}$ acts on W.

Let π be an everywhere unramified cuspidal automorphic representation of G with coefficient field E_π, and let $\lambda_\pi : \mathscr{H} \to E_\pi$ be the associated character, whose kernel $\mathfrak{m}_\pi$ is a maximal ideal of $\mathscr{H}$. Let

$$W_\pi = \mathrm{Ann}(\mathfrak{m}_\pi) \subset W \tag{0.3}$$

be the λ_π-eigenspace of W. This is an E_π-vector space. Let $\mathcal{I}_{\mathrm{Eis}} \subset \mathscr{H}$ be the Eisenstein ideal (cf. [3]). Informally speaking, this is the annihilator of the Eisenstein spectrum in the space of automorphic functions $\mathcal{A}$. Define

$$W_{\mathrm{Eis}} = \mathrm{Ann}(\mathcal{I}_{\mathrm{Eis}}).$$

Theorem 0.1. *We have an orthogonal decomposition of $\mathscr{H}$-modules*

$$W = W_{\mathrm{Eis}} \oplus \left(\bigoplus_\pi W_\pi\right), \tag{0.4}$$

where π runs over the finite set of everywhere unramified cuspidal automorphic representation of G, and W_π is an E_π-vector space of dimension at most one.

The $\mathbb{Q}$-bilinear pairing $(\cdot,\cdot)$ on W_π can be lifted to an E_π-bilinear symmetric pairing

$$(\cdot,\cdot)_\pi : W_\pi \times W_\pi \longrightarrow E_\pi \tag{0.5}$$

where for $w, w' \in W_\pi$, $(w,w')_\pi$ is the unique element in E_π such that $\mathrm{Tr}_{E_\pi/\mathbb{Q}}(e\cdot(w,w')_\pi) = (ew, w')$ for all $e \in E_\pi$.

0.3. **Taylor expansion of L-functions.** Let π be an everywhere unramified cuspidal automorphic representation of G with coefficient field E_π. The standard L-function $L(\pi,s)$ is a polynomial of degree $4(g-1)$ in $q^{-s-1/2}$ with coefficients in E_π, where g is the genus of X. Let $\pi_{F'}$ be the base change to F', and let $L(\pi_{F'},s)$ be its standard L-function. This L-function is a product of two L-functions associated to cuspidal automorphic representations of G over F:

$$L(\pi_{F'},s) = L(\pi,s)L(\pi\otimes\eta_{F'/F},s),$$

where

$$\eta_{F'/F} : F^\times\backslash\mathbb{A}^\times/\mathbb{O}^\times \longrightarrow \{\pm1\}$$

is the character corresponding to the étale double cover X' via class field theory. The function $L(\pi_{F'}, s)$ satisfies a functional equation

$$L(\pi_{F'}, s) = \epsilon(\pi_{F'}, s) L(\pi_{F'}, 1-s),$$

where $\epsilon(\pi_{F'}, s) = q^{-8(g-1)(s-1/2)}$. Let $L(\pi, \mathrm{Ad}, s)$ be the adjoint L-function of π. Denote

$$\mathscr{L}(\pi_{F'}, s) = \epsilon(\pi_{F'}, s)^{-1/2} \frac{L(\pi_{F'}, s)}{L(\pi, \mathrm{Ad}, 1)}, \tag{0.6}$$

where the the square root is understood as $\epsilon(\pi_{F'}, s)^{-1/2} := q^{4(g-1)(s-1/2)}$. In particular, we have a functional equation:

$$\mathscr{L}(\pi_{F'}, s) = \mathscr{L}(\pi_{F'}, 1-s).$$

Consider the Taylor expansion at the central point $s = 1/2$:

$$\mathscr{L}(\pi_{F'}, s) = \sum_{r \geq 0} \mathscr{L}^{(r)}(\pi_{F'}, 1/2) \frac{(s-1/2)^r}{r!},$$

i.e.,

$$\mathscr{L}^{(r)}(\pi_{F'}, 1/2) = \frac{d^r}{ds^r}\Big|_{s=0} \left(\epsilon(\pi_{F'}, s)^{-1/2} \frac{L(\pi_{F'}, s)}{L(\pi, \mathrm{Ad}, 1)} \right).$$

If r is odd, by the functional equation we have

$$\mathscr{L}^{(r)}(\pi_{F'}, 1/2) = 0.$$

Since $\mathscr{L}(\pi_{F'}, s) \in E_\pi[q^{-s-1/2}, q^{s-1/2}]$, we see that

$$\mathscr{L}^{(r)}(\pi_{F'}, 1/2) \in E_\pi \cdot (\log q)^r. \tag{0.7}$$

Then our main result in [3] relates the r-th Taylor coefficient to the self-intersection number of the π-component of the Heegner–Drinfeld cycle $\theta^\mu_*[\mathrm{Sht}^\mu_T]$ on the stack Sht'^μ_G.

Theorem 0.2. *Let π be an everywhere unramified cuspidal automorphic representation of G with coefficient field E_π. Let $[\mathrm{Sht}^\mu_T]_\pi \in W_\pi$ be the projection of the image of $\theta^\mu_*[\mathrm{Sht}^\mu_T] \in \widetilde{W}$ in W to the direct summand W_π under the decomposition* (0.4)*. Then we have an equality in E_π*

$$\frac{1}{2(\log q)^r} |\omega_X| \mathscr{L}^{(r)}(\pi_{F'}, 1/2) = \Big([\mathrm{Sht}^\mu_T]_\pi, \quad [\mathrm{Sht}^\mu_T]_\pi \Big)_\pi,$$

where ω_X is the canonical divisor, and $|\omega_X| = q^{-2g+2}$.

Remark 0.3. When $r = 0$, this formula is equivalent to the special case of Waldspurger formula [2] for unramified π, relating the automorphic period integral to the central value of the L-function of $\pi_{F'}$

$$\Big| \int_{T(F)\backslash T(\mathbb{A})} \varphi(t) dt \Big|^2 = \frac{1}{2} |\omega_X| \mathscr{L}(\pi_{F'}, 1/2),$$

where $\varphi \in \pi^K$ is normalized such that the Petersson inner product $(\varphi, \varphi) = 1$, and the measure on $G(\mathbb{A})$ is such that $\mathrm{vol}(K) = 1$, and the measure on $T(\mathbb{A})$ is such that the maximal compact open subgroup has volume one.

Remark 0.4. In [3] we only consider the everywhere unramified situation where the L-function has nonzero Taylor coefficients in even degrees only. But the same construction with slight modifications should work in the ramified case as well, where the L-function may have nonzero Taylor coefficients in odd degrees. The case $r = 1$ would then give an analog of the Gross–Zagier formula [1] in the function field case.

References

[1] B. Gross, D. Zagier, *Heegner points and derivatives of L-series.* Invent. Math. 84 (1986), no. 2, 225–320.
[2] J. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie.* Compositio Math. 54 (1985), no. 2, 173–242.
[3] Zhiwei Yun, Wei Zhang, *Shtukas and the Taylor expansion of L-functions*, in preparation, 2015.