The Gross–Kohnen–Zagier Theorem over Totally Real Fields

Xinyi Yuan, Shou-Wu Zhang, Wei Zhang

Department of Mathematics
Columbia University
New York, NY 10027

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1 Introduction

On a modular curve $X_0(N)$, Gross–Kohnen–Zagier prove [2] that certain generating series of Heegner points are modular forms of weight $3/2$ with values in Jacobian as a consequence of their formula for Néron-Tate height pairing of Heegner points. Such a result is an analogue of an earlier result of Hirzebruch–Zagier ([5]) on intersection numbers of Shimura curves on Hilbert modular surfaces, and has been extended to orthogonal Shimura varieties in various settings:

- cohomological cycles over totally real fields by Kudla–Millson ([8]) by using their theory of cohomological theta lifting;

- divisor classes in Picard group over $\mathbb{Q}$ by Borchesds ([1]) as an application of his construction of singular theta lifting;

- high-codimensional Chow cycles over $\mathbb{Q}$ by one of us, Wei Zhang ([14]) as a consequence of his criterion of modularity by induction on the codimension.

The main result of this paper is a further extension of the modularity to Chow cycles on orthogonal Shimura varieties over totally real fields. For planning applications of our result, we would like to mention our ongoing work on the Gross–Zagier formula [4] and the Gross–Kudla conjecture on triple product L-series [3] over totally real fields. Our result is also necessary for extending work of Kudla, Rapoport, and Yang [9] to totally real fields.

Different from the work of Gross–Kohnen–Zagier and Borcherds, our main ingredients in the proof are some product formulae, and the modularity of Kudla–Millson. In codimension one case, our result is new only in the case of Shimura curves and their products, as Kudla–Milson's result already implies the modularity in Chow groups when the first Betti numbers of ambient Shimura varieties vanish. Both the modularity and product formulae for certain special cycles are proposed by Steve Kudla in [6, 7]. In the following, we will give details of his definitions and our results.

Let F be a totally real field of degree $d = [F : \mathbb{Q}]$ with a fixed real embedding ι. Let V be a vector space over F with an inner product $\langle\cdot,\cdot\rangle$ which is non-degenerate with signature $(n, 2)$ on $V_{\iota,\mathbb{R}}$ and signature $(n+2, 0)$ at all other real places. Let G denote the reductive group $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSpin}(V)$.

Let $D \subset \mathbb{P}(V_{\mathbb{C}}^{\vee})$ be the Hermitian symmetric domain for $G(\mathbb{R})$ as follows:

$$D = \{v \in V_{\iota,\mathbb{C}} : \quad \langle v, v\rangle = 0,\ \langle v, \bar{v}\rangle < 0\}/\mathbb{C}^{\times}$$

where the quadratic form extends by $\mathbb{C}$-linearity, and $v \longrightarrow \bar{v}$ is the involution on $V_{\mathbb{C}} = V \otimes_F \mathbb{C}$ induced by complex conjugation on $\mathbb{C}$. Then, for any open compact subgroup K of $G(\widehat{\mathbb{Q}})$, we have a Shimura variety with $\mathbb{C}$-points

$$M_K(\mathbb{C}) = G(\mathbb{Q})\backslash D \times G(\widehat{\mathbb{Q}})/K.$$

It is known that $M_K(\mathbb{C})$ has a canonical model M_K over F as a quasi-projective variety for K sufficiently small. In our case, M_K is actually complete if $F \neq \mathbb{Q}$. Let $\mathcal{L}_D$ be the bundle of lines corresponding to points on

D. Then $\mathcal{L}_D$ descends to an ample line bundle $\mathcal{L}_K \in \mathrm{Pic}(M_K) \otimes \mathbb{Q}$ with $\mathbb{Q}$ coefficients.

For an F-subspace W of V with positive definite inner product at all real place of F, and an element $g \in G(\widehat{\mathbb{Q}})$, we define a *Kudla's cycle* $Z(W, g)_K$ represented by points $(z, hg) \in D \times G(\widehat{\mathbb{Q}})$, where $z \in D_W$ is in the subset of lines in D perpendicular to W, and $h \in G_W(\widehat{\mathbb{Q}})$ is in the subgroup of elements in $G(\widehat{\mathbb{Q}})$ fixing every points in $\widehat{W} = W \otimes \widehat{\mathbb{Q}}$. The cycle $Z(W, g)$ depends only the K-class of the F-subspace $g^{-1}W$ of $\widehat{V} := V \otimes_F \widehat{F}$.

For a positive number r, an element $x = (x_1, \cdots, x_r) \in K\backslash V(F)^r$, and an element $g \in G(\widehat{\mathbb{Q}})$, we define a *Kudla's Chow cycle* $Z(x, g)_K$ in M_K as follows: let W be the subspace of $V(F)$ generated by components x_i of x, then

$$Z(x, g)_K := \begin{cases} Z(W, g)_K c_1(\mathcal{L}_K^\vee)^{r-\dim W} & \text{if } W \text{ is positive definite} \\ 0 & \text{otherwise} \end{cases}$$

For any Bruhat-Schwartz function $\phi \in \mathcal{S}(V(\widehat{F})^r)^K$, we define *Kudla's generating function* with coefficients in the Chow group $\mathrm{Ch}(M_K) \otimes \mathbb{Q}$ as follows:

$$Z_\phi(\tau) = \sum_{x \in G(\mathbb{Q})\backslash V^r} \sum_{g \in G_x(\widehat{\mathbb{Q}})\backslash G(\widehat{\mathbb{Q}})/K} \phi(g^{-1}x) Z(x, g)_K q^{T(x)}, \qquad \tau = (\tau_k) \in (\mathcal{H}_r)^d$$

where $\mathcal{H}_r$ is the Siegel upper-half plane of genus r, and $T(x) = \frac{1}{2}\langle x_i, x_j \rangle$ is the intersection matrix

$$q^{T(x)} = \exp(2\pi i \sum_{k=1}^{d} \tau_k \iota_k(T(x))),$$

where $\iota_1 := \iota, \cdots \iota_d$ are all real embeddings of F. Notice that $Z_\phi(\tau)$ does not depend on the choice of K when we consider the sum in the direct limit of $\mathrm{Ch}(M_K)$ via pull-back maps of cycles.

Theorem 1.1 (Product formula). *Let $\phi_1 \in \mathcal{S}(V(\widehat{F})^{r_1})$, and $\phi_2 \in \mathcal{S}(V(\widehat{F})^{r_2})$ be two Bruhat-Schwarts functions. Then in Chow group:*

$$Z_{\phi_1}(\tau_1) \cdot Z_{\phi_2}(\tau_2) = Z_{\phi_1 \otimes \phi_2}\left(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}\right).$$

For a linear functional ℓ on $\mathrm{Ch}^r(M_K)\otimes\mathbb{Q}$, we define a series with complex coefficients:

$$\ell(Z_\phi)(\tau) = \sum_{x\in G(\mathbb{Q})\backslash V^r} \sum_{g\in G_x(\widehat{\mathbb{Q}})\backslash G(\widehat{\mathbb{Q}})/K} \phi(g^{-1}x)\ell(Z(x,g)_K)q^{T(x)}.$$

Theorem 1.2 (Modularity)**.** *Let ℓ be a linear functional on $\mathrm{Ch}^r(M_K)\otimes\mathbb{Q}$ such that the generating function $\ell(Z_\phi(\tau))$ is convergent. Then $\ell(Z_\phi(\tau))$ is a Siegel modular form of weight $\frac{n}{2}+1$, i.e., it satisfies the following equation*

$$\ell(Z_\phi)(\gamma\tau) = \ell(Z_{\omega(\gamma^{-1})\phi})(\tau)\det j(\gamma,\tau)^{n/2+1}, \qquad \gamma\in \mathrm{Sp}_r(F).$$

Here $\gamma\tau$ is the usual action of $\gamma\in\mathrm{Sp}_r(F_\infty)$ on the Siegel domain via the natural projection $\mathrm{Mp}_r(\mathbb{R})\longrightarrow\mathrm{Sp}_r(\mathbb{R})$, and $\omega(\gamma)\phi$ is the Weil representation on $\mathcal{S}(V(\widehat{F})^r)$.

Here we use the following identifications:

$$\mathrm{Mp}_r(\mathbb{R}) = \left\{(g,\det(j(g,\tau))^{1/2}): \quad g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{Sp}_r(\mathbb{R}),\ j(g,\tau)=cz+d\right\},$$

$$\mathrm{Mp}_r(\mathbb{A}) = (\mathrm{Mp}_r(F_\infty)\times\mathrm{Mp}_r(\widehat{F}))/\pm(1,-1).$$

Then the preimage $\mathrm{Mp}_r(F)$ of $\mathrm{Sp}_r(F)$ in $\mathrm{Mp}(\mathbb{A})$ has a unique splitting:

$$\mathrm{Mp}_r(F) = \{\pm1\}\times\mathrm{Sp}_r(F).$$

And the Weil representation is attached to the additive character of $\mathbb{A}_F$ defined by $\psi_F=\psi_{\mathbb{Q}}\circ Tr^F_{\mathbb{Q}}$ for the standard character $\psi_{\mathbb{Q}}$ of $\mathbb{A}_{\mathbb{Q}}$.

Remarks

1. We conjecture that the series $\ell(Z_\phi)$ is convergent for all ℓ. A good example is the functional coming from a cohomological class as follows. For a cohomological cycle $\alpha\in H^{2r}(M_k,\mathbb{Q})$, we may define a functional ℓ_α by taking intersection pairing between cohomological class $[Z]$ of $Z\in\mathrm{Ch}^r(M_K)$ and α:

$$\ell_\alpha(Z) := [Z]\cdot\alpha.$$

In this case, generating series $\ell_\alpha(Z_\phi)$ is convergent and modular by the fundamental work of Kudla–Milson ([8]).

2. Let $N^r(M_K)_{\mathbb{Q}}$ and $\mathrm{Ch}^r(M_K)^0_{\mathbb{Q}}$ be the image and kernel of the class map $\mathrm{Ch}^r(M_K)_{\mathbb{Q}} \longrightarrow H^{2r}(M_K)$ respectively. We expect there is a canonical decomposition of modules over the Hecke algebra of M_K:

$$\mathrm{Ch}^r(M_K)_{\mathbb{Q}} \simeq \mathrm{Ch}^r(M_K)^0_{\mathbb{Q}} \oplus N^r(M_K)_{\mathbb{Q}}.$$

In this way, for any $\alpha \in \mathrm{Ch}^r(M_K)^0_{\mathbb{Q}}$, we can define a functional ℓ_α by taking (a conjectured) Beilinson–Bloch's height pairing between projection Z_0 of $Z \in \mathrm{Ch}^r(M_K)$ and α:

$$\ell_\alpha(Z) := Z^0 \cdot \alpha.$$

The convergence problem is reduced to estimating height paring.

3. Beilison and Bloch has conjectured that the cohomologically trivial cycles $\mathrm{Ch}^r(M_K)^0 \otimes \mathbb{Q}$ will map injectively to the r-th intermediate Jacobian. Thus when $H^{2r-1}(M_K, \mathbb{Q}) = 0$, combination of this conjecture with Kudla–Millson's work implies the modularity of the general series $\ell(Z_\phi)$.

For Kudla divisors, we have an unconditional result:

Theorem 1.3. *For any $\phi \in \mathcal{S}(\widehat{V})$, the generating function $Z_\phi(\tau)$ of Kudla divisor classes is convergent and defines a modular form of weight $\frac{n}{2}+1$.*

Remark

The Shimura variety M_K has vanishing Betti number $h^1(M_K)$ unless M_K is a Shimura curve or the product of two Shimura curves. In this case, $\mathrm{Ch}^1(M_K)^0_{\mathbb{Q}} = 0$ and the modularity in Chow group $\mathrm{Ch}^1(M_K) \otimes \mathbb{Q}$ follows from Kudla-Milson's modularity for cohomology group $H^2(M_K, \mathbb{Q})$.

Now we would like to describe the contents of paper. In §1-2, we prove some intersection formulae for Kudla cycles in Chow groups and then some product formulae for generating series. The modularity Theorem 1.2-3 will be proved in §3. For modularity for divisors (Theorem 1.3), we use Kudla-Millson's modularity for generating functions of cohomological classes and an embedding trick that relies on the vanishing of the first Betti number of our Shimura varieties by results of Kumaresan and Vogan-Zuckerman. For modularity of high-codimensional cycles, we use an induction method in [14].

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2 Intersection formulae

Our aim is to study the intersections of Kudla cycles $Z(W,g)_K$ in Chow group $\mathrm{Ch}^*(M_K)$ of cycles modulo the rational equivalence. We first prove some scheme-theoretic formula and then some intersection formulae in Chow groups.

First we need a more intrinsic definition of Kudla cycles. We say a F-vector subspace W of $\widehat{V}$ is *admissible* if the inner product on W takes F-rational values and is positive definite.

Lemma 2.1. *An F-vector subspace W of $\widehat{V}$ is admissible if and only if $W = gW'$ with W' is a subspace of V and $g \in G(\widehat{\mathbb{Q}})$.*

Proof. Indeed, for any element $w \in W$ with non-zero norm, the F-rational number $\|w\|^2$ is locally a norm of vectors in V_v for every place v of F. Thus, it is a norm of $v \in V$ by Hasse-Minkowski theorem (See Serre [10], P41, Theorem 8). Now we apply Witt's thorem (see Serre [10], Page 31, Theorem 3) to get an element $g \in G(\widehat{\mathbb{Q}})$ such that $gw = v$. Replacing W by gW we may assume that $v = w$. Let V_1 be the orthogonal complement of v in V and W_1 be the orthogonal complement of v in W. Then we may use induction to embed W_1 into V_1. This induces an obvious embedding from W to V. $\square$

For an admissible subspace $W = g^{-1}W'$, $W' \subset V$ and $g \in G(\widehat{\mathbb{Q}})$, we have a well defined Kudla cycle $Z(W)_K := Z(W', g)_K$. For an open subgroup K' of K, the pull-back of cycle $Z(W)_K$ on $M_{K'}$ has a decomposition

$$Z(W)_K = \sum_k Z(k^{-1}W)_{K'} \tag{2.1}$$

where k runs though a set of representatives of the coset $K_W \backslash K/K'$ with K_W the stabilizer of W.

Proposition 2.2. *Let $Z(W_1)_K$ and $Z(W_2)_K$ be two Kudla cycles. The scheme-theoretic intersection is the union of $Z(W)$ indexed by admissible classes W in $K\backslash(KW_1 + KW_2)$.*

Proof. Assume that $W_i = g_i^{-1}V_i$ with $V_i \subset V$. Then the scheme theoretic intersection is represented by $(z, g) \in D \times G(\widehat{\mathbb{Q}})$ such that for some $\gamma \in G(\mathbb{Q})$, $k \in K$,

$$z \in D_{V_1} \cap \gamma D_{V_2}, \qquad g \in G_{V_1}(\widehat{\mathbb{Q}})g_1 \cap \gamma G_{V_2}(\widehat{\mathbb{Q}})g_2 k.$$

It is easy to see that

$$\gamma D_{V_2} = D_{\gamma V_2}, \quad \gamma G_{V_2} = G_{\gamma V_2} \cdot \gamma.$$

Thus we can rewrite the above condition as

$$z \in D_{V_1} \cap D_{\gamma V_2} = D_{V_1+\gamma V_2}, \qquad g \in G_{V_1}(\widehat{\mathbb{Q}})g_1 \cap G_{\gamma V_2}(\widehat{\mathbb{Q}})\gamma g_2.,$$

It follows that the intersection is a union of $Z(V_1 + \gamma V_2, g)_K$ indexed by $\gamma \in G(\mathbb{Q})$ and $g \in G(\widehat{\mathbb{Q}})$ such that

$$g \in G_{V_1+\gamma V_2}(\widehat{\mathbb{Q}})\backslash(G_{V_1}(\widehat{\mathbb{Q}})g_1K \cap G_{\gamma V_2}(\widehat{\mathbb{Q}})\gamma g_2K)/K.$$

For such a g, we may write

$$g = h_1g_1k_1 = h_2\gamma g_2k_2$$

with elements in the corresponding components. Then

$$g^{-1}(V_1 + \gamma V_2) = k_1^{-1}g_1^{-1}V_1 + k_2^{-1}g_2^{-1}V_2 = k_1^{-1}W_1 + k_2^{-1}W_2.$$

Thus the intersection is parameterized by admissible classes in

$$K\backslash(KW_1 + KW_2).$$

□

The following lemma gives the uniqueness of admissible class with fixed generators under small levels:

Proposition 2.3. *Let $x_1, \cdots x_r$ be a basis of an admissible subspace W of V. Then there is an open normal subgroup K' in K such that for any $k \in K$, the only possible admissible class in*

$$K'\backslash\sum K'k^{-1}(x_i)$$

is $\sum_i k^{-1}(x_i)$, where (x_i) denote the subspace Fx_i of V.

Proof. We proceed the proof in several steps.

Step 0: let us reduce to the case $k = 1$. Assume that K' is a normal subgroup. Then we have an bijection of classes:

$$K'\backslash\sum K'k^{-1}(x_i) \longrightarrow K'\backslash\sum_i K'(x_i), \quad t \mapsto kt.$$

Thus we *may assume* $k = 1$ *to prove the Proposition.*

Step 1: we will work on congruence group for a fixed lattice. Pick up a lattice $V_{\mathbb{Z}}$ stable under K and taking integral valued inner products. Then for each positive integer N, we have open subgroups $K(N)$ of $G(\widehat{\mathbb{Q}})$ consisting of elements h such that

$$hx - x \in NV_{\mathbb{Z}}, \qquad \forall x \in V_{\mathbb{Z}}.$$

Now we take $K' = K(N)$ for N big so that K' is a normal subgroup of K. Assume that for some $h_i \in K(N)$, the class $\sum_i Fh_ix_i$ is admissible. We are reduced to show that *there is a* $k \in K'$ *such that* $kx_i = h_ix$ *when* N *is sufficiently large.*

Step 2: let us reduce to a problem of extending maps. Without loss of generality, assume that $x_1, \cdots, x_r \in V_{\mathbb{Z}}$ and generate $W_{\mathbb{Z}} := W \cap V_{\mathbb{Z}}$. Then we have the following properties for the inner product of h_ix_i:

- for some $t_{i,j} \in \mathbb{Z}$

$$(h_ix_i, h_jx_j) = (x_i, x_j) + Nt_{i,j};$$

- Schwartz inequality (as the pairing on $\sum_i \mathbb{Q}h_ix_i$ is positive definite):

$$|(h_ix_i, h_jx_j)| \le \|h_ix_i\|\|h_jx_j\| = \|x_i\|\|y_j\|.$$

It follows that for N big, $t_{i,j} = 0$. In other words, there is a an isometric embedding

$$\xi : \quad W \longrightarrow V, \qquad x_i \mapsto h_ix_i.$$

Thus we reduce to *extend this embedding to an isomorphism* $k : \widehat{V} \longrightarrow \widehat{V}$ *by an* $k \in K$ *for* N *sufficiently large.*

Step 3: work with an orthogonal basis. Write $W = g^{-1}W'$ and take an orthogonal basis $f_1, \cdots f_{n+2}$ of $V_{\mathbb{Z}}$ over $\mathbb{Z}$ such that $f_1, \cdots, f_m$ is s basis $W' \cap V_{\mathbb{Z}}$. Write $e_i = g^{-1}f_i$ and $e_i' = \xi(e_i)$ for $1 \le i \le m$. Notice that e_i is an integral combination of x_i, thus $e_i - e_i' \in N'\widehat{V}_{\mathbb{Z}}$ for an integer N' which can be arbitrarily larger as N goes to infinity. Thus we are in a situation to find an element $k \in K$ such that $ke_i = \xi e_i = e_i'$ for i between 1 and m. We reduce question to *find local component of* k_p *for each* p.

Step 4: work with good primes. Let S be a finite set of primes in $\mathbb{Z}$ consisting the factors of $2N$, and the norms of e_i's. If p is not in S, we claim that one of $(e_1 \pm e_1')/2$ has invertible norm. Otherwise, the sum of their norms, $(\|e_1\|^2 + \|e_2\|^2)/2$ is in $p\mathbb{Z}_p$. This is contradiction because e_1 and e_1' have the same norm. Thus we may have a decomposition $V_{\mathbb{Z}_p}$ into a sum of $\mathbb{Z}_p(e_1 \pm e_1')/2$ and its complement V'. We may take k_1 which is ± 1 on the first fact and ∓ 1 on the second factor. Then $k_1 \in G(\mathbb{Z}_p)$ such that $k_1 e_1' = e_1$. Now we may replace e_i' by $k_1 e_i$ and then reduce to the case where $e_1 = e_1'$. We may continue this process for $\mathbb{Z}_p e_1^\perp$ etc until all $e_m = e_m'$. In other words, we find that an $k_p \in G(\mathbb{Z}_p)$ such that $k_p e_i = e_i'$.

Step 5: work with bad prime. If $p \notin S$, we may replace N by Np^ℓ so that the order α of p in N be arbitrarily large. We define e_i' for $i > m$ by induction such that $\langle e_i', e_j' \rangle = \langle e_i, e_j \rangle$ for all $j \leq i$, and e_i is close to e_i'. This is done by applying Schmidt process for the elements:

$$e_1', \cdots, e_m', e_{m+1}, \cdots, e_{n+2}.$$

More precisely, assume that $e_1', \cdots, e_{i-1}'$ are defined then we define e_i' by

$$e_i'' := e_i - \sum_{j=1}^{i-1} \frac{(e_i, e_j')}{(e_j', e_j')} e_j',$$

$$e_i' := \sqrt{\frac{(e_i, e_i)}{(e_i'', e_i'')}} \cdot e_i''.$$

Notice that when the order of p in N is sufficiently large, e'' is arbitrarily close to e_i thus $(e_i, e_i)/(e_i'', e_i'')$ is arbitrarily close to 1. Thus the square root is well defined. In summary we find a $k_p \in K(p^\beta)$ for β arbitrarily large when $\mathrm{ord}_p(N)$ is arbitrarily large.

□

As an application, we want to decompose the cycle $Z(W, g)_K$ as complete intersection after rasing levels K.

Proposition 2.4. *Let $x_1, \cdots x_r$ be a basis of W over F. Then there is an open normal subgroup K' in K such that the pull-back of $Z(W)_{K'}$ is a (rational) multiple of unions of the complete intersection*

$$\sum_{k \in K' \backslash K} \prod_i Z(k^{-1} x_i)_{K'}.$$

Proof. For an open subgroup K' of K, the cycle $Z(W)_K$ has a decomposition

$$Z(W)_K = \sum_{k \in K_W \backslash K/K'} Z(k^{-1}W)_{K'}.$$

Here K_W is the subgroup of K consists of elements fixing every elements in W.

We want to compare the right hand side with $\sum_{k \in K/K'} \prod_i Z(k^{-1}x_i)_{K'}$. By Proposition 2.2, the components of $\prod_i Z(k^{-1}x_i)_{K'}$ correspond to the admissible classes in

$$K' \backslash \sum_i K'k^{-1}(x_i).$$

By Proposition 2.3, when K' is small, the only admissible class in the above coset is $\sum k^{-1}(x_i) = k^{-1}W$. Thus

$$\prod_i Z(x_i)_{K'} = \sum_j Z(k_j^{-1}W)_{K'}$$

for some $k_j \in K$. Now we translate both sides by $k \in K/K'$ to obtain

$$\sum_{k \in K/K'} \prod_i Z(k^{-1}x_i)_{K'} = c_1 \sum_{k \in K/K'} Z(k^{-1}W)_{K'} = c_2 Z(W)_K$$

where c_1 and c_2 are some positive rational numbers. □

By comparing the codimensions, we conclude that the intersection of $Z(W_1)_K$ and $Z(W_2)_K$ in Proposition 2.2 is proper if and only if $k_1W_1 \cap k_2W_2$ is 0 for all admissible class $k_1W_1 + k_2W_2$. In this case, the set theoretic intersection gives the intersection in Chow group. In the following we want to study what happen if the intersection is not proper. First, we need to express the canonical bundle of M_K in terms of $\mathcal{L}_K$:

Lemma 2.5. *Let $\omega_K = \det \Omega_K^1$ denote the canonical bundle on M_K. Then for K small, there is a canonical isomorphism*

$$\omega_K \simeq \mathcal{L}_K^n \otimes \det V^\vee.$$

Proof. We need only to prove the statements in the lemma for the bundle $\mathcal{L}_D$ on D with an isomorphism

$$\omega_D \simeq \mathcal{L}_D^n \otimes \det V^\vee$$

which is equivariant under the action of $G(\mathbb{R})$. Fix one point p on D corresponding to one point $v \in V_{\iota,\mathbb{C}}$, then $V_{\iota,\mathbb{R}}$ has an orthogonal basis given by $\mathrm{Re}(v), \mathrm{Im}(v), e_1, \cdots e_n$ and $V_{\iota,\mathbb{C}}$ has a basis $v, \bar{v}, e_1, \cdots, e_n$. After a rescaling, we may assume that $\langle \omega, \bar{\omega} \rangle = -1$, and $\langle e_i, e_i \rangle = 1$. Then we can define local coordinates $z = (z_1, \cdots, z_n)$ near p such that the vector v extend to a section of $\mathcal{L}_D$ in a neighborhood of p:

$$v_z := v + \frac{1}{2} \sum_i z_i^2 \bar{v} + \sum_i z_i e_i.$$

For a point $p \in D$ corresponding to a line ℓ in $V_\mathbb{C}$, the tangent space of D at p is canonically isomorphic to $\mathrm{Hom}(\ell, \ell^\perp/\ell)$. In term of coordinators z for $\ell = \mathbb{C}v$, this isomorphism takes $\frac{\partial}{\partial z_i} \otimes v$ to the class of e_i in $\ell^\perp/\ell$. In terms of bundles, one has an equivaraiant isomorphism:

$$T_D \simeq \underline{Hom}(\mathcal{L}_D, \mathcal{L}_D^\perp/\mathcal{L}_D) = \mathcal{L}_D^\perp \otimes \mathcal{L}_D^\vee/\mathcal{O}_D.$$

Write $\omega_D = \det T_D^\vee$ for the canonical bundle on D. Then we have an equivariant isomorphism

$$\omega_D = (\det \mathcal{L}_D^\perp)^\vee \otimes \mathcal{L}_D^{1+n}.$$

In terms of coordinator z, this isomorphism is given by

$$dz_1 \cdots dz_n \otimes (e_1 \wedge e_2 \cdots e_n \wedge v) \mapsto v^{\otimes(n+1)}.$$

Notice that the pairing $\langle \cdot, \cdot \rangle$ induces an equivariant isomorphism between $\mathcal{L}_D^\vee$ and $V_D/\mathcal{L}_D^\perp$ which is represented by $\mathbb{C}\bar{v}$ in our base of $V_\mathbb{C}$. This gives an isomorphism $\det \mathcal{L}_D^\perp \simeq \mathcal{L}_D \otimes \det V$ which is given by

$$e_1 \wedge e_2 \cdots e_n \wedge v \mapsto v \otimes (e_1 \wedge e_2 \cdots e_n \wedge v \wedge \bar{v}.$$

Thus we have a canonical isomorphism

$$\omega_D \simeq \mathcal{L}_D^n \otimes \det V^\vee$$

which is given by

$$dz_1 \wedge dz_2 \cdots dz_n \mapsto v^n \otimes (e_1 \wedge e_2 \cdots e_n \wedge v \wedge \bar{v}).$$

This completes the proof of the lemma. □

Now we have a version of Proposition 2.2 in Chow group. For positive number r and an element $x = (x_1, \cdots, x_r) \in K\backslash V(\widehat{F})^r$, recall that the Kudla cycle $Z(x)_K$ in M_K is defined as follows: let W be the subspace of $V(\widehat{F})$ generated by components x_i of x. Then we define

$$Z(x)_K = \begin{cases} Z(W)_K c_1(\mathcal{L}^\vee)^{r-\dim W} & \text{if } W \text{ is admissible} \\ 0 & \text{otherwise} \end{cases}$$

Proposition 2.6. *Let $Z(W_1)_K$ and $Z(W_2)_K$ be two Kudla cycles. The intersection in Chow group is given as a sum of $Z(W)_K$ indexed by admissible class W in*

$$K\backslash(KW_1 + KW_2).$$

Proof. First we treat the case where W_2 is one dimensional. Then the set theoretic intersection is indexed by admissible classes $k_1W_1 + k_2W_2$ in $K\backslash KW_1 + KW_2$. There is nothing need to prove if this intersection is proper. Otherwise, $k_2W_2 \subset k_1W_1$ and $Z_1 \subset Z_2$ for some components Z_1 of $Z(W_1)$ and component Z_2 of $Z(W_2)$. Let Z be a connected component of M_K containing Z_2. Let i denote the embedding $i : Z_2 \longrightarrow Z$. Then the intersection in Chow group has an expression:

$$Z_1 \cdot Z_2 = i_*(Z_1 \cdot i^* c_1(\mathcal{O}(Z_2))).$$

Let I be the ideal sheaf of Z_2 on Z. Then $\mathcal{O}(Z_2) = I^{-1}$ and $i^*c_1(Z(W_2)) = -c_1(i^*I)$ is the first Chern class of the bundle i^*I^{-1}. From the exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_Z|_{Z_2} \longrightarrow \Omega_{Z_2} \longrightarrow 0$$

we obtain the following isomorphism from the determinant

$$i^*(I) \otimes \omega_{Z_2} \simeq i^*\omega_Z.$$

Thus, we have shown the following equality in Chow group:

$$Z_1 \cdot Z_2 = i_*(Z_1 \cdot c_1(\omega_{Z_2} \otimes i^*\omega_Z)). \tag{2.2}$$

Now we use the canonical isomorphism in the Proposition 2.5:

$$\omega_{Z_i} \simeq \mathcal{L}_{Z_i}^{\dim Z_i}, \quad \mathcal{L}_Z|_{Z_i} = \mathcal{L}_{Z_i}$$

to conclude that

$$i^*\mathcal{L}^\vee \simeq \omega_{Z_2} \otimes i^*\omega_Z \tag{2.3}$$

Combining equations (2.2) and (2.3), we obtain that

$$Z_1 \cdot Z_2 = i_*(Z_1 \cdot i^* c_1(\mathcal{L})) = Z_1 \cdot c_1(\mathcal{L}^\vee).$$

Thus we have the proposition when W_2 is one dimensional. Now we want to prove the proposition for general case. We use Proposition 2.4 to write $Z(W_2)_K$ as a sum:

$$Z(W_2)_K = c \sum_{k \in K/K'} \prod_i Z(k^{-1}x_i)_{K'}.$$

By working on the intersections of $Z(k^{-1}x_i)$ with schema theoretic component of $Z(W_1)\prod_{j<i} Z(k^{-1}x_i)$, we found that the intersection of $Z(W_1)$ and $Z(W_2)$ in Chow group in level K' is simply the sum of

$$Z(W)c_1(\mathcal{L}^\vee)^{\dim W_1 + \dim W_2 - \dim W}$$

where W runs through admissible classes in

$$\coprod_{k \in K' \backslash K} K' \backslash (KW_1 + \sum_i K'k'(x_i)).$$

In other words, in level K', the Chow intersection is the Zariski intersection with correction by powers of first Chern class of $c_1(\mathcal{L}^\vee)$. As $\mathcal{L}$ is invariant under pull-back, and the dimension does not change under push-forward, we have the same conclusion in level K. □

Proposition 2.4 is still true in this case.

Proposition 2.7. *Let $x = (x_1, \cdots, x_r) \in K\backslash\widehat{V}^r$. Then there is an open normal subgroup K' in K such that in Chow group, the pull-back of $Z(x)_K$ is a (rational) multiple of a sum of complete intersections*

$$\sum_{k \in K' \backslash K} \prod_i Z(k^{-1}x_i)_{K'}$$

Proof. By Proposition 2.3, we may choose K' such that for any $k \in K$, the only admissible class in

$$K' \backslash \sum K'k^{-1}(x_i)$$

is $\sum Fk^{-1}x_i$. By Proposition 2.6, the product $\prod_i Z(k^{-1}x_i)_{K'}$ is simply $Z(k^{-1}x)_{K'}$. Their sum is simply $Z(x)_K$. □

3 Product formulae

In this section, we want to apply the formulae in the last section to obtain some product formula for Kudla's generating series. The first one is the product formula in Theorem 1.1 which has been conjectured by Kudla [7] and then the pull-back formula for embedding of Shimura varieties.

Proof of product formula

By definition,

$$Z_{\phi_1}(\tau_1) \cdot Z_{\phi_2}(\tau_2) = \sum_{(x_1,x_2)} Z(x_1)_K Z(x_2)_K \phi_1(x_1)\phi_2(x_2) q^{T(x_1)} q^{T(x_2)}.$$

By Proposition 2.6,

$$Z(x_1)_K Z(x_2)_K = \sum_W Z(W)$$

where W runs through the admissible classes in

$$K\backslash K(x_1) + K(x_2)$$

where (x_i) denote the subspaces of $\widehat{V}$ generated by components x_{ij} of x_i. It is clear that such W is generated by αx_{1i} and βx_{2i} for some α, β in K. Thus we write $x = (\alpha x_1, \beta x_2)$. On the other hand, it is easy to see that for such an x,

$$\phi_1(x_1)\phi_2(x_2) q^{T(x_1)} q^{T(x_2)} = (\phi_1 \otimes \phi_2)(x) q^{T(x)}.$$

Thus we have shown the following:

$$Z_{\phi_1}(\tau_1) \cdot Z_{\phi_2}(\tau_2) = \sum_x Z(x)_K (\phi_1 \otimes \phi_2)(x) q^{T(x)} = Z_{\phi_1 \otimes \phi_2}\left(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}\right).$$

A pull-back formula

The rest of this section is devoted to prove a pull-back formula for generating functions for Kudla cycles with respect to an embedding of Shimura varieties. First let us describe the generating series as a function on adelic points. Recall that $\mathrm{Mp}_r(\mathbb{R})$ is the double cover of $\mathrm{Sp}_r(\mathbb{R})$. Let K' denote the preimage of the compact subgroup

$$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : \quad a + ib \in U(r) \right\}$$

of $\mathrm{Sp}_r(\mathbb{R})$. The group K' has a character $\det^{1/2} := j(\cdot, i_r)^{1/2}$ whose square descends to the determinant character of $U(n)$. Write

$$Z_\phi(g') = Z_{\omega(g'_f)\phi}(g' i_r) j(g', i_r)^{-n/2-1}.$$

Then $Z_\phi(g')$ has a Fourier expansion

$$Z_\phi(g') = \sum_{x \in K\backslash V(\widehat{\mathbb{Q}})^r} (\omega_f(g'_f)\phi)(x) Z(x)_K W_{T(x)}(g'_\infty)$$

where $W_t(g'_\infty)$ is the t-th "holomorphic" Whittaker function on $\mathrm{Mp}_r(\mathbb{R})$ of weight $\frac{r}{2}+1$: for each $g' \in \mathrm{Mp}_r(\mathbb{R})$ with Iwasawa decomposition

$$g = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix} k, \qquad a \in \mathrm{GL}_r(\mathbb{R})^+, \quad k \in K'$$

we have

$$W_t(g) = |\det(a)|^{n/2+1} e(\mathrm{tr} t\tau) \det(k)^{n/2+1}.$$

Here

$$\tau = u + ia \cdot^t a.$$

Now the modularity of Z_ϕ is equivalent to that

$$Z_\phi(g') = Z_\phi(\gamma g'), \qquad \forall \gamma \in \mathrm{Sp}_r(F).$$

Let $W \subseteq V$ be positive definite F-subspace of dimension d and let W' be its orthogonal complement. Then we have a decomposition $\mathcal{S}(V(\mathbb{A})^r) = \mathcal{S}(W'(\mathbb{A})^r) \otimes \mathcal{S}(W(\mathbb{A})^r)$. Consider the embedding map:

$$i: \qquad M_{K,W} = G_W(\mathbb{Q})\backslash D_W \times G_W(\widehat{\mathbb{Q}})/K_W \to M_K = G(\mathbb{Q})\backslash D \times G(\widehat{\mathbb{Q}})/K,$$

where $K_W = G_W(\widehat{Q}) \cap K$. Then we have a pull-back map

$$i^*: \qquad \mathrm{Ch}^r(M_K) \to \mathrm{Ch}^r(M_{K,W}).$$

Now we prove a pull-back formula for

$$i^*(Z_\phi)(g') = \sum_{x \in K\backslash V(\widehat{F})^r} \omega(g'_f)\phi_f(x) i^*(Z(x)_K) W_{T(x)}(g'_\infty).$$

Proposition 3.1. *Let $\phi = \phi_1 \otimes \phi_2 \in \mathcal{S}(V(\mathbb{A})^r) = \mathcal{S}(W'(\mathbb{A})^r) \otimes \mathcal{S}(W(\mathbb{A})^r)$ and suppose that ϕ_1, ϕ_2 are K-invariant. Then, we have an equality in the Chow group*

$$i^*(Z_\phi)(g') = Z_{\phi_1}(g')\theta_{\phi_2}(g') \tag{3.1}$$

where $\theta_{\phi_1}(g')$ is the generating function with coefficients in $\mathrm{Ch}^r(M_{K,W}, \mathbb{C})$

$$Z_{\phi_1}(g') = \sum_{y \in K_W \backslash W'(\widehat{F})^r} \omega_f(g'_f)\phi(y)Z(y)_K W_{T(y)}(g'_\infty)$$

and

$$\theta_{\phi_2}(g') = \sum_{z \in W(F)^r} \omega(g')(\phi_2 \otimes \phi_{2\infty})(z)$$

is the usual theta function, where $\phi_{2\infty}$ is the standard spherical function on $W'(\mathbb{R})$.

Proof. Let $x \in K \setminus V(\widehat{F})^r$. By Proposition 2.6, the intersection of $Z(x)_K$ and $Z(W)_K$ is indexed by admissible classes in $K \setminus KW + Kx$. For an admissible class (W, kx), the projection, denoted by z, of kx to $\widehat{W}$ must lie in W by the definition of admissibility. Thus, $y := kx - z \in W'(\widehat{F})^r$. And conversely, for $y + z \in W'(\widehat{F})^{r,ad} \oplus W(F)^r$, $(W, y + z)$ must be admissible.

Therefore, we have in the Chow group of M_W the following identity:

$$i^* Z(x)_K = \sum_{(y,z)} Z(y)_{K_W} \tag{3.2}$$

where the sum is over all admissible $y \in K_W \setminus W'(\widehat{F})^r$, and all $z \in W(F)^r$ such that

$$K_W(y + z) = K_W y + z \supseteq Kx.$$

By the discussion above, we have

$$\begin{aligned}
&i^* Z_\phi(g') \\
=&\sum_{x \in K \backslash V(\widehat{F})^r} \omega(g'_f)\phi(x) i^* Z(x)_K W_{T(x)}(g'_\infty) \\
=&\sum_{y \in K_W \backslash W'(\widehat{F})^r} \omega(g'_f)\phi_1(y) Z_{W'}(y) W_{T(y)}(g'_\infty) \sum_{z \in W(F)^r} \omega(g')\phi_2(z) W_{T(z)}(g'_\infty) \\
=&Z_{\phi_1}(g')\theta_{\phi_2}(g').
\end{aligned}$$

This completes the proof of the proposition. ∎

4 Modularity in Chow groups

In this section, we want to prove the modularity (Theorem 1.2 and 1.3) of generating series for a linear functional on Chow groups. We will first treat the case of codimension one. Quite different from Borcherds' proof in [1], our proof does not use Borcherds "singular theta lifting", which is actually unavailable on totally real field except $F = \mathbb{Q}$. Roughly speaking, the modularity for large n follows from Kudla–Millson's modularity for cohomological class and the vanishing of the first Betti number of our Shimura varieties. For small n, we use a pull-back trick which deduce the desired modularity from that of large n.

Proof of Theorem 1.3

Suppose that ϕ is K-invariant. The group of cohomologically trivial line bundles, up to torsion, is parameterized by the connected component of Picard variety of M_K. The tangent of the Picard is $H^1(M_K, \mathcal{O})$. For $n \geq 3$, $\dim_{\mathbb{C}} H^1(M_K, \mathcal{O}) = 0$ since it is half of the first Betti number of M_K, which is zero by Kumaresan's vanishing theorem and Vogan-Zuckerman's explicit computation in [11], Theorem 8.1. Thus, the cycle class map is injective up to torsion and the theorem follows from the modularity of Kudla–Millson (Theorem 2 in [8]) where the statement extends obviously to the adelic setting by their proof.

Now we assume that $n \leq 2$. We can embed V into a higher dimension quadratic space $V' = V \oplus W$ such that $\dim_F V' \geq 5$ and with the desired signature at archimedean places. By Proposition 3.1, for any $\phi' \in \mathcal{S}(\widehat{W})$, we have

$$i^* Z_{\phi\otimes\phi'}(g') = Z_\phi(g')\theta_{\phi'}(g').$$

Since both $Z_{\phi\otimes\phi'}(g')$ and the usual theta function $\theta_{\phi'}(g')$ are convergent and $\mathrm{SL}_2(F)$-invariant, we deduce the convergence of $Z_\phi(g')$ and invariance under $\mathrm{SL}_2(F)$, provided that, for each g', we can choose ϕ' such that $\theta_{\phi'}(g') \neq 0$. But we can make such choice since, otherwise, for some g', $\theta_{\phi'}(g') = 0$ for all choices of ϕ'. This would imply that $\theta_{\phi'}(g'g_f) = \theta_{\omega(g_f)\phi'}(g') = 0$ for any $g_f \in \mathrm{Mp}(\widehat{F})$. Contradiction! This complete the proof of the theorem.

Remark

When $dim_F V = 3$, the theorem above generalize Gross-Kohnen-Zagier theorem about Heegner points on modular curves to CM points on Shimura curves. The pull-back trick was already used, as explained in Zagier's paper [12] and the introduction of [2], to deduce Gross–Kohnen–Zagier theorem in a special case from the theorem of Hirzebruch–Zagier [5]. There, a key ingredient is the simply connectedness of Hilbert modular surfaces.

Combining their computation of Néron-Tate pairing and a result of Waldspurger, Gross–Kohnen–Zagier in [2] also proved that eigen-components of Heegner divisors on $X_0(N)$ are co-linear in the Mordell-Weil group. This can be viewed as a "multiplicity one" result. We can give a representation-theoretical proof of this result along the same line as in [14]. Let B be a quaternion algebra over F such that it ramifies at exactly at one archimedean place of F. Let V be the trace free subspace of B. Together with the reduced norm, we obtain a three dimensional quadratic space, and $G = \mathrm{GSpin}(V) = B^\times$. Let M_K be the Shimura curve for an open compact subgroup $K \subseteq G(\mathbb{A}_f)$. Let ξ be the Hodge class defined as in [13]. Consider the subspace $\mathcal{M}_K$ of $\mathrm{Jac}(M_K)(F)$ generated by CM-divisors $Z(x)_K - \deg(Z(x)_K)\xi$ for all $x \in K \setminus \widehat{V}$. Consider the direct limit $\mathcal{M}$ of $\mathcal{M}_K$ for all K and consider it as a $G(\mathbb{A}_f)$-module. For a $G(\mathbb{A}_f)$-module π_f, let σ_f be the representation of $\mathrm{GL}_2(\mathbb{A}_f)$ associated by Jacquet-Langlands correspondence. Let $\sigma_{\infty,(2,2,\ldots,2)}$ be the homomorphic discrete series of $\mathrm{GL}_2(F_\infty)$ of parallel weight $(2, 2, \ldots, 2)$.

Theorem 4.1. *For a $G(\mathbb{A}_f)$-module π_f with trivial central character,*

$$dim\, Hom_{G(\mathbb{A}_f)}(\mathcal{M}, \pi_f) \le 1.$$

If $Hom_{G(\mathbb{A}_f)}(\mathcal{M}, \pi_f)$ is non-trivial, the product $\sigma = \sigma_f \otimes \sigma_{\infty,(2,2,\ldots,2)}$ is a cuspidal automorphic representation of $GL_2(\mathbb{A})$.

Proof. For the first assertion, we sketch the proof and the complete detail can be found in [14] Section 6. Let ρ_f be the representation of $\widetilde{SL}_2(\mathbb{A}_f)$ defined by the local Howe's duality for the pair $(SO(V), \widetilde{SL}_2)$ by the work of Waldspurger. Let $\rho_{\infty,(3/2,\ldots,3/2)}$ be the homomorphic discrete series of $\widetilde{SL}_2(F_\infty)$ of parallel weight $(3/2, \ldots, 3/2)$. Note that we have equivariance of Hecke action on the space $\mathcal{M}$ and the space $\mathcal{S}(V(\mathbb{A}_f))$. In our case, Theorem 1.3 actually implies that generating functions valued in $\mathcal{M}$ are all cuspidal forms. Then, $Hom_{G(\mathbb{A}_f)}(\mathcal{M}, \pi_f)$ vanishes unless $\rho = \rho_f \otimes \rho_{\infty,(3/2,\ldots,3/2)}$ is a cuspidal automorphic representation of $\widetilde{SL}_2(\mathbb{A})$, and the dimension of $Hom_{G(\mathbb{A}_f)}(\mathcal{M}, \pi_f)$ is bounded by the multiplicity of ρ in the space of cuspidal automorphic forms on $\widetilde{SL}(2)$. The "multiplicity one" for cuspidal automorphic representations on $\widetilde{SL}(2)$ holds by Waldspurger's work. For the second assertion, the automorphy of ρ implies that of σ again by Waldspurger's work. □

High-codimensional cycles

In the following we want to prove the modularity in Theorem 1.2 along the same line as in [14]. Now that we have already assumed the convergence of generating series, we only need to verify the automorphy.

Step 0: Modularity when $r = 1$.

When $r = 1$, the assertion is implied by Theorem 1.3. And we actually know that generating functions converge for all linear functionals ℓ.

Step 1: Invariance under Siegel parabolic subgroup.

It is easy to see that the series $Z_\phi(g)$ is invariant under the Siegel parabolic subgroup of $\mathrm{Sp}_r(F)$. It suffices to consider the γ of the form

$$n(u) := \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}, \qquad m(a) = \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix}$$

By definition, we have

$$\omega(n(u)g'_f)\phi(x)W_{T(x)}(n(u)g'_\infty) = \omega(g')\phi(x)W_{T(x)}(g'_\infty)$$

Thus every term in $Z_\phi(g)$ is invariant under $n(u)$. Also by definition,

$$\omega(m(a)g'_f)\phi(x)W_{T(x)}(m(a)g'_\infty) = \omega(g'_f)\phi(xa)W_{T(xa)}(g'_\infty)$$

Since $Z(x)_K = Z(xa)_K$, thus the sum does not change after a substitution $x \to xa$.

Step 2: Invariance under w_1.

We want to show that $Z_\phi(g)$ is invariant under w_1, the image of $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ under the embedding of SL_2 into Sp_{2r}. This is the key step of the proof.

Firstly, we can rewrite the sum as

$$Z_\phi(\tau) = \sum_{y \in K \backslash \widehat{V}^{r-1}} \sum_{x \in K_y \backslash \widehat{V}} \phi(x,y) Z(x,y)_K q^{T(x,y)}$$

where K_y is the stabilizer of y. One can write

$$Z(x,y)_K = \sum_{x_1,x_2} i_{y*} Z(x_1)_{K_y}$$

where

$$i_y : M_{K,y} \to M_K$$

and the sum is over all

$$x_1 \in y^\perp := \left\{ z \in \widehat{V} : \quad \langle z, y_i \rangle = 0, i = 1, 2, ..., r-1 \right\}$$

and $x_2 \in Fy := \sum_{i=1}^{r-1} Fy_i$ satisfying that $K_y(x_1 + x_2) = K_y x$ (also see the equation 3.2).

Thus, the sum becomes

$$Z_\phi(\tau) = \sum_{y \in K \backslash \widehat{V}^{r-1,ad}} \sum_{x_1 \in K_y \backslash y^\perp} \sum_{x_2 \in Fy} \phi(x_1 + x_2, y) i_{y*}(Z(x_1)_{K_y}) q_1^{T(x_1)} \xi^{\langle x_2, y \rangle} q_2^{T(y)}$$

where

$$\xi^{\langle x_2, y \rangle} = \exp(2\pi i \sum_{k=1}^{d} \sum_{i=1}^{r-1} (z_{k,i} \langle x_2, y_i \rangle)),$$

and we have a natural decomposition

$$q^{T(x)} = q_1^{T(x_1)} \xi^{\langle x_2, y \rangle} q_2^{T(y)}, \quad \tau_k = \begin{pmatrix} \tau_{k,1} & z_k \\ {}^t z_k & \tau_{k,2} \end{pmatrix}, k = 1, ..., d.$$

where $\tau_{k,1} \in \mathcal{H}_1$, $z_k \in \mathbb{C}^{r-1}$, $\tau_{k,2} \in \mathcal{H}_{r-1}$. Here, we simply write $Z(x,y)$, $Z(x_1)$ etc. to mean actually cycle classes shifted by appropriate powers of tautological line bundles on ambient Shimura varieties. But all these tautological bundles are compatible with pull-backs (with respect to various compact subgroups K, K_y), or restrictions (from M_K to $M_{K,y}$). We therefore suppress them in the following exposition to avoid messing up notations.

For fixed y, applying the modularity for divisors (proved in Step 0) to $\phi(x_1 + x_2, y)$ as a function of x_1, we know that under the substitution $\tau \mapsto w_1^{-1}\tau$,

$$\sum_{x_1 \in K_y \backslash y^\perp} \phi(x_1 + x_2, y) Z(x_1)_{K_y} q_1^{T(x_1)}$$

becomes

$$\sum_{x_1 \in K_y \backslash y^\perp} \widehat{\phi}^1(x_1+x_2, y) Z(x_1)_{K_y} q_1^{T(x_1)}$$

where $\widehat{\phi}^1(x_1+x_2, y)$ is the partial Fourier transformation with respect to x_1. Note that here we implicitly used the convergence of this partial series (by Theorem 1.3). By Poisson summation formula, we also know that, under the same substitution,

$$\sum_{x_2 \in Fy} \widehat{\phi}^1(x_1+x_2, y) Z(x_1)_{K_y} \xi^{\langle x_2, y\rangle} q_2^{T(y)}$$

becomes

$$\sum_{x_2 \in Fy} \widehat{\phi}^{1,2}(x_1+x_2, y) Z(x_1)_{K_y} \xi^{\langle x_2, y\rangle} q_2^{T(y)}.$$

Note that $\omega(w_1)\phi(x,y) = \widehat{\phi}^x(x,y)$ is the partial Fourier transformation with respect to the first coordinate x. It is easy to see that for $x = x_1 + x_2$, $\widehat{\phi}^{1,2}(x_1+x_2, y) = \widehat{\phi}^x(x,y)$, too. This proves that

$$Z_\phi(w_1^{-1}\tau) = Z_{\omega(w_1)\phi}(\tau).$$

This prove that $Z_\phi(g')$ is invariant under w_1.

Step 3: Invariance under $\mathrm{Sp}_{2r}(F)$.

We claim that the Siegel parabolic subgroup and w_1 generate $Sp_{2r}(F)$. In fact, $\mathrm{SL}_2(F)^r$ and the Siegel parabolic subgroup generate $\mathrm{Sp}_{2r}(F)$. Obviously, one needs just one copy of $\mathrm{SL}_2(F)$ since others can be obtained by permutations which are in the Siegel parabolic subgroup. Further, one copy of $SL_2(F)$ can be generated by w_1 and the Siegel parabolic subgroup. This proves the claim. Thus we have finished the proof of Theorem 1.2 by Step 0, 1 and 2.

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