WEIL REPRESENTATION AND ARITHMETIC FUNDAMENTAL LEMMA

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Abstract. We study a partially linearized version of the relative trace formula for the arithmetic Gan–Gross–Prasad conjecture for the unitary group $\mathrm{U}(V)$. The linear factor in this relative trace formula provides an SL_2-symmetry which allows us to prove by induction the arithmetic fundamental lemma over $\mathbb{Q}_p$ when p is odd and $p \geq \dim V$.

Contents

1. Introduction

The theorem of Gross and Zagier [15] relates the Néron–Tate heights of Heegner points on modular curves to the central derivative of certain L-functions. The arithmetic Gan–Gross–Prasad conjecture [10, 47, 40] is a generalization of this theorem to higher-dimensional Shimura varieties. This conjecture is inspired by the (usual) Gan–Gross–Prasad conjecture relating period integrals on classical groups to special values of Rankin–Selberg tensor product L-functions. In [21] Jacquet and Rallis proposed a relative trace formula (RTF) approach to this last conjecture in the case of unitary groups and there have been much progress along this direction in the past years. Inspired by their approach, in [47] the author proposed a relative trace formula approach to the arithmetic Gan–Gross–Prasad conjecture. This approach reduces the problem to certain

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local statements, notably the arithmetic fundamental lemma (AFL) conjecture formulated by the author in [47], and the arithmetic transfer (AT) conjecture formulated by Rapoport, Smithling, and the author [38, 39]. The AFL and AT conjectures relate the special values of the derivative of orbital integrals to arithmetic intersection numbers on a Rapoport–Zink formal moduli space (RZ space) of p-divisible groups,

$$\partial\mathrm{Orb}\big(\gamma, \mathbf{1}_{S(O_{F_0})}\big) = -\operatorname{Int}(g)\cdot\log q,$$

cf. the precise statement of Conjecture 3.8 for the AFL conjecture.

The goal of this paper is to give a proof of the AFL conjecture over $F_0 = \mathbb{Q}_p$ when $p \geq n$, for an open dense subset of regular semisimple elements (i.e., the set of "strongly regular semisimple elements" in the sense of [46]), cf. Theorem 15.1. This restriction is harmless for the relative trace formula approach to the arithmetic Gan–Gross–Prasad conjecture.

In fact, we also obtain a proof of the Jacquet–Rallis fundamental lemma (FL) conjecture over p-adic field, a theorem due to Yun [46] and Gordan [14] for p large, which is an identity between two orbital integrals

$$\mathrm{Orb}(\gamma, \mathbf{1}_{S(O_{F_0})}) = \mathrm{Orb}(g, \mathbf{1}_{K_0}),$$

cf. the precise statement of Conjecture 2.4. The idea is similar to the proof of the AFL and is easier to explain. For our proof of the FL, the main input is a study of a "partially linearized" version of the Jacquet–Rallis RTF, which we call a semi-Lie algebra version. This is closely related to the RTF of Yifeng Liu to the Fourier–Jacobi period/cycles [30, 31]. The advantage of the linearization is to gain more "symmetry", i.e., there is an "action" on the RTF (changing test functions) by SL_2 under the Weil representation. The SL_2-modularity plays the role in the global setting of the Fourier transform in the local harmonic analysis, a crucial ingredient in [48] to prove the smooth transfer conjecture of Jacquet–Rallis.

Now we give a little more detail of our approach. Let F_0 be a totally real number field, and F a CM quadratic extension of F_0. Let V be an F/F_0-hermitian space with $\dim_F V = n$. Consider the (diagonal) action of $\mathrm{U}(V)$ on the product $\mathrm{U}(V) \times V$. For unexplained notation, we refer the reader to Notation §1.2 and the main body of the paper. To any Schwartz function $\Phi \in \mathcal{S}((\mathrm{U}(V) \times V)(\mathbb{A}_0))$, we can associate a kernel function

$$\mathcal{K}_\Phi(g) = \sum_{(x,u)\in(\mathrm{U}(V)\times V)(F_0)} \Phi(g^{-1}(x,u)), \quad g \in \mathrm{U}(V)(\mathbb{A}),$$

which is left invariant under $\mathrm{U}(V)(F_0)$. Then, as one usually does in the theory of relative trace formula, one may study the distribution on $(\mathrm{U}(V) \times V)(\mathbb{A}_0)$,

$$\mathbb{I}(\Phi) = \int_{[\mathrm{U}(V)]} \mathcal{K}_\Phi(g)\, dg.$$

Here $[G]\colon = G(F_0)\backslash G(\mathbb{A}_0)$ for an algebraic group G over F_0. Similarly, one can start with the (diagonal) action of GL_{n,F_0} on the product $S_n \times V_n'$ where $V_n' = \mathrm{M}_{1,n} \times \mathrm{M}_{n,1}$ is the product of the space of column and row vectors, cf. §2. To any Schwartz function $\Phi' \in \mathcal{S}((S_n \times V_n')(\mathbb{A}_0))$, we have a similar kernel function $\mathcal{K}_{\Phi'}$ and a distribution

$$\mathbb{J}(\Phi') = \int_{[\mathrm{GL}_{n,F_0}]} \mathcal{K}_{\Phi'}(g)\, \eta_{F/F_0} \circ \det(g)\, dg.$$

By the smooth transfer between Φ and Φ' through their orbital integrals (relative to the group actions here), one can match the distributions $\mathbb{I}$ and $\mathbb{J}$.

Now, due to the presence of the linear factors V and V_n' respectively, the Weil representation ω of $\mathrm{SL}_2(\mathbb{A}_0)$ acts on $\mathcal{S}((\mathrm{U}(V) \times V)(\mathbb{A}_0))$ and $\mathcal{S}((S_n \times V_n')(\mathbb{A}_0))$, hence on the distributions $\mathbb{I}$ and $\mathbb{J}$,

$$\mathbb{I}(h, \Phi)\colon = \mathbb{I}(\omega(h)\Phi), \quad \text{and} \quad \mathbb{J}(h, \Phi')\colon = \mathbb{J}(\omega(h)\Phi')$$

where $h \in \mathrm{SL}_2(\mathbb{A}_0)$. Moreover, the action is "modular" in the sense that $h \mapsto \mathbb{I}(h, \Phi)$ and $\mathbb{J}(h, \Phi')$ are left invariant under $\mathrm{SL}_2(F_0)$, as an application of the Poisson summation formula. In other words, we may enrich the kernel function to a two-variable one

$$\mathcal{K}_\Phi(g, h) = \sum_{(x,u)\in(\mathrm{U}(V)\times V)(F_0)} \omega(h)\Phi(g^{-1}(x,u)), \quad g \in \mathrm{U}(V)(\mathbb{A}),\ h \in \mathrm{SL}_2(\mathbb{A}_0).$$

The natural question now is how the Weil representation fits into the comparison of the two distributions. From [48] and [44] one can deduce that the Weil representation commutes with smooth transfer, cf. Theorem A.1 in the appendix.

Both distributions $\mathbb{I}$ and $\mathbb{J}$ can be expanded as a sum over orbital integrals. Then the SL_2-modularity amounts to certain recursive relations between the orbital integrals appearing in $\mathbb{I}$ and $\mathbb{J}$. One may hope that the recursive relations are ample enough to allow us to extract identities such as the aforementioned fundamental lemma, starting from some simple identities that can be verified directly. This resembles the situation in the geometric approach (cf. [36], [46]) where one also needs to verify some simple cases directly as a starting point before applying the "perverse continuation principle".

The idea does not work directly to yield a proof of the Jacquet–Rallis FL; however, it does work if we take two additional inputs. The first input is to consider a "slice" of the semi-Lie algebra version. Here by a slice we mean the sliced representation at a semisimple element for the action of $\mathrm{U}(V)$ on $\mathrm{U}(V)\times V$, which is a good approximation of the action on a neighborhood of the orbit. In our case, we choose a regular semisimple element g_0 of $\mathrm{U}(V)$ and consider $(g_0, 0) \in \mathrm{U}(V)\times V$, a (relative) semisimple element. Then the sliced representation at $(g_0, 0) \in \mathrm{U}(V)\times V$ is isomorphic to the induced action of the stabilizer, a maximal torus T_0, on V (times the trivial action on the Lie algebra of T_0). In terms of harmonic analysis, this leads us to introducing a kernel function for each g_0,

$$\mathcal{K}_{\Phi,g_0}(g) = \sum_{x\in \mathrm{U}(V)(F_0)g_0, u\in V(F_0)} \Phi(g^{-1}(x,u)), \quad g \in \mathrm{U}(V)(\mathbb{A}).$$

Here the sum runs only over a subset of $\mathrm{U}(V)(F_0)$-orbits on $(\mathrm{U}(V)\times V)(F_0)$. Similarly we define a distribution

$$\mathbb{I}_{g_0}(\Phi) = \int_{[\mathrm{U}(V)]} \mathcal{K}_{\Phi,g_0}(g)\, dg.$$

This still keeps the action of $\mathrm{SL}_2(\mathbb{A}_0)$ under the Weil representation ω

$$\mathbb{I}_{g_0}(h, \Phi) = \mathbb{I}_{g_0}(\omega(h)\Phi), \quad h \in \mathrm{SL}_2(\mathbb{A}_0). \tag{1.1}$$

We have the similar construction for $S_n \times V_n'$. Clearly by varying g_0 we have refined the relations between the orbital integrals appearing in $\mathbb{I}$ and $\mathbb{J}$. In the local situation, this sliced version was utilized in [48] to prove the existence of smooth transfer by an induction argument. Here we are exploiting the global analog, i.e., the $\mathrm{SL}_2(F_0)$-modularity of (1.1) and its counterpart for $\mathbb{J}$.

Another input is to impose that $\mathrm{U}(V)$ is compact at archimedean places, and at the same time to plug in the Gaussian test functions, cf. §12. This simplifies the spectra of the SL_2-automorphic forms $\mathbb{I}_{g_0}(\cdot, \Phi)$ and its counterpart on $S_n \times V_n'$, to the extent that the spectra are finite. In fact, in our case, they lie in a finite dimensional vector space corresponding to classical holomorphic modular forms with known levels and weights.

The two inputs allow us to deduce the Jacquet–Rallis fundamental lemma by induction on the dimension of V, at least for $\mathbb{Q}_p$ when $p \geq \dim V$. During the preparation of this paper, the author learned that Beuzart-Plessis [2] has given a purely local proof of the Jacquet–Rallis fundamental lemma for all F/F_0, by induction and using a more precise version (i.e., a local relative trace formula) of the compatibility between the *local* Weil representation (mainly the Fourier transform) and smooth transfer.

Now that we have explained our approach to the FL, let us move to the AFL conjecture. We have indicated that the extra symmetry is the SL_2-modularity of the kernel function, which follows from the Poisson summation formula. In the arithmetic setting, the extra symmetry is a version of the modularity of generating series of special divisors in the arithmetic Chow groups of the integral models of unitary Shimura varieties (e.g. in the recent work of Bruinier–Howard–Kudla–Rapoport–Yang [6]).

To take advantage of the modularity, we consider the semi-Lie algebraic version of the AFL conjecture, which has appeared in Mihatsch's thesis [34, §8] and in Liu's work [31, Conjecture 1.11]. In the semi-Lie algebraic version, we consider the intersection numbers of the Kudla–Rapoport divisors (KR divisors, for short) [25] and the (derived) fixed point locus of an automorphism of the RZ space. We show in §3 that there is an inductive structure similar to the smooth transfer and the fundamental lemma. More precisely, it is possible to reduce the special case when the KR divisor is (formally) smooth to the AFL in one-dimension lower. This is still hardly useful if we only work on the local moduli space. Therefore we introduce a global version of the fixed point locus, called "the derived CM cycle", or "the fat big CM cycle", being a "thickened" variant of the "big CM cycle" in the work of Bruinier–Kudla–Yang [8] and Howard [19]. The naively defined CM cycle may have dimension larger than expected. However, we note that it is a union of connected components of the fixed point locus of a Hecke correspondence (over the integral model), cf. §7.5. Therefore there is a natural derived structure on the naive CM cycle, and the derived CM cycle has virtue dimension one.

By the modularity of generating series of special divisors mentioned above, we obtain a modular form (with known level and weight) by taking the (arithmetic) intersection numbers (cf. (9.4)) of a fixed (derived) CM cycle with special divisors, cf. §9.2. The rest is then similar to the proof of the FL conjecture. The resulting modular form is the arithmetic analog of (1.1). By induction, together with a special case of the AFL (cf. Prop. 3.10), one may assume that the ξ-th Fourier coefficients are known if ξ is prime to a certain finite set of places. The desired equality for *all* Fourier coefficients then follows from the modularity of the generating series and a density principle for the Fourier coefficients of holomorphic modular forms (essentially the newform theory, or more precisely, the classical "Ihara lemma" over complex numbers, cf. Lemma 13.6). Finally, one deduces the AFL conjecture from the global identity, together with a local constancy property of the intersection numbers on RZ spaces, cf. Theorem 5.5.

In our approach, it is important to understand the archimedean local intersection (i.e., the values of Green's function, cf. §10), and correspondingly the derivatives of the archimedean orbital integrals for Gaussian test functions (cf. §12). After subtracting the archimedean terms, the intersection numbers and derivative of orbital integrals at non-archimedean places all lie in $\mathbb{Q}$-linear span of $\log p$ for a finite set of primes p. One can then separate the contribution from different primes by the linear independence of logarithms of prime numbers.

We have restricted the paper to the case $F_0 = \mathbb{Q}$ since in a few places there are missing ingredients in the literature and some of them are subtle. However, we have tried to present most of the arguments in the general totally real field case, especially in the analytic side of RTF.

We would like to point out some earlier works related to the AFL conjecture. The author proved the AFL for low ranks of the unitary group ($n = 2$ and 3) in [47]. Rapoport, Terstiege and the author [42] proved it for arbitrary rank $n \leq p$ and *minuscule* group elements. A Lie algebraic version (in the case of artinian intersection) was studied by Mihatsch in [33, 34], simplifying the proof and generalizing the result in [47]. Finally, in the minuscule case, Li and Zhu in [28] have given a simplified proof of [42]; recently, He, Li, and Zhu [18] have also removed the restriction on the residue characteristics.

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1.2. **Notation.**

Notation on algebra.

- $\mathbb{R}_+$: the set of positive real numbers.
- Let F be a field of character zero. For a reductive group H acting on an affine variety X, we say that a point $x \in X(F)$ is
  - H-semisimple if Hx is Zariski closed in X (when F is a local field, equivalently, $H(F)x$ is closed in $X(F)$ for the analytic topology);
  - H-regular if the stabilizer H_x of x has trivial stabilizer.

  And we say that x is *regular semisimple* if it is regular and semisimple. We denote by $X(F)_{\rm rs}$ the set of regular semisimple elements, and $[X(F)]_{\rm rs}$ the set of regular semisimple $H(F)$-orbits.

- For global fields, unless otherwise stated, F denotes a CM number field and F_0 denotes its (maximal) totally real subfield of index 2. We denote by $a \mapsto \overline{a}$ the nontrivial automorphism of F/F_0. Let $F_{0,+}$ (resp., $F_{0,\geq 0}$) the set of totally positive (resp., semi-positive) elements in F_0.
- We use the symbols v and v_0 to denote places of F_0, and w and w_0 to denote places of F. We write $F_{0,v}$ for the v-adic completion of F_0, and we set $F_v := F \otimes_{F_0} F_{0,v}$; thus F_v is isomorphic to $F_{0,v} \times F_{0,v}$ or to a quadratic field extension of $F_{0,v}$ according as v is split or non-split in F. We write $O_{F_{0,v}} \subset F_{0,v}$ for the ring of integers. We use analogous notation for other fields in place of F_0 and other finite places in place of v.
- Unless otherwise stated, we write $\mathbb{A}$, $\mathbb{A}_0$, and $\mathbb{A}_F$ for the adele rings of $\mathbb{Q}$, F_0, and F, respectively. We systematically use a subscript f for the ring of finite adeles, and a superscript p for the adeles away from the prime number p.
- For an abelian scheme A over a locally noetherian scheme S on which the prime number p is invertible, we write $\mathrm{T}_p(A)$ for the p-adic Tate module of A (regarded as a smooth $\mathbb{Z}_p$-sheaf on S) and $\mathrm{V}_p(A) := \mathrm{T}_p(A) \otimes \mathbb{Q}$ for the rational p-adic Tate module (regarded as a smooth $\mathbb{Q}_p$-sheaf on S). When S is a $\mathbb{Z}_{(p)}$-scheme, we similarly write $\widehat{\mathrm{V}}^p(A)$ for the rational prime-to-p Tate module of A. When S is a scheme in characteristic zero, we write $\widehat{\mathrm{V}}(A)$ for the full rational Tate module of A.
- We use a superscript $\circ$ to denote the operation $- \otimes_{\mathbb{Z}} \mathbb{Q}$ on groups of homomorphisms of abelian schemes, so that for example $\mathrm{Hom}^\circ(A, A') := \mathrm{Hom}(A, A') \otimes_{\mathbb{Z}} \mathbb{Q}$.
- All Chow groups and K-groups have $\mathbb{Q}$-coefficients.
- Given a discretely valued field L, we denote the completion of a maximal unramified extension of it by $\breve{L}$.
- We write 1_n for the $n \times n$ identity matrix. Let $\mathrm{M_{n,m}(R)}$ denote the R-module of $n \times m$-matrices with coefficients in a ring R.
- For a vector space V over a field F, a quadratic form $\mathfrak{q} : V \to F$ has an associated symmetric bilinear pairing defined by
$$\langle x, y \rangle = \mathfrak{q}(x+y) - \mathfrak{q}(x) - \mathfrak{q}(y), \quad x, y \in V. \tag{1.2}$$
In particular,
$$\langle x, x \rangle = 2\mathfrak{q}(x). \tag{1.3}$$
For a vector space V over a quadratic extension F of a field F_0, an F/F_0-hermitian pairing $\langle \cdot, \cdot \rangle : V \times V \to F$ induces a symmetric bi-F_0-linear pairing by $(x, y) \mapsto \frac{1}{2} \operatorname{tr}_{F/F_0} \langle x, y \rangle \in F_0$. In particular, the corresponding quadratic form on V (viewed as an F_0-vector space) is
$$\mathfrak{q}(x) = \langle x, x \rangle \in F_0 \tag{1.4}$$
We then denote by V_ξ the set of vectors $x \in V$ with $\mathfrak{q}(x) = \xi$.
- For a F/F_0-hermitian space V over a non-archimedean local field, and an O_F-lattice $\Lambda \subset V$ (of full rank), we denote by $\Lambda^\vee$ its dual lattice under the hermitian form.

Notation on automorphic forms.

- Fix a non-trivial additive character $\psi : F_0 \backslash \mathbb{A}_0 \to \mathbb{C}^\times$. In the case $F_0 = \mathbb{Q}$ we take the standard one.
- For an algebraic variety X over a local field F, we denote $\mathcal{S}(X(F))$ by the space of Schwartz functions on $X(F)$. When F is non-archimedean, this is the same as the space of locally constant functions with compact support. Similarly, for an algebraic variety X over a global field F, we denote $\mathcal{S}(X(\mathbb{A}))$ by the space of Schwartz functions on $X(\mathbb{A})$.

- Congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$

$$\Gamma(N)=\left\{\gamma\in\mathrm{SL}_2(\mathbb{Z})\middle|\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix}\mod N\right\}$$
$$\Gamma_1(N)=\left\{\gamma\in\mathrm{SL}_2(\mathbb{Z})\middle|\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\equiv\begin{pmatrix}*&*\\0&1\end{pmatrix}\mod N\right\}$$
$$\Gamma_0(N)=\left\{\gamma\in\mathrm{SL}_2(\mathbb{Z})\middle|\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\equiv\begin{pmatrix}*&*\\0&*\end{pmatrix}\mod N\right\}$$

- $\mathcal{H}=\{\tau=b+ia\in\mathbb{C}\mid a>0\}$: the complex upper half plane.
- $\mathcal{A}_{\mathrm{hol}}(\Gamma,k)$: the space of holomorphic modular forms of level Γ, weight k, for Γ where $\Gamma(N)\subset\Gamma\subset\mathrm{SL}_2(\mathbb{Z})$. For any subfield $L\subset\mathbb{C}$, we denote by $\mathcal{A}_{\mathrm{hol}}(\Gamma,k)_L$ the L-vector space consisting of $f\in\mathcal{A}_{\mathrm{hol}}(\Gamma,k)$ whose Fourier coefficients in the q-expansion at the cusp ∞ all lie in L. The $\mathbb{C}$-vector space $\mathcal{A}_{\mathrm{hol}}(\Gamma,k)$ has a $\mathbb{Q}$-structure via the q-expansion at the cusp ∞, i.e., $\mathcal{A}_{\mathrm{hol}}(\Gamma,k)=\mathcal{A}_{\mathrm{hol}}(\Gamma,k)_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{C}$. For any L-vector space W, we have an L-vector space
$$\mathcal{A}_{\mathrm{hol}}(\Gamma,k)_L\otimes_L W. \tag{1.5}$$
We will view this vector space as the space of formal power series in $q^{1/N}$ with coefficients in W
$$\sum_{\xi\geq 0,\xi\in\frac{1}{N}\mathbb{Z}}A_\xi q^\xi,\quad A_\xi\in W$$
where there exist elements $f_i\in\mathcal{A}_{\mathrm{hol}}(\Gamma,k)_L$ indexed by a finite set I whose q-expansion at the cusp ∞ are given by $\sum_{\xi\geq 0,\xi\in\frac{1}{N}\mathbb{Z}}a_\xi(f_i)q^\xi\in L[\![q^{1/N}]\!]$, and elements $w_i\in W, i\in I$, such that
$$A_\xi=\sum_{i\in I}a_\xi(f_i)w_i,\quad\text{for all }\xi.$$
- $\mathcal{A}_{\mathrm{hol}}(\mathrm{SL}_2(\mathbb{A}_0),K,k)$: the space of automorphic forms (with moderate growth) on $\mathrm{SL}_2(\mathbb{A}_0)$, invariant under $K\subset\mathrm{SL}_2(\mathbb{A}_f)$, and parallel weight k under the action of $\prod_{v\in\mathrm{Hom}(F_0,\mathbb{R})}\mathrm{SO}(2,\mathbb{R})$, holomorphic (i.e., annihilated by the element $\frac{1}{2}\begin{pmatrix}i&1\\1&-i\end{pmatrix}$ in the complexifed Lie algebra of $\mathrm{SL}_2(F_{0,v})\simeq\mathrm{SL}_2(\mathbb{R})$ for every $v\in\mathrm{Hom}(F_0,\mathbb{R})$). This is a finite dimensional vector space over $\mathbb{C}$, and it has a basis over $\mathbb{Q}$ via the q-expansion at the cusp ∞. For any L-vector space W, we may define
$$\mathcal{A}_{\mathrm{hol}}(\mathrm{SL}_2(\mathbb{A}_0),K,k)_L\otimes_L W, \tag{1.6}$$
similar to $\mathcal{A}_{\mathrm{hol}}(\Gamma,k)_L\otimes_L W$ as above.
- To a function $\phi\in\mathcal{A}_{\mathrm{hol}}(\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}),K,k)$, and $h_f\in\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q},f})$, we associate a function
$$\phi^\flat_{h_f}\in\mathcal{A}_{\mathrm{hol}}(\Gamma,k)$$
where $\Gamma=h_fKh_f^{-1}\cap\mathrm{SL}_2(\mathbb{Q})$, by
$$b+ai\in\mathcal{H}\longmapsto|a|^{-k/2}\phi(h_\infty,h_f), \tag{1.7}$$
where $h_\infty=\begin{pmatrix}1&b\\&1\end{pmatrix}\begin{pmatrix}a^{1/2}&\\&a^{-1/2}\end{pmatrix}$. When $h_f=1$, we simply write it as $\phi^\flat$.
- $\mathcal{A}_{\mathrm{exp}}(\mathrm{SL}_2(\mathbb{A}_0),K,k)$ when $F_0=\mathbb{Q}$: this is essentially the space $\mathcal{A}^!_k(\rho_L^\vee)$ in [9]. This is an infinite dimensional vector space over $\mathbb{C}$.
- For $\xi\in\mathbb{R}$ and $k\in\mathbb{Z}$, the weight-k Whittaker function on $\mathrm{SL}_2(\mathbb{R})$ is defined by
$$W^{(k)}_\xi(h)=|a|^{k/2}e^{2\pi i\xi(b+ai)}\chi_k(\kappa_\theta), \tag{1.8}$$
where we write $h\in\mathrm{SL}_2(\mathbb{R})$ according to the Iwasawa decomposition
$$h=\begin{pmatrix}1&b\\&1\end{pmatrix}\begin{pmatrix}a^{1/2}&\\&a^{-1/2}\end{pmatrix}\kappa_\theta,\quad a\in\mathbb{R}_+,\quad b\in\mathbb{R}, \tag{1.9}$$

and

$$\kappa(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(2, \mathbb{R}). \tag{1.10}$$

Here the weight k-character of $\mathrm{SO}(2, \mathbb{R})$, for $k \in \mathbb{Z}$, is defined by

$$\chi_k(\kappa_\theta) = e^{ik\theta}. \tag{1.11}$$

Part 1. Local theory

2. FL and variants

2.1. **Group-theoretic setup.** Let F_0 be a field, and F a quadratic semisimple F_0-algebra.

Let

$$e := (0, \dots, 0, 1) \in \mathrm{M}_{n,1}(F) = F^n,$$

be a column vector, and $e^* \in \mathrm{M}_{1,n}(F) \simeq \mathrm{M}_{n,1}(F)^* = (F^n)^*$ the transpose of e. Consider the embedding of algebraic groups over F,

$$\begin{array}{ccc} \mathrm{GL}_{n-1} & \longrightarrow & \mathrm{GL}_n \\ \gamma_0 & \longmapsto & \mathrm{diag}(\gamma_0, 1) \end{array}; \tag{2.1}$$

this identifies GL_{n-1} with the subgroup of points γ in GL_n such that $\gamma e = e$ and $e^*\gamma = e^*$.

We introduce the algebraic group G' over F_0 and its subgroups,

$$\begin{aligned} G' &:= \mathrm{Res}_{F/F_0}(\mathrm{GL}_{n-1} \times \mathrm{GL}_n), \\ H'_1 &:= \mathrm{Res}_{F/F_0} \mathrm{GL}_{n-1}, \\ H'_2 &:= \mathrm{GL}_{n-1} \times \mathrm{GL}_n. \end{aligned}$$

Here H'_1 is embedded diagonally, and H'_2 is embedded in the obvious way. We consider the natural right action of $H'_1 \times H'_2$ on G',

$$(h_1, h_2) \cdot \gamma = h_1^{-1} \gamma h_2.$$

Consider the symmetric space

$$S := S_n := \{\, g \in \mathrm{Res}_{F/F_0} \mathrm{GL}_n \mid g\overline{g} = 1_n \,\}, \tag{2.2}$$

and its tangent space at 1_n, called "the Lie algebra" of S_n,

$$\mathfrak{s} := \mathfrak{s}_n := \big\{\, y \in \mathrm{Res}_{F/F_0} \mathrm{M}_n \bigm| y + \overline{y} = 0 \,\big\}. \tag{2.3}$$

Set

$$H' := \mathrm{GL}_{n-1}.$$

Then H' acts on S_n and $\mathfrak{s}_n$ by conjugation

$$h \cdot \gamma = h^{-1}\gamma h.$$

We also consider a variant (arising from the Fourier–Jacobi period [10, 30]). Let

$$V'_{n-1} = F_0^{n-1} \times (F_0^{n-1})^*, \tag{2.4}$$

and consider the (diagonal) action of H' on the product $S_{n-1} \times V'_{n-1}$,

$$h \cdot (\gamma, (u_1, u_2)) = (h^{-1}\gamma h, (h^{-1}u_1, u_2 h)).$$

The action of H' on the its Lie algebra $\mathfrak{s}_{n-1} \times V'_{n-1}$ is defined similarly.

Next let $V^\sharp$ be an F/F_0-hermitian space of dimension $n \geq 2$. We fix a non-isotropic vector $u_0 \in V^\sharp$, which we call the *special vector*. We denote by V the orthogonal complement of u_0 in $V^\sharp$. We define the algebraic group G over F_0 and its subgroups,

$$\begin{aligned} G &:= \mathrm{U}(V^\sharp), \\ H &:= \mathrm{U}(V), \\ G_V &:= H \times G. \end{aligned} \tag{2.5}$$

We have the natural action of $H \times H$ on G_V, and the conjugation action of H on G. We also consider the adjoint action of H on the Lie algebra $\mathfrak{g} = \mathfrak{u}(V^\sharp)$ of G. When $\dim V = 1$, the Lie algebra $\mathfrak{u}(\mathbb{V})$ is denoted by $\mathfrak{u}(1)$, which is canonically isomorphic to F^-, the (-1)-eigenspace of F under the Galois conjugation.

The variant (arising from the RTF for the Fourier–Jacobi period [30]) is the (diagonal) action of $H = \mathrm{U}(V)$ on the product $\mathrm{U}(V) \times V$ and on the product $\mathfrak{u}(V) \times V$ defined similarly. Here the factor V is viewed as a vector space (affine space) of dimension $2(n-1)$.

2.2. **Orbit matching.** There is a natural bijection of orbit spaces of *regular semisimple* elements,

$$\coprod_V \left[(\mathrm{U}(V^\sharp)(F_0)\right]_{\mathrm{rs}} \xrightarrow{\ \sim\ } \left[S_n(F_0)\right]_{\mathrm{rs}}, \tag{2.6}$$

and

$$\coprod_V \left[(\mathrm{U}(V) \times V)(F_0)\right]_{\mathrm{rs}} \xrightarrow{\ \sim\ } [(S_{n-1} \times V'_{n-1})(F_0)]_{\mathrm{rs}}, \tag{2.7}$$

cf. [47] and [30], where the disjoint union runs over the set of isometric classes of F/F_0-hermitian spaces V, and the larger space $V^\sharp = V \oplus F \cdot u_0$ is then determined uniquely by demanding the special vector u_0 to have norm one (or any fixed number in $F_0^\times$ when varying V). Here the left (resp. right) hand sides denote the orbits under the action of the group $\mathrm{U}(V)$ (resp. GL_{n-1}). The bijections define a *matching relation* between regular semisimple orbits. In both cases, there are also similar injections for orbits on the Lie algebras:

$$\coprod_V \left[(\mathfrak{u}(V^\sharp)(F_0)\right]_{\mathrm{rs}} \xrightarrow{\ \sim\ } \left[\mathfrak{s}_n(F_0)\right]_{\mathrm{rs}}, \tag{2.8}$$

and

$$\coprod_V \left[(\mathfrak{u}(V) \times V)(F_0)\right]_{\mathrm{rs}} \xrightarrow{\ \sim\ } [(\mathfrak{s}_{n-1} \times V'_{n-1})(F_0)]_{\mathrm{rs}}. \tag{2.9}$$

We recall how the map (2.7) is defined. Choose an F-basis for V and complete it to a basis for $V^\sharp$ by adjoining u_0. This identifies V with F^{n-1} and $V^\sharp$ with F^n in such a way that u_0 corresponds to the column vector $e := (0, \dots, 0, 1)$ in F^n, and hence determines embeddings of groups $\mathrm{U}(V^\sharp) \hookrightarrow \mathrm{Res}_{F/F_0} \mathrm{GL}_n$. An element $g \in \mathrm{U}(V)(F_0)_{\mathrm{rs}}$ and an element $\gamma \in S_n(F_0)_{\mathrm{rs}}$ are said to *match* if these two elements, when considered as elements in $\mathrm{Res}_{F/F_0} \mathrm{GL}_n(F_0)$, are conjugate under $\mathrm{Res}_{F/F_0} \mathrm{GL}_{n-1}$. The matching relation is independent of the choice of embeddings and induces a bijection [47, §2]. Similarly, we view elements in $(S_{n-1} \times V'_{n-1})(F_0)$ as elements in $\mathrm{Res}_{F/F_0} \mathrm{M}_{n,n}(F_0)$ by

$$(\gamma, (u_1, u_2)) \longmapsto \begin{pmatrix} \gamma & u_1 \\ u_2 & 0 \end{pmatrix}.$$

And we view elements $(g, u) \in (\mathrm{U}(V) \times V)(F_0)$ as elements in $\mathrm{Res}_{F/F_0} \mathrm{M}_{n,n}(F_0)$

$$(g, u) \longmapsto \begin{pmatrix} g & u \\ u^* & 0 \end{pmatrix}.$$

Here we view $u \in V(F_0)$ as the corresponding element in $\mathrm{Hom}(V^\perp, V)$ sending $u_0 \in V^\perp = F \cdot u_0$ to u, and u^* is the element in $\mathrm{Hom}(V, V^\perp) = \mathrm{Hom}(V, F \cdot u_0)$ defined by $u' \mapsto \langle u', u\rangle u_0$. Then, an element $(g, u) \in (\mathrm{U}(V) \times V)(F_0)_{\mathrm{rs}}$ and an element $(\gamma, (u_1, u_2)) \in \left(S_{n-1}(F_0) \times F_0^{n-1} \times (F_0^{n-1})^*\right)_{\mathrm{rs}}$ are said to *match* if these two elements, when considered as elements in $\mathrm{Res}_{F/F_0} \mathrm{M}_{n,n}(F_0)$, are conjugate under $\mathrm{Res}_{F/F_0} \mathrm{GL}_{n-1}$.

Equivalently, $(g, u) \in (\mathrm{U}(V) \times V)(F_0)_{\mathrm{rs}}$ matches $(\gamma, (u_1, u_2)) \in \left(S_{n-1}(F_0) \times F_0^{n-1} \times (F_0^{n-1})^*\right)_{\mathrm{rs}}$ if and only if the following invariants are equal

$$\det(\lambda\, \mathbf{1}_{n-1} + g) = \det(\lambda\, \mathbf{1}_{n-1} + \gamma), \quad \text{and} \quad \langle g^u, u\rangle = u_2 \gamma^i u_1, \quad 0 \leq i \leq n-1.$$

Here $\det(\lambda\, \mathbf{1}_{n-1} + g) \in F[\lambda]$ is the characteristic polynomial of g.

2.3. **Orbital integral matching: smooth transfer.** We recall orbital integrals [39, §2.2]. Now let F/F_0 be a quadratic extension of local fields (the split $F = F_0 \times F_0$ is similar and simpler). To simplify the exposition we consider the non-archimedean case, though the archimedean case requires very little change. Then there are exactly two isometric classes of F/F_0-hermitian spaces of dimension $n-1$, denoted by V_0 and V_1. When F/F_0 is unramified, we will assume that V_0 has a self-dual lattice. Then the orbit bijections are now

$$\big[(\mathrm{U}(V_0^\sharp)(F_0)\big]_{\mathrm{rs}} \coprod \big[(\mathrm{U}(V_1^\sharp)(F_0)\big]_{\mathrm{rs}} \xrightarrow{\sim} \big[S_n(F_0)\big]_{\mathrm{rs}},$$

and

$$\big[(\mathrm{U}(V_0)\times V_0)(F_0)\big]_{\mathrm{rs}} \coprod \big[(\mathrm{U}(V_1)\times V_1)(F_0)\big]_{\mathrm{rs}} \xrightarrow{\sim} [(S_{n-1}\times V'_{n-1})(F_0)]_{\mathrm{rs}}.$$

For $\gamma \in S_n(F_0)_{\mathrm{rs}}$, $f' \in \mathcal{S}(S_n(F_0))$, and $s \in \mathbb{C}$, we define

$$\mathrm{Orb}(\gamma, f', s) := \int_{\mathrm{GL}_{n-1}(F_0)} f'(h^{-1}\gamma h)|\det h|^s \eta(h)\, dh, \tag{2.10}$$

where $|\ |$ denotes the normalized absolute value on F_0, where we set

$$\eta(h) := \eta(\det h).$$

We define the special values

$$\mathrm{Orb}(\gamma, f') := \omega(\gamma)\,\mathrm{Orb}(\gamma, f', 0) \quad \text{and} \quad \partial\mathrm{Orb}(\gamma, f') := \omega(\gamma)\,\frac{d}{ds}\Big|_{s=0}\mathrm{Orb}(\gamma, f', s), \tag{2.11}$$

where the transfer factor $\omega(\gamma)$ is to be explicated below by (2.14). Here, we have included the transfer factor in the special values of the orbital integrals, different from [39, §2.2].

For $(\gamma, u') \in (S_{n-1}\times V'_{n-1})_{\mathrm{rs}}(F_0)$, $\Phi' \in \mathcal{S}((S_{n-1}\times V'_{n-1})(F_0))$, and $s \in \mathbb{C}$, we define

$$\mathrm{Orb}((\gamma, u'), \Phi', s) := \int_{\mathrm{GL}_{n-1}(F_0)} \Phi'(h\cdot(\gamma, u'))|\det h|^s \eta(h)\, dh, \tag{2.12}$$

and define their special values similar to (2.11), replacing the transfer factor $\omega(\gamma)$ by $\omega(\gamma, u')$ to be explicated below by (2.15).

On the unitary side, for $g \in \mathrm{U}(V^\sharp)(F_0)_{\mathrm{rs}}$ and $f \in \mathcal{S}(\mathrm{U}(V^\sharp)(F_0))$, we define

$$\mathrm{Orb}(g, f) := \int_{\mathrm{U}(V)(F_0)} f(h^{-1}gh)\, dh.$$

For $(g, u) \in (\mathrm{U}(V)\times V)(F_0)_{\mathrm{rs}}$ and $\Phi \in \mathcal{S}((\mathrm{U}(V)\times V)(F_0))$, we define

$$\mathrm{Orb}((g,u), \Phi) := \int_{\mathrm{U}(V)(F_0)} \Phi(h\cdot(g,u))\, dh. \tag{2.13}$$

Finally, we define an explicit transfer factors, cf. [39, §2.4]. First fix an extension $\widetilde{\eta}$ of the quadratic character η from $F_0^\times$ to $F^\times$ (not necessarily of order 2). If F is unramified, then we take the natural extension $\widetilde{\eta}(x) = (-1)^{v(x)}$. For S_n, we take the transfer factor

$$\omega(\gamma) := \widetilde{\eta}\big(\det(\gamma)^{-\lfloor n/2\rfloor}\det(\gamma^i e)_{0\le i\le n-1}\big), \quad \gamma \in S_n(F_0)_{\mathrm{rs}}. \tag{2.14}$$

For $(\gamma, u') \in \big(S_{n-1}\times V'_{n-1}\big)(F_0)_{\mathrm{rs}}$ where $u' = (u_1, u_2) \in V_{n-1}(F_0) = F_0^{n-1}\times (F_0^{n-1})^*$, we take

$$\omega(\gamma, u') := \widetilde{\eta}\big(\det(\gamma)^{-\lfloor (n-1)/2\rfloor}\det(\gamma^i u_1)_{0\le i\le n-2}\big). \tag{2.15}$$

Similarly we define transfer factors on $\mathfrak{s}_n$ and $\mathfrak{s}_{n-1}\times V'_{n-1}$.

Definition 2.1. A function $f' \in \mathcal{S}(S_n(F_0))$ and a pair of functions $(f_0, f_1) \in \mathcal{S}(\mathrm{U}(V_0^\sharp)(F_0)) \times \mathcal{S}(\mathrm{U}(V_1^\sharp)(F_0))$ are *transfers* of each other if for each $i \in \{0,1\}$ and each $g \in \mathrm{U}(V_i^\sharp)(F_0)_{\mathrm{rs}}$,

$$\mathrm{Orb}(g, f_i) = \mathrm{Orb}(\gamma, f')$$

whenever $\gamma \in S(F_0)_{\mathrm{rs}}$ matches g.

Definition 2.2. A function $\Phi' \in \mathcal{S}((S_{n-1} \times V_{n-1})(F_0))$ and a pair of functions $(\Phi_0, \Phi_1) \in \mathcal{S}((\mathrm{U}(V_0) \times V_0)(F_0)) \times \mathcal{S}((\mathrm{U}(V_1) \times V_1)(F_0))$ are *transfers* of each other if for each $i \in \{0, 1\}$ and each $(g, u) \in (\mathrm{U}(V_i) \times V_i)(F_0)_{\mathrm{rs}}$,

$$\mathrm{Orb}((g, u), \Phi_i) = \mathrm{Orb}((\gamma, u'), \Phi') \tag{2.16}$$

whenever $(\gamma, u') \in \left(S_{n-1} \times V'_{n-1}\right)(F_0)_{\mathrm{rs}}$ matches (g, u).

Remark 2.3. For the archimedean F/F_0, we need to consider all isometric classes of F/F_0-hermitian spaces V of the same fixed dimension.

The definitions made above easily extend verbatim to the setting of the full Lie algebras $\mathfrak{u}(V) \times V$ and $\mathfrak{s}_{n-1} \times V'_{n-1}$. Finally, we remark that the definitions extend to the archimedean local field extension $F/F_0 = \mathbb{C}/\mathbb{R}$, where one only needs to replace the pair of functions (Φ_0, Φ_1) by a tuple of functions $\{\Phi_V\}_V$ indexed by the set of isometric classes of F/F_0-hermitian spaces V, as in (2.7) and (2.9). We will not repeat the detail here.

2.4. **Review of the FL conjecture.** We review the FL conjecture, cf. [21, 47, 39]. Let F/F_0 be an unramified quadratic extension of p-adic field for an *odd* prime p. Assume furthermore that the special vectors $u_i \in V_i$ have norm one (or any fixed unit in O_{F_0}). Then the hermitian space $V_i^\sharp$ is again split for $i = 0$ and non-split for $i = 1$. We write $G_i = \mathrm{U}(V_i^\sharp)$, $\mathfrak{g}_i = \mathrm{Lie}\, G_i$, and $H_i = \mathrm{U}(V_i)$. Fix a self-dual O_F-lattice

$$\Lambda_0 \subset V_0,$$

which exists and is unique up to $H_0(F_0)$-conjugacy. Let

$$\Lambda_0^\sharp := \Lambda_0 \oplus O_F u_0 \subset V_0^\sharp,$$

which is again self-dual. We denote by

$$K_0 \subset H_0(F_0)$$

the stabilizer of Λ_0, and by

$$K_0^\sharp \subset G_0(F_0) \quad \text{and} \quad \mathfrak{k}_0^\sharp \subset \mathfrak{g}_0(F_0)$$

the respective stabilizers of Λ_0. Then K_0 and $K_0^\sharp$ are both hyperspecial maximal subgroups.

We normalize the Haar measures on the groups

$$\mathrm{GL}_{n-1}(F_0), \quad \text{and} \quad \mathrm{U}(V_0)(F_0)$$

by assigning each of the respective subgroups

$$\mathrm{GL}_{n-1}(O_{F_0}), \quad \text{and} \quad K_0$$

measure one.

With respect to these normalizations, the Jacquet–Rallis Fundamental lemma conjecture is the following statement, cf. [39, §3]. Note that the semi-Lie algebra version below is essentially the Fourier–Jacobi case arising from the relative trace formula of Yifeng Liu [30].

Conjecture 2.4 (Jacquet–Rallis Fundamental lemma conjecture)**.**

(a) (The group version) *The characteristic function* $\mathbf{1}_{S_n(O_{F_0})} \in \mathcal{S}(S_n(F_0))$ *transfers to the pair of functions* $(\mathbf{1}_{K_0}, 0) \in \mathcal{S}(G_0(F_0)) \times \mathcal{S}(G_1(F_0))$.

(b) (The Lie algebra version) *The characteristic function* $\mathbf{1}_{\mathfrak{s}_n(O_{F_0})} \in \mathcal{S}(\mathfrak{s}_n(F_0))$ *transfers to the pair of functions* $(\mathbf{1}_{\mathfrak{k}_0}, 0) \in \mathcal{S}(\mathfrak{g}_0(F_0)) \times \mathcal{S}(\mathfrak{g}_1(F_0))$.

(c) (The semi-Lie algebra version) *The characteristic function* $\mathbf{1}_{(S_{n-1} \times V'_{n-1})(O_{F_0}))} \in \mathcal{S}((S_{n-1} \times V'_{n-1})(F_0))$ *transfers to the pair of functions* $(\mathbf{1}_{(G_0 \times V_0)(O_{F_0})}, 0) \in \mathcal{S}((G_0 \times V_0)(F_0)) \times \mathcal{S}((G_1 \times V_1)(F_0))$.

Remark 2.5. We note that the equal characteristic analog of the FL conjecture was proved by Z. Yun for $p > n$, cf. [46]; J. Gordon deduced the p-adic case for p large, but unspecified, cf. [14].

It is straightforward to check a special case.

Proposition 2.6. *The semi-Lie algebra version FL holds for $(g, u) \in (G_0 \times V_0)(F_0)_{\mathrm{rs}}$ when g is regular semisimple (i.e. $F[g]$ is a product of fields with total degree equal to $\dim V$) and generates a maximal order $O_F[g]$ (in $F[g]$).*

Proof. This is easy to check, e.g., [46, Lemma 2.5.5] for the Lie algebra version; but the argument is the same for the semi-Lie algebra version. □

Proposition 2.7. *Fix F/F_0. Assume that $q \geq n$ where q denotes the cardinality of the residue field of O_{F_0}. Then*

(i) In Conjecture 2.4, all three parts are equivalent.

(ii) In Conjecture 2.4, part (a) *for S_n implies part* (c) *when $\dim V_0 = n$ and, in the regular semisimple orbit $(g, u) \in G_0 \times V_0$, the norm of u is a unit.*

Proof. The fact that part (a) implies part (b) was shown in [30, Thm 5.15] using Cayley map under the assumption $q \geq n$; the reverse implication can be shown by the same argument.

For the other assertions, since we will prove a similar statement for the AFL conjecture where the situation is more delicate, we omit the argument here and only point out that the proof of Proposition 4.12 also works here.

□

3. AFL and variants

3.1. **The AFL conjecture and variants.** For any $n \geq 1$, we recall the construction of the Rapoport–Zink formal moduli scheme $\mathcal{N}_n = \mathcal{N}_{n,F/F_0}$ associated to unitary groups, cf. [39, §4]. For $\operatorname{Spf} O_{\breve{F}}$-schemes S, we consider triples (X, ι, λ), where

- X is a p-divisible group of absolute height $2nd$ and dimension n over S,
- ι is an action of O_F such that the induced action of O_{F_0} on $\operatorname{Lie} X$ is via the structure morphism $O_{F_0} \to \mathcal{O}_S$, and
- λ is a principal (O_{F_0}-relative) polarization.

Here $d := [F_0 : \mathbb{Q}_p]$. Hence $(X, \iota|_{O_{F_0}})$ is a formal O_{F_0}-module of relative height $2n$ and dimension n. We require that the Rosati involution Ros_λ on O_F is the non-trivial Galois automorphism in $\mathrm{Gal}(F/F_0)$, and that the *Kottwitz condition* of signature $(n-1, 1)$ is satisfied, i.e.

$$\operatorname{char}\big(\iota(a) \mid \operatorname{Lie} X\big) = (T - a)^{n-1}(T - \overline{a}) \in \mathcal{O}_S[T] \quad \text{for all} \quad a \in O_F. \tag{3.1}$$

An isomorphism $(X, \iota, \lambda) \xrightarrow{\sim} (X', \iota', \lambda')$ between two such triples is an O_F-linear isomorphism $\varphi\colon X \xrightarrow{\sim} X'$ such that $\varphi^*(\lambda') = \lambda$.

Over the residue field $\overline{k}$ of $O_{\breve{F}}$ there is a unique such triple $(\mathbb{X}_n, \iota_{\mathbb{X}_n}, \lambda_{\mathbb{X}_n})$ such that $\mathbb{X}_n$ is supersingular, up to O_F-linear quasi-isogeny compatible with the polarization. Then $\mathcal{N}_n$ represents the functor over $\operatorname{Spf} O_{\breve{F}}$ that associates to each S the set of isomorphism classes of quadruples $(X, \iota, \lambda, \rho)$ over S, where the final entry is an O_F-linear quasi-isogeny of height zero defined over the special fiber,

$$\rho\colon X \times_S \overline{S} \longrightarrow \mathbb{X}_n \times_{\operatorname{Spec} \overline{k}} \overline{S},$$

such that $\rho^*((\lambda_{\mathbb{X}_n})_{\overline{S}}) = \lambda_{\overline{S}}$ (a *framing*). The formal scheme $\mathcal{N}_n$ is smooth over $\operatorname{Spf} O_{\breve{F}}$ of relative dimension $n-1$.

For $n \geq 2$, define the product $\mathcal{N}_{n-1,n} := \mathcal{N}_{n-1} \times_{\operatorname{Spf} O_{\breve{F}}} \mathcal{N}_n$. It is a (locally Noetherian) formal scheme of (formal) dimension $2(n-1)$, formally smooth over $\operatorname{Spf} O_{\breve{F}}$.

When $n = 1$, we have the (unique up to isomorphism) formal O_F-module $\mathbb{E}$ (with signature $(1, 0)$) over $\overline{k}$ and its canonical lift $\mathcal{E}$ over $O_{\breve{F}}$, as well as the "conjugate" objects $\overline{\mathbb{E}}$ and $\overline{\mathcal{E}}$ (with signature $(0, 1)$). For $n \geq 2$, there is a natural closed embedding of formal schemes

$$\begin{aligned} \delta_{\mathcal{N}}\colon \qquad \mathcal{N}_{n-1} &\longrightarrow \mathcal{N}_n \\ (X, \iota, \lambda, \rho) &\longmapsto \big(X \times \mathcal{E}, \iota \times \iota_{\mathcal{E}}, \lambda \times \lambda_{\mathcal{E}}, \rho \times \rho_{\mathcal{E}}\big), \end{aligned}$$

where we set $\mathbb{X}_1 = \overline{\mathbb{E}}$ and inductively take

$$\mathbb{X}_n = \mathbb{X}_{n-1} \times \mathbb{E} \tag{3.2}$$

as the framing object for $\mathcal{N}_n$. Let

$$\Delta_{\mathcal{N}}\colon \mathcal{N}_{n-1} \xrightarrow{(\mathrm{id}_{\mathcal{N}_{n-1}},\delta_{\mathcal{N}})} \mathcal{N}_{n-1}\times_{\operatorname{Spf} O_{\breve F}} \mathcal{N}_n = \mathcal{N}_{n-1,n}$$

be the graph morphism of $\delta_{\mathcal{N}}$. Then

$$\Delta := \Delta_{\mathcal{N}}(\mathcal{N}_{n-1})$$

is a closed formal subscheme of half the formal dimension of $\mathcal{N}_{n-1,n}$. Note that

$$\mathrm{Aut}^\circ(\mathbb{X}_n,\iota_{\mathbb{X}_n},\lambda_{\mathbb{X}_n}) \cong \mathrm{U}\big(\mathbb{V}_n\big)(F_0), \tag{3.3}$$

where the left-hand side is the group of self-framings of $\mathbb{X}_n$, and where $\mathbb{V}_n$ is the hermitian space attached to $\mathbb{X}_n$ induced by the principle polarization

$$\mathbb{V}_n = \mathrm{Hom}^\circ_{O_F}(\mathbb{E},\mathbb{X}_n).$$

More concretely

$$\mathrm{U}\big(\mathbb{V}_n\big)(F_0) = \{g \in \mathrm{End}_F(\mathbb{V}_n) \mid gg^* = \mathrm{id}\}\,.$$

Here we denote by $g^* = \mathrm{Ros}_{\lambda_{\mathbb{X}_n}}(g)$ the Rosati involution. Then the group $\mathrm{U}\big(\mathbb{V}_n\big)(F_0)$ acts naturally on $\mathcal{N}_n$ by acting on the framing:

$$g\cdot(X,\iota,\lambda,\rho) = (X,\iota,\lambda,g\circ\rho).$$

Furthermore $\mathbb{V}_n$ contains a natural special vector u_0 given by the inclusion of $\mathbb{E}$ in $\mathbb{X}_n = \mathbb{X}_{n-1}\times\mathbb{E}$ via the second factor. The norm of u_0 is 1. Then $\mathbb{V}_n$ is a non-split hermitian space of dimension n. Therefore, in the setting of §2.3, we can choose an identifications $V_1^\sharp = \mathbb{V}_n$ and $V_1 = \mathbb{V}_{n-1}$ compatible with the natural inclusions on both sides. Hence we obtain an action of $H_1(F_0)$ on $\mathcal{N}_{n-1}$, of $G_1(F_0)$ on $\mathcal{N}_n$, and of $G_{V_1}(F_0)$ on $\mathcal{N}_{n-1,n}$, cf. (2.5); and furthermore the maps $\delta_{\mathcal{N}}$ and $\Delta_{\mathcal{N}}$ are equivariant with respect to the respective embeddings $H_1(F_0)\hookrightarrow G_1(F_0)$ and $H_1(F_0)\hookrightarrow G_{V_1}(F_0)$.

For $g\in G_{V_1}(F_0)_{\mathrm{rs}}$, we denote by $\mathrm{Int}(g)$ the intersection product on $\mathcal{N}_{n-1,n}$ of Δ with its translate $g\Delta$, defined through the derived tensor product of the structure sheaves, cf. (B.3),

$$\mathrm{Int}(g) := \langle\Delta, g\Delta\rangle_{\mathcal{N}_{n-1,n}} := \chi(\mathcal{N}_{n-1,n}, \mathcal{O}_\Delta\otimes^{\mathbb{L}}\mathcal{O}_{g\Delta}). \tag{3.4}$$

We similarly define $\mathrm{Int}(g)$ for $g\in G_1(F_0)_{\mathrm{rs}}$,

$$\mathrm{Int}(g) := \big\langle\Delta,(1\times g)\Delta\big\rangle_{\mathcal{N}_{n-1,n}}. \tag{3.5}$$

In both cases, when g is regular semisimple, the right-hand side of this definition is finite since the (formal) schematic intersection $\Delta\cap g\Delta$ is a proper *scheme* over $\operatorname{Spf} O_{\breve F}$. We refer to the appendix B for the terminology regarding various K-groups of formal schems, following the work of Gillet–Soulé for schemes in [12].

Now we introduce a new variant of the above intersection number $\mathrm{Int}(g)$ via the Kudla–Rapoport special divisors [25]. This variant is closely related to in the AFL conjecture in the context of Fourier–Jacobi cycles in the work of Yifeng Liu [31, Conjecture 1.11]. A special case has also appeared in Mihatsch's thesis [34, §8].

Recall from [25], for every non-zero $u\in\mathbb{V}_n$, Kudla and Rapoport have defined a special divisor $\mathcal{Z}(u)$ in $\mathcal{N}_n$. This is the locus where the quasi-homomorphism $u\colon\mathbb{E}\to\mathbb{X}_n$ lifts to a homomorphism from $\mathcal{E}$ to the universal object over $\mathcal{N}_n$. By [25, Prop. 3.5], $\mathcal{Z}(u)$ is a relative divisor (or empty) whenever $u\neq 0$. Then $\delta_{\mathcal{N}}$ induces an obvious closed embedding

$$\mathcal{N}_{n-1}\xrightarrow{\ \sim\ }\mathcal{Z}(u_0)$$

for the unit norm special vector u_0, which is an isomorphism by [25, Lem. 5.2]. It follows from the definition that if $g\in\mathrm{U}(\mathbb{V}_n)(F_0)$, then

$$g\mathcal{Z}(u) = \mathcal{Z}(gu). \tag{3.6}$$

Every $g\in\mathrm{U}(\mathbb{V}_n)$ induces an automorphism of $\mathcal{N}_n$. Let $\Gamma_g\subset\mathcal{N}_n\times_{\operatorname{Spf} O_{\breve F}}\mathcal{N}_n$ be the graph and define the (naive) fixed point locus $\mathcal{N}_n^g$ as the (formal) schematic intersection (i.e, fiber product of formal schemes)

$$\mathcal{N}_n^g\colon = \Gamma_g\cap\Delta_{\mathcal{N}_n}. \tag{3.7}$$

We also form a derived fixed point locus ${}^{\mathbb{L}}\mathcal{N}_n^g$, i.e., the derived tensor product

$$ {}^{\mathbb{L}}\mathcal{N}_n^g :\ = \Gamma_g \cap^{\mathbb{L}} \Delta_{\mathcal{N}_n} := \mathcal{O}_{\Gamma_g} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_n \times \mathcal{N}_n}} \mathcal{O}_{\Delta_{\mathcal{N}_n}}. \tag{3.8} $$

Being supported on $\Delta_{\mathcal{N}_n}$, we view this derived tensor product as an element in the K-group $K_0'(\mathcal{N}_n^g)$ of coherent sheaves on $\mathcal{N}_n^g$. Note that $K_0'(\mathcal{N}_n^g) \simeq K_0^{\mathcal{N}_n^g}(\mathcal{N}_n)$ by the regularity of $\mathcal{N}_n$. Then ${}^{\mathbb{L}}\mathcal{N}_n^g$ lies in the filtration $F^{n-1}K_0^{\mathcal{N}_n^g}(\mathcal{N}_n)$ under the codimension filtration $F^iK_0^{\mathcal{N}_n^g}(\mathcal{N}_n)$, cf. (B.2).

For a pair $(g,u) \in (\mathrm{U}(\mathbb{V}_n) \times \mathbb{V}_n)(F_0)_{\mathrm{rs}}$, we define, cf. (B.3),

$$ \mathrm{Int}(g,u) := \langle\, {}^{\mathbb{L}}\mathcal{N}_n^g, \mathcal{Z}(u) \rangle_{\mathcal{N}_n} := \chi\left(\mathcal{N}_n,\ {}^{\mathbb{L}}\mathcal{N}_n^g \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_n}} \mathcal{O}_{\mathcal{Z}(u)}\right). \tag{3.9} $$

Similar to (3.4) and (3.5), when (g,u) is regular semisimple, $\mathcal{N}_n^g \cap \mathcal{Z}(u)$ is a proper *scheme* over $\mathrm{Spf}\, O_{\breve F}$ and hence the right-hand side of this definition is finite. The number $\mathrm{Int}(g,u)$ depends only on its $\mathrm{U}(\mathbb{V}_n)(F_0)$-orbit.

Remark 3.1. By the projection formula for the closed immersion $\Delta : \mathcal{N}_n \to \mathcal{N}_n \times \mathcal{N}_n$, we obtain an equality in $K_0'(\mathcal{N}_n \times \mathcal{N}_n)$,

$$ \mathrm{R}\Delta_*({}^{\mathbb{L}}\mathcal{N}_n^g \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_n}} \mathcal{O}_{\mathcal{Z}(u)}) = \mathcal{O}_{\Gamma_g} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_n\times\mathcal{N}_n}} \mathcal{O}_{\Delta(\mathcal{Z}(u))}, $$

where we have used $\mathrm{R}\Delta_*(\mathcal{O}_{\mathcal{Z}(u)}) = \mathcal{O}_{\Delta(\mathcal{Z}(u))}$ for a closed immersion. Therefore, an equivalent definition of the intersection number (3.9) is

$$ \mathrm{Int}(g,u) = \chi\left(\mathcal{N}_n \times \mathcal{N}_n,\ \mathcal{O}_{\Gamma_g} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_n\times\mathcal{N}_n}} \mathcal{O}_{\Delta_{\mathcal{Z}(u)}}\right). $$

This also appears in the AFL in the context of Fourier–Jacobi cycles in [31].

Conjecture 3.2 (Arithmetic fundamental lemma conjecture)**.**

(a) (The group version) *Suppose that $\gamma \in S_n(F_0)_{\mathrm{rs}}$ matches an element $g \in \mathrm{U}(\mathbb{V}_n)(F_0)_{\mathrm{rs}}$. Then*

$$ \partial\mathrm{Orb}\big(\gamma, \mathbf{1}_{S_n(O_{F_0})}\big) = -\,\mathrm{Int}(g)\cdot \log q. $$

(b) (The semi-Lie algebra version) *Suppose that $(\gamma,u') \in (S_n \times V_n')(F_0)_{\mathrm{rs}}$ matches an element $(g,u) \in (\mathrm{U}(\mathbb{V}_n)\times\mathbb{V}_n)(F_0)_{\mathrm{rs}}$. Then*

$$ \partial\mathrm{Orb}\big((\gamma,u'), \mathbf{1}_{(S_n\times V_n')(O_{F_0})}\big) = -\,\mathrm{Int}(g,u)\cdot \log q. $$

Remark 3.3. We refer to [39, §4] for the homogeneous (group) version of AFL involving the intersection numbers (3.4).

Remark 3.4. Mihatsch [33] has pointed out that a naive formulation of Lie algebra version of AFL is not well behaved (unless the formal schematic intersection is artinian). Therefore the semi-Lie algebraic version seems to be the best possible linearization of the AFL conjecture.

Definition 3.5. A regular semisimple element $(g,u) \in (\mathrm{U}(V)\times V)(F_0)$ is called strongly regular semisimple ("srs" for short) if g is semisimple. A regular semisimple element $g \in \mathrm{U}(V^\sharp)(F_0)$ (w.r.t. $V^\sharp = V \oplus F\,u_0$) is called strongly regular semisimple if so is $(g,u_0) \in (\mathrm{U}(V^\sharp)\times V^\sharp)(F_0)$ (equivalently $g \in \mathrm{U}(V^\sharp)(F_0)$ is regular semisimple w.r.t. the conjugation action of $\mathrm{U}(V^\sharp)$ on itself).

Definition 3.6. A regular semisimple element $(\gamma,u') \in (S_{n-1}\times V_{n-1}')(F_0)$ is called strongly regular semisimple ("srs" for short) if γ is semisimple (w.r.t. the conjugation action of GL_{n-1,F_0}). A regular semisimple element $\gamma' \in S_n$ is called strongly regular semisimple (w.r.t. $F^n = F^{n-1}\oplus Fu_0$) if so is $(\gamma,u_0') \in (S_n\times V_n')(F_0)$.

Remark 3.7. On the Lie algebras the notion of "strongly regular semisimple" has appeared in [46].

Conjecture 3.8 (Arithmetic fundamental lemma conjecture for strongly regular semisimple elements)**.**

(a) (The group version) *Suppose that $\gamma \in S_n(F_0)_{\mathrm{srs}}$ matches an element $g \in \mathrm{U}(\mathbb{V}_n)(F_0)_{\mathrm{srs}}$. Then*

$$ \partial\mathrm{Orb}\big(\gamma, \mathbf{1}_{S_n(O_{F_0})}\big) = -\,\mathrm{Int}(g)\cdot \log q. $$

(b) (The semi-Lie algebra version) *Suppose that* $(\gamma, u') \in (S_{n-1} \times V'_{n-1})(F_0)_{\rm srs}$ *matches an element* $(g, u) \in (\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1})(F_0)_{\rm srs}$. *Then*

$$\partial \mathrm{Orb}\big((\gamma, u'), \mathbf{1}_{(S_{n-1}\times V'_{n-1})(O_{F_0})}\big) = -\operatorname{Int}(g, u)\cdot \log q.$$

3.2. **Basic properties of the derived fixed point locus $^{\mathbb{L}}\mathcal{N}_n^g$.**

Lemma 3.9. *Let $g \in \mathrm{U}(\mathbb{V}_n)(F_0)$ be regular semisimple (relative to the adjoint action by $\mathrm{U}(\mathbb{V}_n)$).*

(i) The generic fiber of the (naive) fixed point locus $\mathcal{N}_n^g$ is zero dimensional (or empty).

(ii) Let $\mathcal{N}^g_{n,\mathscr{H}}$ (resp., $\mathcal{N}^g_{n,\mathscr{V}}$) be the union of connected components of $\mathcal{N}_n^g$ that are flat (resp., not flat) over $\operatorname{Spf} O_{\breve F}$. *Then*

$$\mathrm{Gr}^{n-1} K_0^{\mathcal{N}_n^g}(\mathcal{N}_n) = \mathrm{Gr}^{n-1} K_0^{\mathcal{N}^g_{n,\mathscr{H}}}(\mathcal{N}_n) \oplus \mathrm{Gr}^{n-1} K_0^{\mathcal{N}^g_{n,\mathscr{V}}}(\mathcal{N}_n).$$

(Here $\mathscr{H}$ stands for "horizontal" and $\mathscr{V}$ for "vertical". See (B.2) *for the graded groups* $\mathrm{Gr}^{n-1}K_0$ *associated to the co-dimension filtration.)*

Proof. For part (i), let us assume that $\mathcal{N}_n^g$ is non-empty. It suffices to show that, over the algebraic closed field $C = \widehat{\overline{\breve F}}$, the set of tuples $(X, \iota, \lambda, \rho)$ over O_C with an action of R is discrete (i.e., there are only finitely many elements in $\mathcal{N}_n(O_C)$ that are reduced to any given quasi-compact subscheme of the reduced scheme $\mathcal{N}_{n,\rm red}$ of $\mathcal{N}_n$). Now $R = O_F[g]$ is an order in a semisimple $\mathbb{Q}_p$-algebra of rank $2nd$, and the p-divisible group X is of height $2nd$, hence has formal complex multiplication by R. Therefore, over O_C, there are only finitely many equivalence classes of such tuple $(X, \iota, \lambda, \rho)$ up to *isogeny*. Fixing one representative $(X_0, \iota_0, \lambda_0, \rho_0)$ for each equivalence class, elements within the equivalence class are bijective to self-dual lattices in the rational Tate module $\mathrm{V}(X_{0,C})$ (endowed with an F/F_0-hermitian space by (ι_0, λ_0)). This shows that the generic fiber of $\mathcal{N}_n^g$ is zero dimensional.

For part (ii), we use Lemma B.1. By part (i), the $\mathcal{N}^g_{n,\mathscr{H}}$ is one dimensional or empty, and hence $\mathcal{N}^g_{n,\mathscr{H}} \cap \mathcal{N}^g_{n,\mathscr{V}}$ is at most zero dimensional. Hence passing to the quotient Gr^{n-1} of the respective groups in Lemma B.1 yields the desired assertion. ∎

According to the direct sum in part (ii), we have a unique decomposition in $\mathrm{Gr}^{n-1} K_0^{\mathcal{N}_n^g}(\mathcal{N}_n)$

$$^{\mathbb{L}}\mathcal{N}_n^g = {}^{\mathbb{L}}\mathcal{N}^g_{n,\mathscr{H}} + {}^{\mathbb{L}}\mathcal{N}^g_{n,\mathscr{V}}, \tag{3.10}$$

where $^{\mathbb{L}}\mathcal{N}^g_{n,\mathscr{H}} \in \mathrm{Gr}^{n-1} K_0^{\mathcal{N}^g_{n,\mathscr{H}}}(\mathcal{N}_n)$, and $^{\mathbb{L}}\mathcal{N}^g_{n,\mathscr{V}} \in \mathrm{Gr}^{n-1} K_0^{\mathcal{N}^g_{n,\mathscr{V}}}(\mathcal{N}_n)$. By Lemma B.2, the first component is represented by the element $\mathcal{O}_{\mathcal{N}^g_{n,\mathscr{H}}}$ (or rather its image in the quotient). Hence we may rewrite the above decomposition as

$$^{\mathbb{L}}\mathcal{N}_n^g = \mathcal{N}^g_{n,\mathscr{H}} + {}^{\mathbb{L}}\mathcal{N}^g_{n,\mathscr{V}}. \tag{3.11}$$

3.3. **A special case of AFL.**

Proposition 3.10. *Let $p > n$. Conjecture 3.8 part* (b) *(i.e., the semi-Lie algebra version AFL) holds for $(g, u) \in (G_1 \times V_1)(F_0)_{\rm srs}$ when $O_F[g]$ is a maximal order (in $F[g]$).*

Proof. This follows from [34, Corollary 10.9] for general F/F_0. When $F_0 = \mathbb{Q}_p$, this can also be deduced from [19]. ∎

4. Relation between the two versions of AFL

4.1. **Orbits in $\mathrm{U}(\mathbb{V}_n)$.** We recall that the Cayley map is the rational morphism

$$\begin{aligned} \mathfrak{c} = \mathfrak{c}_n : \qquad \mathfrak{u}(\mathbb{V}_n) &\longrightarrow \mathrm{U}(\mathbb{V}_n) \\ x &\longmapsto -\tfrac{1-x}{1+x}. \end{aligned} \tag{4.1}$$

Here $\frac{1+x}{1-x} = (1-x)^{-1}(1+x) = (1+x)(1-x)^{-1}$. Its inverse is

$$\mathfrak{c}^{-1}(g') = \frac{1+g'}{1-g'}.$$

By definition $\mathbb{V}_n = \mathrm{Hom}(\mathbb{E}, \mathbb{X}_n)$ and $\mathbb{X}_n = \mathbb{X}_{n-1} \times \mathbb{E}$, we decompose

$$\mathbb{V}_n = \mathbb{V}_{n-1} \oplus \mathrm{End}(\mathbb{E}) = \mathbb{V}_{n-1} \oplus F\, u_0.$$

Accordingly, write $g' \in \mathrm{U}(\mathbb{V}_n)$ in the matrix form

$$\mathbb{X}_{n-1} \times \mathbb{E} \xrightarrow{g' = \begin{pmatrix} h & u \\ w^* & d \end{pmatrix}} \mathbb{X}_{n-1} \times \mathbb{E} \tag{4.2}$$

where $*$ denotes the map induced by polarizations on $\mathbb{X}_{n-1}$ and $\mathbb{E}$, and

$$h \in \mathrm{End}(\mathbb{X}_{n-1}), \quad u, w \in \mathbb{V}_{n-1}, \quad d \in \mathrm{End}(\mathbb{E}).$$

Lemma 4.1. *Let $g' \in \mathrm{U}(\mathbb{V}_n)$ be as in (4.2). Write*

$$x' = \mathfrak{c}_n^{-1}(g') = \begin{pmatrix} x & \widetilde{u} \\ -\widetilde{u}^* & e \end{pmatrix} \in \mathfrak{u}(\mathbb{V}_n), \tag{4.3}$$

and define

$$g \colon = \mathfrak{c}_{n-1}(x) \in \mathrm{U}(\mathbb{V}_{n-1}). \tag{4.4}$$

Then

$$\begin{cases} g = h + (1-d)^{-1} u\, w^*, \\ \widetilde{u} = 2(1-d)^{-1}(1-g)^{-1} u, \\ \det(1-g') = (1-d)\det(1-g), \\ gw = \epsilon_d\, u, \end{cases} \tag{4.5}$$

where we define

$$\epsilon_d \colon = \frac{1-\overline{d}}{1-d}. \tag{4.6}$$

Proof. By definition of $\mathfrak{c}_n^{-1}$, we expand the equality $1 + g' = (1-g')x'$

$$\begin{pmatrix} 1+h & u \\ w^* & 1+d \end{pmatrix} = \begin{pmatrix} 1-h & -u \\ -w^* & 1-d \end{pmatrix} \begin{pmatrix} x & \widetilde{u} \\ -\widetilde{u}^* & e \end{pmatrix}$$

to obtain

$$\begin{cases} 1+h = (1-h)x + u\widetilde{u}^*, \\ w^* = -w^* x - (1-d)\widetilde{u}^*. \end{cases}$$

The second equality yields

$$\widetilde{u}^* = -(1-d)^{-1} w^*(1+x) \Longrightarrow \widetilde{u} = -(1-\overline{d})^{-1}(1-x)w.$$

Plug into the the first equality:

$$1+h = (1-h)x - (1-d)^{-1} u w^*(1+x),$$

and

$$1 + h + (1-d)^{-1} u w^* = (1 - h - (1-d)^{-1} u w^*)x.$$

It follows that

$$g = \mathfrak{c}_{n-1}(x) = h + (1-d)^{-1} u w^*.$$

Now note that the condition for $g' g'^* = 1$ amounts to

$$hh^* + uu^* = 1, \quad hw + \overline{d}u = 0, \quad w^* w + d\overline{d} = 1. \tag{4.7}$$

The last equality follows from

$$\begin{aligned} gw &= hw + (1-d)^{-1} u\, w^* w \\ &= (-\overline{d} + (1 - d\overline{d})(1-d)^{-1}) u \\ &= \frac{1-\overline{d}}{1-d} u. \end{aligned}$$

This in turn yields the last equality. The other equalities then follow easily. $\square$

We now define a rational map by the formulas in Lemma 4.1

$$\begin{aligned} \mathfrak{r}: \mathrm{U}(\mathbb{V}_n) &\longrightarrow \mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1} \\ g' &\longmapsto \left(g, \tfrac{u}{(1-d)\sqrt{\epsilon}}\right), \end{aligned} \tag{4.8}$$

where $\epsilon \in O_{F_0}^{\times}$ is chosen such that $F = F_0[\sqrt{\epsilon}]$. We also define a variant

$$\begin{aligned} \mathfrak{r}^{\natural}: \mathrm{U}(\mathbb{V}_n) &\longrightarrow \mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1} \\ g' &\longmapsto \left(g, \tfrac{\widetilde{u}}{\sqrt{\epsilon}}\right). \end{aligned} \tag{4.9}$$

Following the notation in Lemma 4.1, let $\mathrm{U}(\mathbb{V}_n)^{\circ}$ be the open sub-variety of $\mathrm{U}(\mathbb{V}_n)$ defined by

$$1 - d \neq 0, \quad \text{and} \quad \det(1-g') \neq 0.$$

Let $(\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1} \times \mathfrak{u}(1))^{\circ}$ be the open sub-variety of $\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1} \times \mathfrak{u}(1)$ defined by

$$\det(1-g) \neq 0, \quad \text{and} \quad \det(1+x') \neq 0.$$

Lemma 4.2. *The map $\mathfrak{r}$ together with $e \in \mathfrak{u}(1)$ (cf. (4.3)) induce an isomorphism, equivariant under the action of $\mathrm{U}(\mathbb{V}_{n-1})$,*

$$\begin{aligned} \widetilde{\mathfrak{r}} = (\mathfrak{r}, e)\colon \mathrm{U}(\mathbb{V}_n)^{\circ} &\xrightarrow{\ \sim\ } (\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1} \times \mathfrak{u}(1))^{\circ} \\ g' &\longmapsto (\mathfrak{r}(g'), e). \end{aligned}$$

The same holds if we replace $\mathfrak{r}$ by $\mathfrak{r}^{\natural}$.

Proof. By (4.5) we have

$$\det(1-g') = (1-d)\det(1-g),$$

and by $1-d \neq 0$, it follows that $\det(1-g) \neq 0$. Then the map $x \mapsto \mathfrak{c}(x)$ is well defined since $1 - g = \frac{1}{1+x}$. Therefore the rational map $\widetilde{\mathfrak{r}} = (\mathfrak{r}, e)$ is defined on $\mathrm{U}(\mathbb{V}_n)^{\circ}$

To reverse the map $\widetilde{\mathfrak{r}}$, let $(g, u, e) \in (\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1} \times \mathfrak{u}(1))^{\circ}$. First we send g to $\mathfrak{c}^{-1}(g) = x$ (this is defined since $\det(1-g) \neq 0$). Then we define $\widetilde{u}$ by $\widetilde{u} = 2\sqrt{\epsilon}(1-g)^{-1}u$, cf. (4.5) and (4.8). Finally, we apply Cayley map $\mathfrak{c}$ (4.1) to $\begin{pmatrix} x & \widetilde{u} \\ -\widetilde{u}^* & e \end{pmatrix}$ to obtain g' (the Cayley map is well-defined by the second condition $\det(1+x') \neq 0$ when defining $(\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1} \times \mathfrak{u}(1))^{\circ}$). It is easy to see that the composition of above maps is defined on $(\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1} \times \mathfrak{u}(1))^{\circ}$ and defines an inverse to the rational map $\widetilde{\mathfrak{r}}$. The desired assertion for $\widetilde{\mathfrak{r}}$ follows. It is easy to see the assertion for $\mathfrak{r}^{\natural}$.

$\square$

We may apply the same construction to $\xi g'$ for $\xi \in F^1 = \ker(\mathrm{Nm} : F^{\times} \to F_0^{\times})$:

$$\begin{aligned} \mathfrak{r}_{\xi}: \mathrm{U}(\mathbb{V}_n) &\longrightarrow \mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1}\,. \\ g' &\longmapsto \mathfrak{r}(\xi g') \end{aligned} \tag{4.10}$$

We define the variant $\mathfrak{r}_{\xi}^{\natural}$ similar to (4.9).

Lemma 4.3.

(i) An element $g' \in \mathrm{U}(\mathbb{V}_n)^{\circ}$ is regular semisimple (relative to the action of $\mathrm{U}(\mathbb{V}_{n-1})$ for $\mathbb{V}_n = \mathbb{V}_{n-1} \oplus F\,u_0$) if and only if $\mathfrak{r}(g') = (g, u)$ is regular semisimple as an element in $\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1}$.

(ii) Let $g' \in \mathrm{U}(\mathbb{V}_n)^{\circ}_{\mathrm{srs}}$. Then, for but finitely many $\xi \in F^1$, the element $\xi g' \in \mathrm{U}(\mathbb{V}_n)^{\circ}$ and $\mathfrak{r}_{\xi}(g') \in (\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1})_{\mathrm{srs}}$.

(iii) Let $(g, u) \in (\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1})_{\mathrm{srs}}$. Then, for all but finitely many $e \in \mathfrak{u}(1)$, the element $(g, u, e) \in (\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1} \times \mathfrak{u}(1))^{\circ}$ and $\widetilde{\mathfrak{r}}^{-1}(g, u, e) \in \mathrm{U}(\mathbb{V}_n)^{\circ}_{\mathrm{srs}}$.

Proof. The regular semi-simplicity for $g' \in \mathrm{U}(\mathbb{V}_n)$ is equivalent to the vectors

$$\{g'^i\, u_0 \mid 0 \le i \le n-1\}$$

being a basis of $\mathbb{V}_n$ (as an F-vector space). By the decomposition (4.2), this is equivalent to $\{h^i u, 0 \le i \le n-2\}$ being a basis of $\mathbb{V}_{n-1}$. By (4.5), we can show inductively that, for all $1 \le i \le n-2$, $g^i u - h^i u$ lies in the span of $u, hu, \cdots, h^{i-1}u$. This proves part (i).

Let $P(\lambda) = \det(\lambda + h)$ be the characteristic polynomial of h, and let

$$Q(\lambda) = \det(\lambda + h) \cdot w^*(\lambda + h)^{-1} u,$$

which is a polynomial in λ of degree $n-2$. Then the characteristic polynomial of g' can be written as

$$\det(\lambda + g') = (\lambda + d)P(\lambda) - Q(\lambda). \tag{4.11}$$

Since $g' \in \mathrm{U}(\mathbb{V}_n)^\circ_{\mathrm{srs}}$ (particularly, regular semisimple relative to the $\mathrm{U}(\mathbb{V}_n)$-conjugation action), this polynomial in λ has only simple roots.

Let $\mathfrak{r}_\xi(g') = (g_\xi, u_\xi)$ and now we study how the characteristic polynomial of g_ξ (or equivalently, of $\xi^{-1} g_\xi$) depends on ξ . By (4.5),

$$\det(\lambda + \xi^{-1} g_\xi) = \det\left(\lambda + h + \frac{\xi}{1 - d\xi} u w^*\right).$$

Set

$$t = \frac{\xi}{1 - d\xi}.$$

Then

$$\begin{aligned} \det(\lambda + \xi^{-1} g_\xi) &= \det(\lambda + h) \det\left(1 + t\, u w^* (\lambda + h)^{-1}\right) \\ &= \det(\lambda + h)\left(1 + t\, w^*(\lambda + h)^{-1} u\right) \\ &= \det(\lambda + h) + t\ \det(\lambda + h)\, w^*(\lambda + h)^{-1} u \\ &= P(\lambda) + t\, Q(\lambda). \end{aligned}$$

Here in the second equality we have used the fact that $uw^* \in \mathrm{End}(\mathbb{V}_{n-1})$ is of rank at most one.

Let $R(\xi)$ be the GCD of $P(\lambda)$ and $Q(\lambda)$. By the semi-simplicity of g', the polynomial $R(\lambda)$ is multiplicity free. Fix an algebraic closed field $\Omega \supset F$. Since there are only finitely many $t \in \Omega$ such that $P/R + t\,Q/R$ and R have common roots, the question is reduced to the case $R = 1$ (and possibly smaller n). Now assume $R = 1$. Then $P + t\,Q \in F[t, \lambda]$ is an irreducible (over Ω) polynomial in t, λ, hence defines an irreducible curve C in $\mathbf{A}^2_F$ (the affine plane in t, λ), and t defines a non-constant rational morphism to the projective line $C \to \mathbf{P}^1_F$. The polynomial $P + t\,Q$ has a repeated root precisely when the rational morphism is ramified at t. Hence there are only finitely many such $t \in \Omega$. This proves part (ii).

Part (iii) is proved similarly to part (ii). ∎

4.2. **Reduction of the intersection numbers.** We recall that $\delta : \mathcal{N}_{n-1} \to \mathcal{N}$ is the embedding whose image is the special divisor $\mathcal{Z}(u_0)$ for a unit $u_0 \in \mathrm{End}(\mathbb{E})$ in the above decomposition. Consider

$$\mathcal{N}_{n-1} \times \mathcal{N}_{n-1} \xrightarrow{\ \delta\times\delta\ } \mathcal{N}_n \times \mathcal{N}_n$$

and let $\pi_2 : \mathcal{N}_{n-1} \times \mathcal{N}_{n-1} \longrightarrow \mathcal{N}_{n-1}$ be the projection to the second factor. We have the following pull-back formula for the graph of an automorphism.

Lemma 4.4. *Let $g' \in \mathrm{U}(\mathbb{V}_n)^\circ(F_0)$ be such that $1 - d \in O_F^\times$, and let $(g, u) = \mathfrak{r}(g') \in (\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1})(F_0)$. Then*

$$(\delta \times \delta)^* \Gamma_{g'} = \Gamma_g \cap \pi_2^* \mathcal{Z}(u),$$

where $(\delta\times\delta)^*$ *is the naive pull-back, i.e., the fiber product*

$$\begin{tikzcd} (\delta\times\delta)^*\Gamma_{g'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Gamma_{g'} \arrow[d] \\ \mathcal{N}_{n-1}\times\mathcal{N}_{n-1} \arrow[r, "\delta\times\delta"] & \mathcal{N}_n\times\mathcal{N}_n \end{tikzcd} .$$

Moreover, if u *is non-zero, then the same equality holds in the derived sense (i.e., as elements in* K*-group with support on the schematic intersection) and*

$$\Gamma_g\cap\pi_2^*\mathcal{Z}(u)=\Gamma_g\cap^{\mathbb{L}}\pi_2^*\mathcal{Z}(u).$$

Remark 4.5. By (4.5), we have $gw=\epsilon_d u$. Since $d\neq 1$, $\epsilon_d=\frac{1-\bar{d}}{1-d}$ is a unit in O_F, and hence we may replace $\pi_2^*\mathcal{Z}(u)$ by $\pi_1^*\mathcal{Z}(w)$ in the above statements.

Proof. We prove the natural map on S-points are the identity map. Let (X_1,X_2) be an S-point of $\mathcal{N}_{n-1}\times_{\mathrm{Spf}\,O_{\breve F}}\mathcal{N}_{n-1}$, and let $X_i'=X_i\times\mathcal{E}$ (in the notation we have omitted S and the obvious additional structure ι,λ etc.).

We start from (X_1,X_2) on the graph $\Gamma_{g'}$, i.e., there exists (uniquely) $\varphi':X_1'\to X_2'$ lifting g'. Write φ' in the matrix form

$$X_1\times\mathcal{E}\xrightarrow{\varphi'=\begin{pmatrix}\varphi & \psi\\ \psi'^* & d\end{pmatrix}} X_2\times\mathcal{E}$$

which lifts the diagram (4.2). We then need to construct elements in $\Gamma_g\cap\pi_2^*\mathcal{Z}(u)$. The subtle point is that X_1 and X_2 are different, whereas the $\mathbb{X}_n$ in the target and the source in the map g' of (4.2) are (unfortunately) identified.

First we have $X_2\in\mathcal{Z}(u)$ (note that the u in $\mathfrak{r}(g')=(g,u)$ differs from the u in (4.2) only by a unit $(1-d)\sqrt{\epsilon}$, hence we ignore the difference in this proof). Consider the homomorphism

$$\widetilde{\varphi}:=\varphi+\tfrac{\psi\psi'^*}{1-d}:X_1\longrightarrow X_2$$

This is a lifting of $g\in\mathrm{U}(\mathbb{V}_n)$ by Lemma 4.1, hence we have constructed (X_1,X_2) on $\Gamma_g\cap\pi_2^*\mathcal{Z}(u)$. Again by Lemma 4.1, ψ' lifting $\epsilon_d g^{-1}u$ (and $\epsilon_d=\frac{1-\bar d}{1-d}$ is a unit), hence

$$\psi'=\epsilon_d\,\widetilde{\varphi}^{-1}\psi=\epsilon_d\,\widetilde{\varphi}^*\psi$$

can be recovered from $\widetilde{\varphi}$ and ψ. The desired isomorphism follows.

Now we prove the second part of the lemma. Note that, when u is non-zero, $\mathcal{Z}(u)$ is a relative divisor and Γ_g is the graph of an automorphism. It follows that the dimension of the intersection is as expected. Finally, both $\Gamma_{g'}$ and $\mathcal{N}_{n-1}\times\mathcal{N}_{n-1}$ are local complete intersection in the ambient $\mathcal{N}_n\times\mathcal{N}_n$, and Lemma B.2 shows that higher Tor all vanish. $\square$

Corollary 4.6. *Let* $g'\in\mathrm{U}(\mathbb{V}_n)^\circ(F_0)$ *be such that* $1-d\in O_F^\times$*, and let* $(g,u)=\mathfrak{r}(g')\in(\mathrm{U}(\mathbb{V}_{n-1})\times\mathbb{V}_{n-1})(F_0)$*. Assume further that the vector* $u\neq 0$ *in* $\mathbb{V}_{n-1}$*. Then*

$$\Delta_{\mathcal{N}_{n-1}}\cap^{\mathbb{L}}_{\mathcal{N}_{n-1,n}}(\mathrm{id}\times g')\Delta_{\mathcal{N}_{n-1}}={}^{\mathbb{L}}\mathcal{N}_{n-1}^{g}\cap^{\mathbb{L}}_{\mathcal{N}_n}\mathcal{Z}(u).$$

In particular, if g' *is regular semisimple (hence so is* (g,u) *by Lemma 4.3 (i)), then*

$$\mathrm{Int}(g')=\mathrm{Int}(g,u).$$

Proof. First of all, we have

$$\Delta_{\mathcal{N}_{n-1}}\cap^{\mathbb{L}}_{\mathcal{N}_{n-1,n}}(\mathrm{id}\times g')\Delta_{\mathcal{N}_{n-1}}={}^{\mathbb{L}}\mathcal{N}_{n}^{g'}\cap^{\mathbb{L}}_{\mathcal{N}_n}\delta_{\mathcal{N}_{n-1}}.$$

It follows from Lemma 4.4 that the following two squares are cartesian in the derived sense:

$$\begin{tikzcd} {}^{\mathbb{L}}\mathcal{N}_{n-1}^{g}\cap^{\mathbb{L}}\mathcal{Z}(u) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Gamma_g\cap\pi_2^*\mathcal{Z}(u) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Gamma_{g'} \arrow[d] \\ \mathcal{N}_{n-1} \arrow[r, "\Delta"] & \mathcal{N}_{n-1}\times\mathcal{N}_{n-1} \arrow[r, "\delta\times\delta"] & \mathcal{N}_n\times\mathcal{N}_n \end{tikzcd} .$$

This completes the proof.

□

4.3. **Reduction of orbital integrals.** We again use the Cayley map

$$\begin{aligned}\mathfrak{c} = \mathfrak{c}_n : \mathfrak{s}_n &\longrightarrow S_n \quad . \\ y &\longmapsto -\tfrac{1-y}{1+y}\end{aligned} \tag{4.12}$$

Its inverse is

$$\mathfrak{c}^{-1}(\gamma) = \frac{1+\gamma}{1-\gamma}.$$

Similar to $\mathrm{U}(\mathbb{V}_n)$, we now write $\gamma' \in S_n$ according to the decomposition $F^n = F^{n-1} \oplus Fu_0$:

$$\gamma' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Lemma 4.7. *Let*

$$y' = \mathfrak{c}_n^{-1}(\gamma') = \begin{pmatrix} y & \widetilde{b} \\ \widetilde{c} & e \end{pmatrix} \in \mathfrak{s}_n, \quad \textit{and} \quad \gamma = \mathfrak{c}_{n-1}(y) \in S_{n-1}. \tag{4.13}$$

Then

$$\begin{cases} \gamma = a + (1-d)^{-1}bc, \\ \widetilde{b} = 2(1-d)^{-1}(1-\gamma)^{-1}b, \\ \widetilde{c} = -2c(1-d)^{-1}(1-\gamma)^{-1}, \\ \gamma\bar{b} = \epsilon_d\, b, \end{cases} \tag{4.14}$$

where we recall that $\epsilon_d = \frac{1-\bar{d}}{1-d}$, *cf.* (4.6).

Proof. Similar to the proof of 4.1, we obtain

$$\begin{cases} 1+a = (1-a)y + b\widetilde{c}, \\ c = -cy - (1-d)\widetilde{c}. \end{cases}$$

We obtain

$$\widetilde{c} = -(1-d)^{-1}c(1+y).$$

and

$$1 + a + (1-d)^{-1}bc = (1 - a - (1-d)^{-1}bc)y.$$

It follows that

$$\gamma = \mathfrak{c}_{n-1}(y) = a + (1-d)^{-1}bc.$$

The remaining assertions follow similarly.

□

We now define a rational map by the formulas in Lemma 4.7

$$\begin{aligned}\mathfrak{r} : S_n &\longrightarrow S_{n-1} \times V'_{n-1} \\ \gamma' &\longmapsto \left(\gamma, \left(\tfrac{\widetilde{b}}{\sqrt{\epsilon}}, \tfrac{\widetilde{c}}{\sqrt{\epsilon}} \cdot (1-y^2)^{-1}\right)\right).\end{aligned} \tag{4.15}$$

From (4.14), and the fact that $y \in \mathfrak{s}_{n-1} \Longrightarrow y^2 \in M_{n,n}$, it follows that the last component of $\mathfrak{r}(\gamma')$ indeed lies in $V'_{n-1} = F_0^{n-1} \times (F_0^{n-1})^*$. We also define a variant:

$$\begin{aligned}\mathfrak{r}^\natural : S_n &\longrightarrow S_{n-1} \times V'_{n-1} \\ \gamma' &\longmapsto \left(\gamma, \left(\tfrac{\widetilde{b}}{\sqrt{\epsilon}}, \tfrac{\widetilde{c}}{\sqrt{\epsilon}}\right)\right).\end{aligned} \tag{4.16}$$

Following the notation in Lemma 4.7, let S_n° be the open sub-variety of S_n defined by

$$1 - d \neq 0, \quad \text{and} \quad \det(1-\gamma') \neq 0.$$

Let $\left(S_{n-1} \times V'_{n-1} \times \mathfrak{s}_1\right)^\circ$ be the open sub-variety of $S_{n-1} \times V'_{n-1} \times \mathfrak{s}_1$ defined by

$$\det(1-\gamma) \neq 0, \quad \text{and} \quad \det(1+y') \neq 0.$$

Lemma 4.8. *The map $\mathfrak{r}$ together with $e \in \mathfrak{s}_1$ (cf. (4.13)) induce an isomorphism (between two open sub-varieties), equivariant under the action of* GL_{n-1},

$$\begin{aligned}\widetilde{\mathfrak{r}} = (\mathfrak{r}, e) \colon S_n^\circ &\xrightarrow{\ \sim\ } \big(S_{n-1} \times V'_{n-1} \times \mathfrak{s}_1\big)^\circ\\ \gamma' &\longmapsto (\mathfrak{r}(\gamma'), e).\end{aligned}$$

The same holds if we replace $\mathfrak{r}$ by $\mathfrak{r}^\natural$.

Proof. The proof of Lemma 4.2 still works, and we omit the detail. □

We may apply the same construction to $\xi\gamma'$ for $\xi \in F^1 = \ker(\mathrm{Nm} : F^\times \to F_0^\times)$:

$$\begin{aligned}\mathfrak{r}_\xi : S_n &\longrightarrow S_{n-1} \times V'_{n-1}\\ \gamma' &\longmapsto \mathfrak{r}(\xi\gamma').\end{aligned} \tag{4.17}$$

We define $\mathfrak{r}_\xi^\natural$ similar to (4.16).

Lemma 4.9.
(i) An element $\gamma' \in S_n^\circ$ is regular semisimple if and only if $\mathfrak{r}_\xi(\gamma')$ is regular semisimple as an element in $S_{n-1} \times V'_{n-1}$.
(ii) Let $\gamma' \in S^\circ_{n,\mathrm{srs}}$. Then, for but finitely many $\xi \in F^1$, the element $\xi\gamma' \in S_n^\circ$ and $\mathfrak{r}_\xi(\gamma') \in (S_{n-1} \times V'_{n-1})_{\mathrm{srs}}$.
(iii) Let $(\gamma, u') \in (S_{n-1} \times V'_{n-1})_{\mathrm{srs}}$. Then, for all but finitely many $e \in \mathfrak{s}_1$, the element (γ, u', e) lies in $\big(S_{n-1} \times V'_{n-1} \times \mathfrak{s}_1\big)^\circ$ and $\widetilde{\mathfrak{r}}^{-1}(\gamma, u', e) \in S_n^\circ$ is strongly regular semisimple.

Proof. The same argument as the proof of Lemma 4.3 works here. Hence we omit the detail. □

Lemma 4.10. *If $\gamma' \in S_n(F_0)_{\mathrm{srs}}$ and $g' \in \mathrm{U}(\mathbb{V}_n)(F_0)_{\mathrm{srs}}$ match, then the following pairs also match (whenever they are well-defined for $\xi \in F^1$ under the rational maps):*
- $\mathfrak{r}_\xi^\natural(\gamma') \in (S_{n-1} \times V'_{n-1})(F_0)_{\mathrm{srs}}$ *and* $\mathfrak{r}_\xi^\natural(g') \in (\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1})(F_0)_{\mathrm{srs}}$;
- $\mathfrak{r}_\xi(\gamma') \in (S_{n-1} \times V'_{n-1})(F_0)_{\mathrm{srs}}$ *and* $\mathfrak{r}_\xi(g') \in (\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1})(F_0)_{\mathrm{srs}}$.

Proof. We retain the notation in Lemma 4.3 and Lemma 4.7. We may assume $\xi = 1$. By choosing a basis of $\mathbb{V}_{n-1}$ and of $\mathbb{V}_n = \mathbb{V}_{n-1} \oplus F u_0$, we write $g' \in M_{n,n}(F)$ in matrix form, cf. the discussion on matching orbits in §2.2. Since the inverse Cayley map (cf. (4.1), (4.12)) preserve the matching conditions, $\mathfrak{c}^{-1}(\gamma')$ and $\mathfrak{c}^{-1}(g')$ also match. It follows that the two elements denoted by e in their lower right corner are equal. Moreover, there exists $k \in \mathrm{GL}_{n-1}(F)$ such that

$$\begin{pmatrix} x & \widetilde{u} \\ -\widetilde{u}^* & e \end{pmatrix} = \begin{pmatrix} k^{-1} & \\ & 1 \end{pmatrix} \begin{pmatrix} y & \widetilde{b} \\ \widetilde{c} & e \end{pmatrix} \begin{pmatrix} k & \\ & 1 \end{pmatrix},$$

or equivalently,

$$x = k^{-1}\, y\, k, \quad \widetilde{u} = k^{-1}\, \widetilde{b}, \quad -\widetilde{u}^* = \widetilde{c}\, k.$$

It follows that

$$g = \mathfrak{c}(x) = k^{-1}\, \mathfrak{c}(y)\, k = k^{-1}\gamma\, k,$$

and hence

$$\begin{pmatrix} g & \frac{\widetilde{u}}{\sqrt{\epsilon}} \\ \left(\frac{\widetilde{u}}{\sqrt{\epsilon}}\right)^* & e \end{pmatrix} = \begin{pmatrix} k^{-1} & \\ & 1 \end{pmatrix} \begin{pmatrix} \gamma & \frac{\widetilde{b}}{\sqrt{\epsilon}} \\ \frac{\widetilde{c}}{\sqrt{\epsilon}} & e \end{pmatrix} \begin{pmatrix} k & \\ & 1 \end{pmatrix}.$$

This proves the first part.

By Lemma 4.1 (4.5), $\widetilde{u} = 2(1-d)^{-1}(1-g)^{-1}u$, we obtain

$$u = 2^{-1}(1-d)(1-g)\widetilde{u} = (1-d)(1+x)^{-1}\widetilde{u}.$$

We compute the invariants of (g, u). For $0 \le i \le n-1$,

$$\begin{aligned} u^* g^i u &= (1-d)(1-\overline{d})\widetilde{u}^*(1+x^*)^{-1} g^i (1+x)^{-1}\widetilde{u} \\ &= (1-d)(1-\overline{d})\widetilde{u}^*(1-x^2)^{-1} g^i \widetilde{u}, \end{aligned}$$

where we have used that g and x commute, and $x^* = -x$. In terms of the invariants of $(\gamma, \widetilde{b}, \widetilde{c})$ This last quantity is equal to

$$\begin{aligned} u^* g^i u &= (1-d)(1-\overline{d})\widetilde{u}^*(1-x^2)^{-1} g^i \widetilde{u} \\ &= -(1-d)(1-\overline{d})\widetilde{c}\, k\, (1-x^2)^{-1} g^i\, k^{-1}\widetilde{b} \\ &= -(1-d)(1-\overline{d})\widetilde{c}\,(1-y^2)^{-1}\,\gamma^i\,\widetilde{b}. \end{aligned}$$

Obviously g and γ have the same characteristic polynomial. It follows that $\mathfrak{r}(g') = (g, \frac{u}{\sqrt{\epsilon}(1-d)})$ has the same set of invariants as

$$\left(\gamma, \left(\sqrt{\epsilon}^{-1}\widetilde{b}, \sqrt{\epsilon}^{-1}\widetilde{c}\cdot(1-y^2)^{-1}\right)\right) = \mathfrak{r}(\gamma').$$

This completes the proof of the second part.

□

Lemma 4.11. *Let $\gamma' \in S_n(F_0)_{\mathrm{rs}}$ and $g' \in \mathrm{U}(\mathbb{V}_n)(F_0)_{\mathrm{rs}}$ be a matching pair, and $\xi \in F^1$. Assume that*

$$1 - \xi d \in O_F^\times, \quad \textit{and} \quad \det(1-\xi\gamma') \in O_F^\times. \tag{4.18}$$

Then

$$\begin{aligned} \mathrm{Orb}(\gamma', \mathbf{1}_{S_n(O_{F_0})}, s) &= \mathrm{Orb}\left(\mathfrak{r}_\xi^\natural(\gamma'), \mathbf{1}_{(S_{n-1}\times V'_{n-1})(O_{F_0})}, s\right) \\ &= \mathrm{Orb}\left(\mathfrak{r}_\xi(\gamma'), \mathbf{1}_{(S_{n-1}\times V'_{n-1})(O_{F_0})}, s\right). \end{aligned}$$

Proof. It suffices to prove the assertions for $\xi = 1$. We also consider the orbital integral on the Lie algebra $\mathfrak{s}_n$. Since $\det(1-\gamma') \in O_F^\times$ by assumption (4.18), and the Cayley map is equivariant under the $\mathrm{GL}_{n-1}(F_0)$,

$$h \cdot \mathfrak{c}^{-1}(\gamma') \in \mathfrak{s}_n(O_{F_0}) \quad \text{if and only if} \quad h\cdot\gamma' \in S_n(O_{F_0}).$$

It follows that

$$\mathrm{Orb}(\mathfrak{c}^{-1}(\gamma'), \mathbf{1}_{\mathfrak{s}_n(O_{F_0})}, s) = \mathrm{Orb}(\gamma', \mathbf{1}_{S_n(O_{F_0})}, s).$$

Similarly, by $\det(1+y) = (1-d)^{-1}\det(1-\gamma')$ and (4.18), we know that $\det(1+y) \in O_F^\times$. Therefore,

$$h^{-1} y h \in \mathfrak{s}_{n-1}(O_{F_0}) \quad \text{if and only if} \quad h^{-1}\gamma h \in S_{n-1}(O_{F_0}).$$

It follows that (note that d and e are now in O_F and $\mathfrak{s}_1(O_{F_0})$ respectively)

$$\mathrm{Orb}(\mathfrak{c}^{-1}(\gamma'), \mathbf{1}_{\mathfrak{s}_n(O_{F_0})}, s) = \mathrm{Orb}\left(\mathfrak{r}^\natural(\gamma'), \mathbf{1}_{(S_{n-1}\times V'_{n-1})(O_{F_0})}, s\right).$$

This proves the first equality.

We now simply denote

$$\widetilde{c}' := \widetilde{c}\cdot(1-y^2)^{-1}$$

so that

$$\mathfrak{r}(\gamma') = \left(\gamma, \left(\widetilde{b}/\sqrt{\epsilon}, \widetilde{c}'/\sqrt{\epsilon}\right)\right).$$

Note now that $\det(1-y^2) = \mathrm{Nm}\det(1+y) \in O_{F_0}^\times$ under our assumption. Therefore, when $h^{-1}\gamma h \in S_{n-1}(O_{F_0})$, we have

$$\frac{\widetilde{c}'}{\sqrt{\epsilon}} h \in O_{F_0}^n \quad \text{if and only if} \quad \frac{\widetilde{c}}{\sqrt{\epsilon}} h \in O_{F_0}^n.$$

This immediately implies the second equality.

□

4.4. **Relation between the two versions of AFL.**

Proposition 4.12. *Fix F/F_0. Assume that $q \geq n$ where q denotes the cardinality of the residue field of O_{F_0}. Then*

(i) in Conjecture 3.8, part (a) *for $\mathbb{V}_n$ is equivalent to part* (b) *for $\mathbb{V}_{n-1}$.*

(ii) in Conjecture 3.8, part (a) *for $\mathbb{V}_{n-1}$ implies part* (b) *for $\mathbb{V}_{n-1}$ and $(g,u) \in (\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1})(F_0)_{\mathrm{srs}}$ where the norm of u is a unit.*

Remark 4.13. Similar results hold for Conjecture 3.2 for regular semisimple elements.

Proof. For part (i), let $g' \in \mathrm{U}(\mathbb{V}_n)(F_0)_{\mathrm{srs}}$. We may assume that $d \in O_F$ and the characteristic polynomial of g' has integral coefficients (otherwise both sides of part (a) vanish). Since $q+1 > n$, there exists $\xi \in F^1$ such that $\det(1 - \xi g') \in O_F^\times$ is a unit (looking at the reduction of the characteristic polynomial modulo the uniformizer ϖ_F of O_F). Since both side of part (a) for $\mathbb{V}_n$ are invariant under the substitution $g' \mapsto \xi g'$, we may just assume that g' has the property that $d \in O_F$ and $\det(1 - g') \in O_F^\times$. Then $g' \in \mathrm{U}(\mathbb{V}_n)^\circ(F_0)_{\mathrm{srs}}$, so that we may apply the map $\mathfrak{r}$. By Lemma 4.3, we may adjust $\xi \in F^1$ within the same residue class $\mod \varpi_F$ such that the image $\mathfrak{r}(g') = (g,u)$ lies in $(\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1})(F_0)_{\mathrm{srs}}$. Now, by Corollary 4.6,

$$\mathrm{Int}(g') = \mathrm{Int}(g,u).$$

Now we consider the orbital integral. By Lemma 4.11

$$\partial\mathrm{Orb}(\gamma', \mathbf{1}_{S_n(O_{F_0})}) = \partial\mathrm{Orb}\left(\mathfrak{r}(\gamma'), \mathbf{1}_{(S_{n-1}\times V'_{n-1})(O_{F_0})}\right).$$

Here we refer to [38, Lemma 11.9] for the comparison of the transfer factors. By Lemma 4.10, $\mathfrak{r}(\gamma') \in (S_{n-1} \times V'_{n-1})(F_0)_{\mathrm{srs}}$ and $\mathfrak{r}(g') \in (\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1})(F_0)_{\mathrm{srs}}$ match. This shows that part (b) for $\mathbb{V}_{n-1}$ implies part (a) for $\mathbb{V}_n$.

For the inverse direction, we start from $(g,u) \in (\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1})(F_0)_{\mathrm{srs}}$. Again it suffices to prove part (b) when the invariants of (g,u) are all integers. By multiplying a suitable $\xi \in F^1$, we may assume $\det(1-g) \in O_F^\times$. Then $\det(1+x) \in O_F^\times$. By Lemma 4.3 part (iii), there exists $e \in \mathfrak{u}(1)(O_{F_0})$ such that $\det(1+x') \in O_F^\times$, (g,u,e) lies in $(\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1} \times \mathfrak{u}(1))^\circ$ and $g' = \widetilde{\mathfrak{r}}^{-1}(g,u,e) \in \mathrm{U}(\mathbb{V}_n)^\circ_{\mathrm{srs}}$. Then we may apply Corollary 4.6. Similar procedure proves the desired identity between orbital integrals. This shows that part (a) for $\mathbb{V}_n$ implies part (b) for $\mathbb{V}_{n-1}$.

For Part (ii) we note that for $g \in \mathrm{U}(\mathbb{V}_{n-1})(F_0)_{\mathrm{srs}}$, the pair $(g,u_0) \in (\mathrm{U}(\mathbb{V}_{n-1}) \times \mathbb{V}_{n-1})(F_0)_{\mathrm{srs}}$, and it is easy to see

$$\mathrm{Int}(g) = \mathrm{Int}(g,u_0).$$

One can show that the orbital integrals are equal easily, and we leave the detail to the reader. ☐

5. Local constancy of intersection numbers

This section is not used until §15.

5.1. **Local constancy of the function** $\mathrm{Int}(g,\cdot)$**.** We recall the Bruhat–Tits stratification of the underlying reduced scheme $\mathcal{N}_{n,\mathrm{red}}$ of $\mathcal{N}_n$, following the work of Vollaard–Wedhorn [43]. The scheme $\mathcal{N}_{n,\mathrm{red}}$ admits a stratification by Deligne–Lusztig varieties of dimensions $0,1,\ldots,\lfloor\frac{n-1}{2}\rfloor$, attached to unitary groups in an odd number of variables and to Coxeter elements, with strata parametrized by the vertices of the Bruhat–Tits complex of the special unitary group for the non-split n-dimensional F/F_0-hermitian space $\mathbb{V}_n$. The vertices of the Bruhat–Tits complex is bijective to vertex lattices in $\mathbb{V}_n$ where an O_F-lattice (of full rank) $\Lambda \subset \mathbb{V}_n$ is called a vertex lattice if $\Lambda \subset \Lambda^\vee \subset \varpi^{-1}\Lambda$. The parametrization of the strata by vertex lattices in $\mathbb{V}_n$ is compatible with the action of the group $\mathrm{U}(\mathbb{V}_n)$ on $\mathcal{N}_{n,\mathrm{red}}$ (cf. (3.3)) and on $\mathbb{V}_n$. The type of a vertex lattice Λ is by definition the integer $t(\Lambda) := \dim_k \Lambda^\vee/\Lambda$. Denote by $\mathcal{V}(\Lambda)$ the corresponding (generalized) Deligne–Lusztig variety; it is smooth projective of dimension $\frac{t(\Lambda)-1}{2}$, cf. *loc. cit.*. Note that the type $t(\Lambda)$ is odd in our case because the F/F_0-hermitian space $\mathbb{V}_n$ is non-split.

Let $\Lambda \subset \mathbb{V}_n$ be a vertex lattice of type 3. We define

$$\mathrm{Int}_{\mathcal{V}(\Lambda)}(u) := \chi\big(\mathcal{N}_n, \mathcal{V}(\Lambda) \cap^{\mathbb{L}} \mathcal{Z}(u)\big), \quad u \in \mathbb{V} \setminus \{0\}.$$

The following result may be of some independent interest.

Lemma 5.1. *Let $\Lambda \subset \mathbb{V}_n$ be a vertex lattice of type* 3. *Then*

$$\mathrm{Int}_{\mathcal{V}(\Lambda)} = \sum_{\Lambda \subset \Lambda',\, t(\Lambda')=1} \mathbf{1}_{\Lambda'} - q^2(1+q)\mathbf{1}_{\Lambda}.$$

Proof. This is [27, Lemma 6.2.1]. □

Lemma 5.2. *Let $n \geq 3$, and $\Lambda \subset \mathbb{V}_n$ a vertex lattice of maximal type (i.e., type $2[(n-1)/2]+1$). Let $C \in \mathrm{Ch}_{1,\mathcal{V}(\Lambda)}(\mathcal{N}_{n,\mathrm{red}})$. Then the function*

$$\begin{aligned} \mathrm{Int}_C\colon\ \mathbb{V}_n &\longrightarrow \mathbb{Q} \\ u &\longmapsto \chi(\mathcal{N}_n, C \cap^{\mathbb{L}} \mathcal{Z}(u)) \end{aligned}$$

is locally constant and compactly supported. Here, even though the function is only defined for $u \neq 0$, the local constancy around $u = 0$ is to be interpreted as that the function takes a constant value for all $u \neq 0$ in a neighborhood of $0 \in \mathbb{V}$.

Proof. On the DL variety $\mathcal{V}(\Lambda)$ there is a collection of DL curves $\mathcal{V}(\Lambda')$ for type-3 vertex lattices Λ' nested between Λ and $\Lambda^\vee$

$$\Lambda \subset \Lambda' \subset \Lambda'^\vee \subset \Lambda^\vee.$$

It can be deduced from the computation of Lusztig [32] (for type ${}^2A_{2m}$) that for any $\ell \neq p$, the Tate classes[1] in $\mathrm{H}^2(\mathcal{V}(\Lambda)_{\mathbb{F}}, \mathbb{Q}_\ell)$ is spanned by the classes of these DL curves, cf. [27, Theorem 5.2.2] for a detailed proof. Since the intersection number depends only on the cohomology class of C on $\mathcal{V}(\Lambda)$, it suffices to prove the assertion when C is (the equivalence class of) a DL curve $\mathcal{V}(\Lambda')$. However, in that case, the local constancy follows immediately from Lemma 5.1. This completes the proof.

□

Proposition 5.3. *Fix a regular semisimple element $g \in \mathrm{U}(\mathbb{V}_n)$. Let*

$$\mathbb{V}_{n,g} :=\{u \in \mathbb{V}_n \mid (g,u) \text{ is not regular semisimple}\}.$$

Then the function

$$\begin{aligned} \mathrm{Int}(g,\cdot)\colon\ \mathbb{V}_n \setminus \mathbb{V}_{n,g} &\longrightarrow \mathbb{Q} \\ u &\longmapsto \mathrm{Int}(g,u) = \chi(\mathcal{N}_n,\ {}^{\mathbb{L}}\mathcal{N}_n^g \cap^{\mathbb{L}} \mathcal{Z}(u)) \end{aligned}$$

is locally constant.

Proof. By the decomposition (3.11) we have

$${}^{\mathbb{L}}\mathcal{N}_n^g = \mathcal{N}_{n,\mathscr{H}}^g + {}^{\mathbb{L}}\mathcal{N}_{n,\mathscr{V}}^g \in \mathrm{Gr}^{n-1} K_0^{\mathcal{N}_n^g}(\mathcal{N}_n).$$

Therefore we may represent them by a formal (locally finite) sum $\sum_C \mathrm{mult_C} \cdot [\mathcal{O}_C]$ where C are formal curves (i.e., integral closed formal subschemes of formal dimension one) on $\mathcal{N}_n^g$, and $\mathrm{mult_C} \in \mathbb{Q}$. Let C be an irreducible formal curve on $\mathcal{N}_n^g$. We claim that that the function

$$\begin{aligned} \mathrm{Int}_C\colon\ \mathbb{V}_n \setminus \{0\} &\longrightarrow \mathbb{Q} \\ u &\longmapsto \chi(\mathcal{N}_n,\ C \cap^{\mathbb{L}} \mathcal{Z}(u)) \end{aligned}$$

is locally constant. This would imply the desired local constancy, because the intersection $\mathcal{N}_n^g \cap \mathcal{Z}(u)$ is a proper scheme over $\mathrm{Spf}\, O_{\breve{F}}$, and hence only finitely many irreducible curves contribute to the intersection $\mathrm{Int}(g,u)$.

There are the following (mutually exclusive) possibilities for C

- an irreducible closed formal subscheme of $\mathcal{N}_{n,\mathscr{H}}^g$, which is of formal dimension one and intersects $\mathcal{N}_{n,\mathrm{red}}$ with dimension zero,
- an irreducible closed formal subscheme of $\mathcal{N}_{n,\mathscr{V}}^g$ of formal dimension one that intersects $\mathcal{N}_{n,\mathrm{red}}$ with dimension zero,
- an irreducible closed subscheme of $\mathcal{N}_{n,\mathrm{red}}$.

[1]Here by Tate classes in $H^2(\mathcal{V}(\Lambda)_{\mathbb{F}}, \mathbb{Q}_\ell(1))$ we mean all classes that are fixed by some powers of Frobenius.

For the third type, we have proved an even stronger version in Lemma 5.2. Now we show the locally constancy for the other two types. We take $\widetilde{C}$ to be the normalization of C, and $\widetilde{C} \to \mathcal{N}_n$ the induced morphism. Then

$$\chi(\mathcal{N}_n, C \cap^{\mathbb{L}} \mathcal{Z}(u)) = \chi(\mathcal{N}_n, \mathcal{O}_{\widetilde{C}} \otimes^{\mathbb{L}}_{\mathcal{N}_n} \mathcal{O}_{\mathcal{Z}(u)}).$$

Now $C \cap \mathcal{Z}(u)$ must be zero dimensional (otherwise $C \subset \mathcal{Z}(u)$; however we know that $\mathcal{N}_n^g \cap \mathcal{Z}(u)$ is supported in $\mathcal{N}_{n,\mathrm{red}}$).

Pull back the universal object on $\mathcal{N}_n$ to $\widetilde{C}$ and we obtain a family of unitary p-divisible groups on $\widetilde{C}$. Then $\mathrm{Int}_C(u)$ is the length of the artinian scheme where the special homomorphism u lifts. We prove this on the complete local ring A of $\widetilde{C}$ at a point in the support of $\widetilde{C} \cap \mathcal{Z}(u)$. Here A is isomorphic to $\mathbb{F}[\![t]\!]$ or a finite extension of $\mathrm{Spf}\, O_{\breve{F}}$, and we will simply write t for a uniformizer of A. Let $(\mathcal{X}, \iota, \lambda, \rho)$ be the unitary p-divisible group over $\mathrm{Spf}\, A$. For all $i \geq 0$, $\mathrm{Hom}_{A/(t^i)}(\mathcal{E}, \mathcal{X})$ must be a lattice (of full rank) in $\mathrm{Hom}_{A/(t)}(\mathbb{E}, \mathbb{X}_n) = \mathbb{V}_n$. This implies the desired local constancy.

□

5.2. **Local constancy of the function** $\mathrm{Int}(\cdot, \cdot)$**.**

Lemma 5.4. *Fix* $(g_0, u_0, e_0) \in (\mathrm{U}(\mathbb{V}_n) \times \mathbb{V}_n \times \mathfrak{u}(1))^\circ$ *such that* $g' = \widetilde{\mathfrak{r}}^{-1}(g_0, u_0, e_0) \in \mathrm{U}(\mathbb{V}_{n+1})_{\mathrm{srs}}$ *(cf. Lemma 4.3 for the notation). Then the map (defined on some open subsets of* F_0*-varieties)*

$$\begin{aligned} \mathrm{char}(g_0, \cdot, \cdot)\colon\ \mathbb{V}_n \times \mathfrak{u}(1) &\longrightarrow [\mathrm{U}(\mathbb{V}_{n+1})_{/\!\!/\mathrm{U}(\mathbb{V}_{n+1})}] \\ (u, e) &\longmapsto \mathrm{char\,poly}(\widetilde{\mathfrak{r}}^{-1}(g_0, u, e)) \end{aligned}$$

is submersive (i.e., the induced map on tangent spaces is surjective) at (u_0, e_0)*.*

Here $[\mathrm{U}(\mathbb{V}_{n+1})_{/\!\!/\mathrm{U}(\mathbb{V}_{n+1})}]$ *denotes the categorical quotient (with respect to the adjoint action) and* char poly *denotes the characteristic polynomial.*

Proof. The question is local on the source. Tracing the definition back to (4.8) and Lemma 4.2, we may reduce the question to the Lie algebra version: for a fixed $\begin{pmatrix} x_0 & u_0 \\ -u_0^* & e_0 \end{pmatrix} \in \mathfrak{u}(\mathbb{V}_{n+1})_{\mathrm{srs}}$, the map

$$\mathbb{V}_n \times \mathfrak{u}(1) \longrightarrow [\mathfrak{u}(\mathbb{V}_{n+1})_{/\!\!/\mathrm{U}(\mathbb{V}_{n+1})}]$$

sending (u, e) to the characteristic polynomial of $\begin{pmatrix} x_0 & u \\ -u^* & e \end{pmatrix} \in \mathfrak{u}(\mathbb{V}_{n+1})$ is submersive at (u_0, e_0).

Note that a complete set of generators of invariants relative to the $\mathrm{U}(\mathbb{V}_n)$-action on $\mathfrak{u}(\mathbb{V}_{n+1})$ is given by:

$$\mathrm{char\,poly}(x), \quad e, \quad u^* x^j u, \quad 0 \leq j \leq n-1,$$

where $x' = \begin{pmatrix} x & u \\ -u^* & e \end{pmatrix} \in \mathfrak{u}(\mathbb{V}_{n+1})$ cf. [47]. It is easy to see that an equivalent set is

$$\mathrm{char\,poly}(x), \quad \mathrm{char\,poly}(x').$$

Therefore, it suffices to show the analogous map

$$\mathbb{V}_n \longrightarrow \prod_{i=0}^{n-1} F^{(-1)^j}$$

sending $u \in \mathbb{V}_n$ to the invariants

$$u^* x^j u, \quad 0 \leq j \leq n-1,$$

is submersive at u_0. Here $F^{(-1)^j}$ is the $(-1)^j$-eigenspace of F under the Galois conjugation. Now the assertion follows from the regular semi-simplicity of x_0, which reduces the question to the case $n = 1$, but for the product of field extensions of F. This is routine and we omit the detail.

□

Theorem 5.5. *The function*

$$\begin{aligned}\mathrm{Int}(\cdot,\cdot)\colon\ (\mathrm{U}(\mathbb{V}_n)\times\mathbb{V}_n)(F_0)_{\mathrm{srs}} &\longrightarrow \mathbb{Q}\\ (g,u) &\longmapsto \mathrm{Int}(g,u)\end{aligned}$$

is locally constant. Its support is compact modulo the action of $\mathrm{U}(\mathbb{V}_n)(F_0)$.

Remark 5.6. See the forthcoming work of Mihatsch [35] for a different proof, which also yields the local constancy on the regular semisimple locus.

Proof. We may assume that the invariants of (g,u) are all integers. We now fix such a pair (g,u) and we want to show the local constancy near (g,u).

First, by the argument in the proof of part (i) of Proposition 4.12, there exists $g'=\widetilde{\mathfrak{r}}^{-1}(g,u,e)\in \mathrm{U}(\mathbb{V}_{n+1})^\circ(F_0)_{\mathrm{srs}}$ such that

$$\mathrm{Int}(g')=\mathrm{Int}(g,u). \tag{5.1}$$

In fact, by the same argument the equality holds if we replace (g,u,e) by any element $(g^\sharp,u^\sharp,e^\sharp)$ near it, and g' by the respective image $g'^\sharp$ under the map $\widetilde{\mathfrak{r}}^{-1}$.

On the other hand, we may write

$$\mathrm{Int}(g')=\mathrm{Int}(g',u_0'),$$

where $u_0'\in\mathbb{V}_{n+1}$ is the fixed unit normed vector that induces the embedding $\mathcal{N}_n\hookrightarrow\mathcal{N}_{n+1}$. We now apply Proposition 5.3 to (g',u_0'):

$$\mathrm{Int}(g',u_0')=\mathrm{Int}(g',u'),$$

where $u'\in\mathbb{V}_{n+1}$ is close to u_0'. In particular the equality holds for $u'=hu_0'$ for $h\in\mathrm{U}(\mathbb{V}_{n+1})$ in a small neighborhood of 1. By the invariance under $\mathrm{U}(\mathbb{V}_{n+1})$, we have for $u'=hu_0'$

$$\mathrm{Int}(g',u')=\mathrm{Int}(h^{-1}g'h,u_0').$$

It follows that $\mathrm{Int}(g',u_0')=\mathrm{Int}(h^{-1}g'h,u_0')$ and hence

$$\mathrm{Int}(g')=\mathrm{Int}(h^{-1}g'h) \tag{5.2}$$

for $h\in\mathrm{U}(\mathbb{V}_{n+1})(F_0)$ in a small neighborhood of 1. This shows that, as a function on the quotient $[\mathrm{U}(\mathbb{V}_{n+1})/\!\!/\mathrm{U}(\mathbb{V}_n)](F_0)$, $\mathrm{Int}(g')$ is constant on those elements near g' and having the same characteristic polynomial (as g').

Now we claim that the desired local constancy near $(g,u)\in(\mathrm{U}(\mathbb{V}_n)\times\mathbb{V}_n)(F_0)_{\mathrm{srs}}$ follows from the following two properties

(1) the local constancy in the u-variable (for a fixed g), by Proposition 5.3;

(2) the invariance (5.2) under conjugation by elements h near $1\in\mathrm{U}(\mathbb{V}_{n+1})$.

To show the claim, let $g'^\sharp$ be an element in a small neighborhood of g'. By Lemma 5.4, there exists a neighborhood $\Omega\subset\mathbb{V}_n\times\mathfrak{u}(1)$ of (u,e) such that $g'^\sharp$ is conjugate (by an element $h\in\mathrm{U}(\mathbb{V}_{n+1})(F_0)$ near 1) to $\widetilde{\mathfrak{r}}^{-1}(g,u^\sharp,e^\sharp)$ for some $(u^\sharp,e^\sharp)\in\Omega$. By the invariance (5.2), we have

$$\mathrm{Int}(g'^\sharp)=\mathrm{Int}(\widetilde{\mathfrak{r}}^{-1}(g,u^\sharp,e^\sharp)).$$

By (5.1) (and the remark following it),

$$\mathrm{Int}(\widetilde{\mathfrak{r}}^{-1}(g,u^\sharp,e^\sharp))=\mathrm{Int}(g,u^\sharp).$$

By Proposition 5.3 for the local constancy in u,

$$\mathrm{Int}(g,u^\sharp)=\mathrm{Int}(g,u).$$

Again by (5.1) $\mathrm{Int}(g,u)=\mathrm{Int}(g')$, we obtain $\mathrm{Int}(g'^\sharp)=\mathrm{Int}(g')$. The desired local constancy of $\mathrm{Int}(g,u)$ follows from (5.1) (and the remark following it).

To show the compactness of the support modulo $\mathrm{U}(\mathbb{V}_n)(F_0)$, it suffices to show the *claim*: the support is contained in the union of compact subsets

$$K_\Lambda\times\Lambda\subset(\mathrm{U}(\mathbb{V}_n)\times\mathbb{V}_n)(F_0),$$

where Λ runs over all vertex lattices, and K_Λ is the stabilizer of Λ. Then the desired compactness follows from the fact that the group $\mathrm{U}(\mathbb{V}_n)(F_0)$ acts transitively on the set of vertex lattices of any given type t (and there are only finitely many possible types $t=1,3,\cdots,2[(n-1)/2]+1$).

Now we show the claim. If $\mathrm{Int}(g,u)\neq 0$, then there exists a point $x\in\mathcal{N}_n\left(\overline{k}\right)$ lying on $\mathcal{Z}(u)$ and $\mathcal{N}_n^g$. Let $\mathcal{V}(\Lambda)$ for some vertex lattice Λ be the smallest stratum containing the point x. Then $g\Lambda=\Lambda$ (otherwise the intersection $g\mathcal{V}(\Lambda)\cap\mathcal{V}(\Lambda)$ is non-empty and is a strictly smaller stratum), and $u\in\Lambda$ by [25, Prop. 4.1]. Therefore $(g,u)\in K_\Lambda\times\Lambda$ as desired. □

Part 2. Global theory

6. Shimura varieties and their integral models

In this section we recall the construction of the integral models of certain Shimura varieties, following [6, 40, 41]. In fact we need less precise information than provided in *loc. cit.*. We only need a regular integral model over away from a suitable finite set of places of the full ring of integers. Therefore we will not give the complete detail on the formulation moduli spaces as in *loc. cit.*.

6.1. Shimura variety.

6.1.1. *Shimura data.* Let F be a CM number field with maximal totally real subfield F_0 and nontrivial F/F_0-automorphism $a\mapsto\overline{a}$. Let n be a positive integer. A *generalized CM type of rank* n is a function $r\colon \mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}})\to\mathbb{Z}_{\geq 0}$, denoted $\varphi\mapsto r_\varphi$, such that

$$r_\varphi+r_{\overline{\varphi}}=n\quad\text{for all}\quad\varphi. \tag{6.1}$$

Here $\overline{\varphi}$ denotes the pre-composition of φ by the nontrivial F/F_0-automorphism. When $n=1$, a generalized CM type is the same as a usual CM type (i.e., a half-system Φ of complex embeddings of F), via $\Phi=\{\varphi\in\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}})\mid r_\varphi=1\}$.

Let $(V,(\ ,\ ))$ be an F/F_0-hermitian vector space of dimension n. Fix a CM type Φ of F. Then the signatures of V at the archimedean places determine a generalized CM type r of rank n (and vice versa), by the following recipe

$$\mathrm{sig}V_\varphi=(r_\varphi,r_{\overline{\varphi}}),\quad \varphi\in\Phi,\quad V_\varphi:=V\otimes_{F,\varphi}\mathbb{C}. \tag{6.2}$$

Let $G^{\mathbb{Q}}$ be the group of unitary similitudes of $(V,(\ ,\ ))$,

$$G^{\mathbb{Q}}:=\left\{\,g\in\mathrm{Res}_{F_0/\mathbb{Q}}\mathrm{GU}(V)\;\middle|\;c(g)\in\mathbb{G}_m\,\right\},$$

considered as a linear algebraic group over $\mathbb{Q}$ (with similitude factor in $\mathbb{G}_m$).

Given Φ, r and V, we define Shimura datum [41]. For each $\varphi\in\Phi$, choose a $\mathbb{C}$-basis of V_φ with respect to which the matrix of $(\ ,\ )$ is given by

$$\mathrm{diag}(1_{r_\varphi},-1_{r_{\overline{\varphi}}}). \tag{6.3}$$

The conjugacy class $\{h_{G^{\mathbb{Q}}}\}$ in the Shimura datum is the $G^{\mathbb{Q}}(\mathbb{R})$-conjugacy class of the homomorphism $h_{G^{\mathbb{Q}}}=(h_{G^{\mathbb{Q}},\varphi})_{\varphi\in\Phi}$, where the components $h_{G^{\mathbb{Q}},\varphi}$ are defined with respect to the inclusion

$$G^{\mathbb{Q}}(\mathbb{R})\subset\mathrm{GL}_{F\otimes\mathbb{R}}(V\otimes\mathbb{R})\xrightarrow[\sim]{\Phi}\prod_{\varphi\in\Phi}\mathrm{GL}_{\mathbb{C}}(V_\varphi), \tag{6.4}$$

and where each component is defined on $\mathbb{C}^\times$ by

$$h_{G^{\mathbb{Q}},\varphi}\colon z\longmapsto\mathrm{diag}(z\cdot 1_{r_\varphi},\overline{z}\cdot 1_{r_{\overline{\varphi}}}).$$

Then the reflex field $E(G^{\mathbb{Q}},\{h_{G^{\mathbb{Q}}}\})$ is the reflex field E_r of r, which is the subfield of $\overline{\mathbb{Q}}$ defined by

$$\mathrm{Gal}(\overline{\mathbb{Q}}/E_r)=\left\{\,\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\;\middle|\;\sigma^*(r)=r\,\right\}. \tag{6.5}$$

Now, in addition to the CM type Φ, we also fix a distinguished element $\varphi_0\in\Phi$. We will assume that the generalized CM type r is of strict fake Drinfeld type relative to Φ and φ_0, in the sense of [41], i.e.,

$$r_\varphi=\begin{cases}n-1, & \varphi=\varphi_0;\\ n, & \varphi\in\Phi\smallsetminus\{\varphi_0\}.\end{cases}$$

The first special case is when $n = 1$ and V is totally positive definite, i.e., V has signature $(1, 0)$ at each archimedean place[2]. In this case, we write $Z^{\mathbb{Q}} := G^{\mathbb{Q}}$ (a torus over $\mathbb{Q}$) and $h_{Z^{\mathbb{Q}}} := h_{G^{\mathbb{Q}}}$. The reflex field of $(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})$ is E_Φ, the reflex field of Φ.

Now for general n, we set

$$\widetilde{G} := Z^{\mathbb{Q}} \times_{\mathbb{G}_m} G^{\mathbb{Q}}, \tag{6.6}$$

where the two maps are respectively given by Nm_{F/F_0} and the similitude character. We form a Shimura datum for $\widetilde{G}$ by

$$h_{\widetilde{G}} \colon \mathbb{C}^\times \xrightarrow{(h_{Z^{\mathbb{Q}}}, h_{G^{\mathbb{Q}}})} \widetilde{G}(\mathbb{R}).$$

Then $(\widetilde{G}, \{h_{\widetilde{G}}\})$ has reflex field $E \subset \overline{\mathbb{Q}}$ being the composite $E_\Phi E_r$ (cf. [40]). In particular, the field F is a subfield of E via φ_0.

Note that, by [40, Remark 3.11], over E we have

$$\mathrm{Sh}_{K_{\widetilde{G}}}\big(\widetilde{G}, \{h_{\widetilde{G}}\}\big) \simeq \mathrm{Sh}_{K_{Z^{\mathbb{Q}}}}\big(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\}\big) \times \mathrm{Sh}_{K_G}\big(\mathrm{Res}_{F/F_0} G, \{h_G\}\big), \tag{6.7}$$

for the Shimura variety $\mathrm{Sh}_{K_G}\big(\mathrm{Res}_{F/F_0} G, \{h_G\}\big)$ associated to the unitary group defined in [10].

6.2. **Integral models.**

6.2.1. *The auxiliary moduli problem for $Z^{\mathbb{Q}}$.* We recall the moduli problem $\mathcal{M}_0$ over O_E of [40, §3.2]. Its generic fiber is a disjoint union of copies of the Shimura variety $\mathrm{Sh}_{K_{Z^{\mathbb{Q}}}}\big(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\}\big)$. For a locally noetherian O_E-scheme S, we define $\mathcal{M}_0(S)$ to be the groupoid of triples $(A_0, \iota_0, \lambda_0)$, where

- A_0 is an abelian scheme over S;
- $\iota_0 \colon O_F \to \mathrm{End}(A_0)$ is an O_F-action satisfying the Kottwitz condition:

$$\mathrm{char}\big(\iota(a) \mid \mathrm{Lie}\, A_0\big) = \prod_{\varphi \in \Phi} \big(T - \varphi(a)\big) \quad \text{for all} \quad a \in O_F; \tag{6.8}$$

and

- λ_0 is a principal polarization on A_0 such that the induced Rosati involution via ι_0 coincides with the Galois involution on O_F.

A morphism between two objects $(A_0, \iota_0, \lambda_0)$ and $(A_0', \iota_0', \lambda_0')$ in this groupoid is an O_F-linear isomorphism $\mu_0 \colon A_0 \to A_0'$ under which λ_0' pulls back to λ_0. Then $\mathcal{M}_0$ is a Deligne–Mumford stack, finite and étale over $\mathrm{Spec}\, O_E$ cf. [19, Prop. 3.1.2].

Remark 6.1. In order to avoid the possible emptiness of $\mathcal{M}_0$ in some cases (cf. *loc. cit.*), we assume that F/F_0 is ramified throughout the paper. We let M_0 denote the generic fiber of $\mathcal{M}_0$. Its complex fiber $\mathcal{M}_0 \otimes_{O_E} \mathbb{C}$ is isomorphic to a finite number of copies of $\mathrm{Sh}_{K_{Z^{\mathbb{Q}}}}(Z^{\mathbb{Q}}, \{h_{Z^{\mathbb{Q}}}\})$. The parameterization of the copies is rather subtle, roughly in correspondence to isomorphism classes of one-dimensional F/F_0-hermitian spaces satisfying certain conditions (and the possible emptiness is accounted by the possible non-existence of hermitian spaces). It suffices for our purpose to note that the partition descends over E, and hence we will work with a fixed copy through out the paper. Hence we will suppress this issue in the notation and simply write $\mathcal{M}_0$ for this fixed copy. A detailed discussion is in [40, §3].

6.2.2. *The RSZ integral model for $\Big(\widetilde{G}, \{h_{\widetilde{G}}\}\Big)$.* We will follow [40, 41] to define the moduli interpretation of our Shimura varieties associated to the Shimura datum $\Big(\widetilde{G}, \{h_{\widetilde{G}}\}\Big)$ for certain special level structure. When $F_0 = \mathbb{Q}$, this is closely related to [26, 6]. In fact for this paper we only need to define an integral model over a Zariski open subscheme of $\mathrm{Spec}\, O_E$.

Let $\mathscr{D}_0$ denote the set consisting of all non-archimedean places v of F_0 such that

- the residue characteristic of v is 2, or
- v is ramified in F, or
- v is inert in F where V_v is non-split.

[2]Here we follow the convention of [41], which differs from [40] where the space V is totally *negative* definite.

Let $\mathscr{D}$ be a finite set of non-archimedean places containing $\mathscr{D}_0$, such that $\mathscr{D}$ is pull-back from a set of places $\mathscr{D}_{\mathbb{Q}}$ of $\mathbb{Q}$. Define

$$\mathfrak{d} = \prod_{p | \mathscr{D}_{\mathbb{Q}}} p.$$

We fix an O_F-lattice Λ in V such that for all $v \notin \mathscr{D}$

$$\Lambda_v = \Lambda_v^\vee.$$

There is no requirement at $v \in \mathscr{D}$.

We consider the open compact subgroups

$$K_G^\circ := \{ g \in G(\widehat{\mathbb{Q}}) \mid g(\Lambda \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}) = \Lambda \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \},$$

and

$$K_{\widetilde{G}}^\circ = K_{Z^{\mathbb{Q}}}^\circ \times K_G^\circ \subset \widetilde{G}(\widehat{\mathbb{Q}}),$$

where $K_{Z^{\mathbb{Q}}}^\circ = Z^{\mathbb{Q}}(\widehat{\mathbb{Z}})$ is the unique maximal compact open subgroup of $Z^{\mathbb{Q}}(\widehat{\mathbb{Q}})$.

From now on we will consider a fixed component $\mathcal{M}_0$ of the auxiliary moduli problem in §6.2.1 (cf. Remark 6.1).

Definition 6.2. The functor $\mathcal{M}_{K_{\widetilde{G}}^\circ}(\widetilde{G})$ associates to each locally noetherian $O_E[1/\mathfrak{d}]$-scheme S the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda)$, where

- $(A_0, \iota_0, \lambda_0)$ is an object of $\mathcal{M}_0(S)$;
- A is an abelian scheme over S;
- $\iota \colon O_F \to \mathrm{End}(A)$ is an action satisfying the Kottwitz condition of signature

$$((n-1,1)_{\varphi_0}, (n,0)_{\varphi \in \Phi \smallsetminus \{\varphi_0\}})$$

on O_F; and

- $\lambda : A \to A^\vee$ is a polarization on A whose Rosati involution inducing the Galois involution on O_F with respect to ι.
- We impose the sign condition (cf. [40, §4.1]) and require that the kernel of the polarization $\lambda : A \to A^\vee$ is of the prescribed type defined by the lattice Λ for every p, see [40, §4.1].

A morphism $(A_0, \iota_0, \lambda_0, A, \iota, \lambda) \to (A_0', \iota_0', \lambda_0', A', \iota', \lambda')$ in this groupoid is given by an isomorphism $(A_0, \iota_0, \lambda_0) \xrightarrow{\sim} (A_0', \iota_0', \lambda_0')$ in $\mathcal{M}_0(S)$ and an O_F-linear isomorphism $A \xrightarrow{\sim} A'$ of abelian schemes pulling λ' back to λ.

By [40, Prop. 3.5], the generic fiber of $\mathcal{M}_{K_{\widetilde{G}}^\circ}(\widetilde{G})$ is naturally isomorphic to the canonical model of $\mathrm{Sh}_{K_{\widetilde{G}}^\circ}(\widetilde{G}, \{h_{\widetilde{G}}\})$. By [40, Theorem 5.2], $\mathcal{M}_{K_{\widetilde{G}}^\circ}(\widetilde{G})$ is a Deligne–Mumford stack, the morphism $\mathcal{M}_{K_{\widetilde{G}}^\circ}(\widetilde{G}) \to \operatorname{Spec} O_E[1/\mathfrak{d}]$ is separated of finite type, and smooth of relative dimension $n-1$.

6.2.3. *Level structure.* We will need certain moduli spaces that are more general than $\mathcal{M}_{K_{\widetilde{G}}^\circ}(\widetilde{G})$, defined by level-structure at the finite set of places dividing $\mathfrak{d}$. Let $K_G = \prod_v K_{G,v} \subset K_G^\circ = K_{G,v}^\circ$ be a compact open subgroup and $K_{G,v} = K_{G,v}^\circ$ for $v \nmid \mathfrak{d}$, and accordingly define $K_{\widetilde{G}} = K_{Z^{\mathbb{Q}}}^\circ \times K_G$.

We define the moduli problem $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ analogous to $\mathcal{M}_{K_{\widetilde{G}}^\circ}(\widetilde{G})$.

Definition 6.3. The functor $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ associates to each locally noetherian $O_E[1/\mathfrak{d}]$-scheme S the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta})$, where

- $(A_0, \iota_0, \lambda_0)$ is an object of $\mathcal{M}_0(S)$;
- A is an abelian scheme over S;
- $\iota \colon O_F[1/\mathfrak{d}] \to \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}[1/\mathfrak{d}]$ is an action satisfying the Kottwitz condition of signature

$$((n-1,1)_{\varphi_0}, (n,0)_{\varphi \in \Phi \smallsetminus \{\varphi_0\}})$$

on $O_F[1/\mathfrak{d}]$; and

- $\lambda : A \to A^\vee$ is a prime-to-$\mathfrak{d}$ principle polarization whose Rosati involution inducing the Galois involution on $O_F[1/\mathfrak{d}]$ with respect to ι;

• $\overline{\eta}$ is a $\prod_{v|\mathfrak{d}} K_{G,v}$-orbit of isometries of hermitian modules (as smooth $F_{\mathfrak{d}} = \prod_{v|\mathfrak{d}} F_v$-sheaves on S endowed with its natural hermitian form induced by the polarization)

$$\eta\colon \mathrm{V}_{\mathfrak{d}}(A_0, A) \xrightarrow{\sim} V(F_{0,\mathfrak{d}}) , \tag{6.9}$$

where

$$\mathrm{V}_{\mathfrak{d}}(A_0, A)\colon = \prod_{p|\mathfrak{d}} \mathrm{V}_p(A_0, A), \quad \text{and} \quad \mathrm{V}_p(A_0, A) = \mathrm{Hom}_{F\otimes_{\mathbb{Q}}\mathbb{Q}_p}(V_p(A_0), V_p(A))$$

and

$$V(F_{0,\mathfrak{d}})\colon = \prod_{p|\mathfrak{d}} V \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{v|\mathfrak{d}} V \otimes_{F_0} F_{0,v}. \tag{6.10}$$

More precisely, this is understood in the sense of, e.g., [26, Remark 4.2]. Fixing any geometric point $\overline{s}$ of a connected scheme S, the rational Tate module $\mathrm{V}_{\mathfrak{d}}(A_0, A)$ is a smooth $F_{\mathfrak{d}} = \prod_{v|\mathfrak{d}} F_v$-sheaf on S determined by the rational Tate module $\mathrm{V}_{\mathfrak{d}}(A_{0,\overline{s}}, A_{\overline{s}})$ together with the action of the fundamental group $\pi_1(S, \overline{s})$. Moreover, the polarizations on A_0 and A defines an $F_{\mathfrak{d}}$-valued hermitian forms $\langle\cdot,\cdot\rangle$ on $\mathrm{V}_{\mathfrak{d}}(A_{0,\overline{s}}, A_{\overline{s}})$:

$$\langle x, y\rangle = \lambda_0^{-1} \circ y^\vee \circ \lambda \circ x \in \mathrm{End}_{F_{\mathfrak{d}}} (V_{\mathfrak{d}}(A_{0,\overline{s}})) = F_{\mathfrak{d}}.$$

Then the level structure $\overline{\eta}$ is a $\prod_{v|\mathfrak{d}} K_{G,v}$-orbit of isometries of hermitian modules

$$\eta\colon \mathrm{V}_{\mathfrak{d}}(A_{0,\overline{s}}, A_{\overline{s}}) \xrightarrow{\sim} V(F_{0,\mathfrak{d}}) , \tag{6.11}$$

that is required to be stable under the action of $\pi_1(S, \overline{s})$. The notion of $\prod_{v|\mathfrak{d}} K_{G,v}$-level structure is independent of the choice of the geometric point $\overline{s}$ on S.

• Finally, we also impose the sign condition (cf. [40, §4.1]).

A morphism between two objects $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta})$ and $(A_0', \iota_0', \lambda_0', A', \iota', \lambda', \overline{\eta}')$ is an isomorphism $(A_0, \iota_0, \lambda_0) \xrightarrow{\sim} (A_0', \iota_0', \lambda_0')$ in $\mathcal{M}_0(S)$ and an O_F-linear prime-to-$\mathfrak{d}$ isogeny $A \to A'$, pulling λ' back to λ and $\overline{\eta}'$ back to $\overline{\eta}$.

Similar to $\mathcal{M}_{K^\circ_{\widetilde{G}}}(\widetilde{G})$, the functor $\mathcal{M}_{K^\circ_{\widetilde{G}}}(\widetilde{G})$ is a Deligne–Mumford stack, the morphism $\mathcal{M}_{K^\circ_{\widetilde{G}}}(\widetilde{G}) \to \mathrm{Spec}\, O_E[1/\mathfrak{d}]$ is separated of finite type, and smooth of relative dimension $n-1$. When $\prod_{v|\mathfrak{d}} K_{G,v}$ is small enough, the functor $\mathcal{M}_{K^\circ_{\widetilde{G}}}(\widetilde{G})$ is represented by a quasi-projective scheme.

Finally, we briefly recall the moduli functor $M_{K_{\widetilde{G}}}(\widetilde{G})$ for the canonical model of the Shimura variety $\mathrm{Sh}_{K^\circ_{\widetilde{G}}}(\widetilde{G}, \{h_{\widetilde{G}}\})$ over $\mathrm{Spec}\, E$, for any compact open subgroup $K^\circ_{\widetilde{G}}$ of the form $K_{\widetilde{G}} = K^\circ_{Z^{\mathbb{Q}}} \times K_G$. Similar to Definition 6.3, the functor $M_{K_{\widetilde{G}}}(\widetilde{G})$ associates to each locally noetherian scheme S over $\mathrm{Spec}\, E$ the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta})$, where everything is the same as Definition 6.3 with the following minor change. Now $\iota\colon F \to \mathrm{End}^\circ(A)$ is an action, $\lambda : A \to A^\vee$ is a polarization, and $\overline{\eta}$ is a K_G-orbit of isometries of hermitian modules

$$\eta\colon \widehat{\mathrm{V}}(A_0, A) \xrightarrow{\sim} V(\mathbb{A}_{0,f}) , \tag{6.12}$$

where $\widehat{\mathrm{V}}(A_0, A)\colon = \prod_p \mathrm{V}_p(A_0, A)$, and $V(\mathbb{A}_{0,f}) = V \otimes_{F_0} \mathbb{A}_{F_0,f}$. The rest is the same as Definition 6.3, with the appropriate modification of the definition of morphisms in the groupoid, cf. [40, §3.2].

7. Kudla–Rapoport divisors and the derived CM cycles

In this section we consider two type of special cycles on the integral models of Shimura varieties in the previous section

• the Kudla–Rapoport special divisors [26], and

• the derived CM cycle, which is a variant of the (1-dimensional) "big CM cycle" of Bruinier–Kudla–Yang and Howard [8, 19].

The derived CM cycle is the main novel geometric construction of this paper.

We make the following notational remark: in Part 2 of the paper, all Schwartz functions on totally disconnected topological spaces are $\mathbb{Q}$-valued.

7.1. **The global Kudla–Rapoport divisors on $M_{K_{\widetilde{G}}}(\widetilde{G})$ over** $\operatorname{Spec} E$**.** We first define the global Kudla–Rapoport divisors on the canonical model $M_{K_{\widetilde{G}}}(\widetilde{G})$ over $\operatorname{Spec} E$, introduced at the end of §6, for an arbitrary compact open subgroup $K^\circ_{\widetilde{G}}$ of the form $K_{\widetilde{G}} = K^\circ_{Z^{\mathbb{Q}}} \times K_G$. We follow [26] when $F_0 = \mathbb{Q}$, and [31, Definition 4.21] and [41] for general totally real F_0.

Let $\xi \in F_{0,+}$ and let $\mu \in V(\mathbb{A}_{0,f})/K_G$ be a K_G-orbit.

Definition 7.1. For a locally noetherian scheme S over $\operatorname{Spec} E$, the S-points of the KR cycle $Z(\xi, \mu)$ is the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}, u)$ where

- $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}) \in M_{K_{\widetilde{G}}}(\widetilde{G})(S)$, and
- $u \in \operatorname{Hom}^\circ_F(A_0, A)$ such that $\langle u, u\rangle = \xi$, and $\overline{\eta}(u)$ is a homomorphism in the K_G-orbit μ. Here $\langle\cdot,\cdot\rangle$ denotes the hermitian form induced by the polarization λ_0 and λ:
$$\langle x, y\rangle = \lambda_0^{-1} \circ y^\vee \circ \lambda \circ x \in \operatorname{End}^\circ_F(A_0) \simeq F.$$

A morphism between two objects $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}, u)$ and $(A_0', \iota_0', \lambda_0', A', \iota', \lambda', \overline{\eta}', u')$ is an isomorphism $(A_0, \iota_0, \lambda_0) \xrightarrow{\sim} (A_0', \iota_0', \lambda_0')$ in $\mathcal{M}_0(S)$ and an F-linear isogeny $\varphi : A \to A'$, compatible with λ and λ', and with $\overline{\eta}$ and $\overline{\eta}'$, and such that $u' = u \circ \varphi$.

Forgetting u defines a natural morphism $i : Z(\xi, \mu) \to M_{K_{\widetilde{G}}}(\widetilde{G})$, and we defer to Proposition 7.3 for its properties. In particular, the push-forward defines a class in the Chow group $\mathrm{Ch}^1(M_{K_{\widetilde{G}}}(\widetilde{G}))$. For $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))^{K_G}$,[3] we define
$$Z(\xi, \phi) := \sum_{\mu \in V(\mathbb{A}_{0,f})_\xi / K_G} \phi(\mu)\, Z(\xi, \mu), \tag{7.1}$$
viewed as an element in the Chow group $\mathrm{Ch}^1(M_{K_{\widetilde{G}}}(\widetilde{G}))$. Here
$$V(\mathbb{A}_{0,f})_\xi := \{\mu \in V(\mathbb{A}_{0,f}) \mid \langle \mu, \mu\rangle = \xi\}.$$
Note that (7.1) is a finite sum due to the compactness of the support of ϕ (and $G(\mathbb{A}_{0,f})$ acts transitively on $V(\mathbb{A}_{0,f})_\xi$ when $\xi \neq 0$).

7.2. **The global Kudla–Rapoport divisors on the integral model $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$.** We now consider the moduli function $\mathcal{M} = \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ with level structure at primes dividing $\mathfrak{d}$, cf. Definition (6.3). Here $K^\circ_{\widetilde{G}}$ is of the form $K_{\widetilde{G}} = K^\circ_{Z^{\mathbb{Q}}} \times K_G$ with $K_{G,v} = K^\circ_{G,v}$ for $v \nmid \mathfrak{d}$ where $K^\circ_{G,v}$ is the stabilizer of a self-dual lattice Λ_v.

Let $\xi \in F_{0,+}$ and $\mu \in V(F_{0,\mathfrak{d}})/K_{G,\mathfrak{d}}$. Here $V(F_{0,\mathfrak{d}})$ is as in (6.10).

Definition 7.2. For a locally noetherian scheme S over $\operatorname{Spec} O_E[1/\mathfrak{d}]$, the S-points of the KR cycle $\mathcal{Z}(\xi, \mu)$ is the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}, u)$ where

- $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}) \in \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})(S)$, and
- $u \in \operatorname{Hom}_{O_F}(A_0, A) \otimes_{\mathbb{Z}} \mathbb{Z}[1/\mathfrak{d}]$ such that $\langle u, u\rangle = \xi$, and $\overline{\eta}(u)$ is a homomorphism in the $K_{G,\mathfrak{d}}$-orbit μ. Here $\langle\cdot,\cdot\rangle$ denotes the hermitian form induced by the polarization λ_0 and λ:
$$\langle x, y\rangle = \lambda_0^{-1} \circ y^\vee \circ \lambda \circ x \in \operatorname{End}_{O_F}(A_0) \otimes_{\mathbb{Z}} \mathbb{Z}[1/\mathfrak{d}] \simeq O_F[1/\mathfrak{d}].$$

A morphism between two objects $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}, u)$ and $(A_0', \iota_0', \lambda_0', A', \iota', \lambda', \overline{\eta}', u')$ is an isomorphism $(A_0, \iota_0, \lambda_0) \xrightarrow{\sim} (A_0', \iota_0', \lambda_0')$ in $\mathcal{M}_0(S)$ and an O_F-linear prime-to-$\mathfrak{d}$ isogeny $\varphi : A \to A'$, compatible with λ and λ', and with $\overline{\eta}$ and $\overline{\eta}'$, and such that $u' = u \circ \varphi$.

Forgetting u defines a natural morphism $i : \mathcal{Z}(\xi, \mu) \to \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$.

Proposition 7.3. *(a) The morphism $i : \mathcal{Z}(\xi, \mu) \to \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ is representable, finite and unramified.*

(b) The morphism i defines étale locally a Cartier divisor. Moreover, the morphism $\mathcal{Z}(\xi, \mu) \to \operatorname{Spec} O_E[1/\mathfrak{d}]$ is flat.

[3]Henceforth we will assume all the Schwartz functions (i.e., locally constant with compact support) that are used to define cycles are $\mathbb{Q}$-valued, in order to define elements in various Chow group or K-groups with $\mathbb{Q}$-coefficients.

Proof. When $F_0 = \mathbb{Q}$, part (a) follows from [26, Prop. 2.9], part (b) from [6, §2.5]. For a general totally real F_0, both follow from [31, Prop. 4.22]. $\square$

To a function $\phi_{\mathfrak{d}} \in \mathcal{S}(V_{\mathfrak{d}})^{K_{G,\mathfrak{d}}}$, we associate $\phi = \mathbf{1}_{\Lambda^{\mathfrak{d}}} \otimes \phi_{\mathfrak{d}} \in \mathcal{S}(V(\mathbb{A}_{0,f}))$, where $\Lambda^{\mathfrak{d}} = \prod_{v \nmid \mathfrak{d}} \Lambda_v$. Then we define

$$\mathcal{Z}(\xi, \phi) :\, = \sum_{\mu \in V(F_{0,\mathfrak{d}})_\xi / K_G} \phi_{\mathfrak{d}}(\mu)\, \mathcal{Z}(\xi, \mu), \tag{7.2}$$

viewed as an element in the Chow group $\mathrm{Ch}^1(\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}))$. The generic fiber of $\mathcal{Z}(\xi, \phi)$ is $Z(\xi, \phi)$ defined by (7.1) (specializing to the current level $\widetilde{K}_G$).

7.3. **Special divisors in the formal neighborhood of the basic locus.** We consider the restriction of the KR divisors to the formal completion of $\mathcal{M} = \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ along the basic locus.

Let $\nu \nmid \mathfrak{d}$ be a non-archimedean place of E. Its restriction to F (F_0, resp.) is a place denoted by w_0 (v_0, resp.). Assume that v_0 is *inert*. We recall from [40, §8, in the proof of Thm. 8.15] the non-archimedean uniformization along the basic locus:

$$\mathcal{M}^{\widehat{}}_{O_{\breve{E}_\nu}} :\, = \big(\mathcal{M}_{(\nu)} \otimes_{O_{E,(\nu)}} O_{\breve{E}_\nu}\big)^{\widehat{}} = \widetilde{G}' \Big\backslash \Big[\mathcal{N}' \times \widetilde{G}(\mathbb{A}_f^p)/K_{\widetilde{G}}^p\Big]. \tag{7.3}$$

Here the hat on the left-hand side denotes the completion along the basic locus in the special fiber of $\mathcal{M}_{(\nu)}$. The group $\widetilde{G}'$ is an inner twist of $\widetilde{G}$. More precisely, the group $\widetilde{G}'$ is associated to the "nearby" hermitian space V', that is positive definite at all archimedean places, and isomorphic to V, locally at all non-archimedean places except at v_0. Let $\mathcal{N} \to \operatorname{Spf} O_{\breve{F}_{w_0}}$ be the *relative* RZ space, i.e., the formal scheme of polarized p-divisible groups with action by O_{F,w_0} satisfying the Kottwitz condition of signature $((1, n-1)_{\varphi_0}, (0, n)_{\varphi \in \Phi_{v_0} \smallsetminus \{\varphi_0\}})$. Let $\mathcal{N}_{O_{\breve{E}_\nu}} = \mathcal{N} \widehat{\otimes}_{O_{\breve{F}_{w_0}}} O_{\breve{E}_\nu}$. Then as in *loc. cit.* we may rewrite (7.3) as

$$\mathcal{M}^{\widehat{}}_{O_{\breve{E}_\nu}} = \widetilde{G}'(\mathbb{Q}) \Big\backslash \Big[\mathcal{N}_{O_{\breve{E}_\nu}} \times \widetilde{G}(\mathbb{A}_f^{v_0})/K_{\widetilde{G}}^{v_0}\Big]. \tag{7.4}$$

Here we denote (even though $\widetilde{G}$ is not an algebraic group over F_0)

$$\widetilde{G}(\mathbb{A}_f^{v_0})/K_{\widetilde{G}}^{v_0} = \widetilde{G}(\mathbb{A}_f^p)/K_{\widetilde{G}}^p \times \big(Z^{\mathbb{Q}}(\mathbb{Q}_p)/K_{Z^{\mathbb{Q}},p}\big) \times \prod_{v \in \mathcal{V}_p \smallsetminus \{v_0\}} G(F_{0,v})/K_{G,v}.$$

Here we fix an isomorphism $G'(\mathbb{A}_f^{v_0}) \simeq G(\mathbb{A}_f^{v_0})$.

Note that the uniformization (7.4) induces a projection to a discrete set (in fact an abelian group)

$$\mathcal{M}^{\widehat{}}_{O_{\breve{E}_\nu}} \longrightarrow Z^{\mathbb{Q}}(\mathbb{A}) \backslash \big(Z^{\mathbb{Q}}(\mathbb{A}_f)/K_{Z^{\mathbb{Q}}}\big). \tag{7.5}$$

This gives a partition of the formal scheme $\mathcal{M}^{\widehat{}}_{O_{\breve{E}_\nu}}$, each fiber is naturally isomorphic to

$$\mathcal{M}^{\widehat{}}_{O_{\breve{E}_\nu},0} := G'(\mathbb{Q}) \Big\backslash \Big[\mathcal{N}_{O_{\breve{E}_\nu}} \times G(\mathbb{A}_f^{v_0})/K_G^{v_0}\Big]. \tag{7.6}$$

Recall that we have the local KR divisors $\mathcal{Z}(u)$ on $\mathcal{N}$ for each $u \in V' \otimes F_{0,v_0} \simeq \mathrm{Hom}^\circ(\mathbb{E}, \mathbb{X}_n)$, the hermitian space of local special homomorphisms (w.r.t. some fixed framing objects $\mathbb{E}$ and $\mathbb{X}_n$ in the uniformization (7.4) above). For a pair $(u, g) \in V'(F_0) \times G(\mathbb{A}_f^{v_0})/K_G^{v_0}$ with $u \neq 0$, we define the product divisor on $\mathcal{N}_{O_{\breve{E}_\nu}} \times G(\mathbb{A}_f^{v_0})/K_G^{v_0}$

$$\mathcal{Z}(u, g)_{K_G^{v_0}} = \mathcal{Z}(u) \times \mathbf{1}_{g\, K_G^{v_0}}, \tag{7.7}$$

and its image in the quotient (7.6)

$$[\mathcal{Z}(u, g)]_{K_G^{v_0}} = \sum \mathcal{Z}(u', g')_{K_G^{v_0}}, \tag{7.8}$$

where the sum is over (u', g') in the $G'(\mathbb{Q})$-orbit of the pair (u, g) (for the diagonal action of $G'(\mathbb{Q})$ on $V'(F_0) \times G(\mathbb{A}_f^{v_0})/K_G^{v_0}$).

Proposition 7.4. *Let $\xi \in F_{0,+}$. The restriction of the special divisor $\mathcal{Z}(\xi, \phi)$ to each fiber of the above projection* (7.5) *is the sum*

$$\sum_{(u,g)\in G'(\mathbb{Q})\backslash (V'(F_0)_\xi \times G(\mathbb{A}_f^{v_0})/K_G^{v_0})} \phi^{v_0}(g^{-1}u)\cdot [\mathcal{Z}(u,g)]_{K_G^{v_0}}, \tag{7.9}$$

viewed as a divisor on (7.6)*. This is a finite sum.*

Remark 7.5. This is similar to the description of the special divisors over the complex number, cf. (8.8).

Proof. This follows from the proof of [26, Proposition 6.3], also cf. [31, §4.2] □

7.4. **Fat big CM cycles.** We introduce a fat variant of the "big CM cycle" of [8, 19] on our moduli space $\mathcal{M} = \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ with level structure at primes dividing $\mathfrak{d}$ (cf. Definition 6.3).

Let F_0' be a totally real extension of F_0 of degree n. Then $F' = F \otimes_{F_0} F_0'$ is a CM extension of F_0'.

$$\begin{tikzcd}
 & F' = F\otimes_{F_0} F_0' \arrow[dl, dash] \arrow[d, dash] \\
F_0' \arrow[d, dash] & F \arrow[dl, dash] \\
F_0 \arrow[d, dash] & \\
\mathbb{Q} &
\end{tikzcd}$$

Fix an F/F_0-hermitian space V as before. Consider a 1-dimensional F'/F_0'-hermitian space $(W, (\cdot,\cdot)_{F_0'})$ such that

$$(\mathrm{R}_{F'/F}W,\ \mathrm{tr}_{F'/F}(\cdot,\cdot)_{F_0'}) \xrightarrow{\ \sim\ } (V,\ (\cdot,\cdot))\ . \tag{7.10}$$

Here $\mathrm{R}_{F'/F}W$ denotes the "restriction of scalar" of W, i.e., to view it as an F-vector space. To choose such an isometry is the same as to choose an embedding (as F_0-algebraic groups)

$$\mathfrak{i} : F'^1 \longrightarrow G = \mathrm{U}(V), \tag{7.11}$$

where

$$F'^1 = \ker(\mathrm{Nm} : F'^\times \longrightarrow F_0'^\times) = \mathrm{U}(W). \tag{7.12}$$

Let R be an $O_F[1/\mathfrak{d}]$-order in F', i.e., a locally free $O_F[1/\mathfrak{d}]$-subalgebra of F' of rank $[F' : F]$. Let $\mathrm{Ram}(R)$ be the set of primes $v \nmid \mathfrak{d}$ of F_0 where R is non-maximal (i.e., $R \otimes_{O_{F_0}} O_{F_0,v}$ is not a product of DVRs). We make the following important hypothesis throughout the rest of the paper:

The order R is monogenic $R = O_F[1/\mathfrak{d}, g_0]$ for some element $g_0 \in F'^1$.

Denote by $\mathrm{char}_F(g_0) \in O_F[1/\mathfrak{d}][T]$ the characteristic polynomial of g_0 (acting on F' as an F-vector space). Then it is irreducible of degree equal to $[F' : F]$.

Definition 7.6. The functor $\mathcal{CM}_{R=O_F[1/\mathfrak{d},g_0]}$ associates to each locally noetherian $O_E[1/\mathfrak{d}]$-scheme S the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}, \varphi)$ where $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}) \in \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ and $\varphi \in \mathrm{End}_{O_F}(A) \otimes \mathbb{Z}[1/\mathfrak{d}]$ such that

- the characteristic polynomial $\mathrm{char}_F(g_0)$ annihilates the endomorphism φ;
- φ is compatible with λ, i.e., $\varphi^*\lambda = \lambda$, or equivalently, the Rosati involution sends φ to φ^{-1}; and

- φ preserves the level structure $\overline{\eta}$, i.e., we have a commutative diagram

$$\begin{tikzcd} \mathrm{V}_{\mathfrak{d}}(A_0, A) \arrow[r, "\eta_1"] \arrow[d, "\varphi"] & V(F_{0,\mathfrak{d}}) \arrow[d, "\mathrm{id}"] \\ \mathrm{V}_{\mathfrak{d}}(A_0, A) \arrow[r, "\eta_2"] & V(F_{0,\mathfrak{d}}), \end{tikzcd}$$

where $\eta_i \in \overline{\eta}$.

Morphisms are defined in the obvious way.

Remark 7.7. We warn the reader that the stack $\mathcal{CM}_{R=O_F[1/\mathfrak{d},g_0]}$ depends not only on R but also the characteristic polynomial $\mathrm{char}_F(g_0)$, and therefore the notation may be a little misleading.

We have a natural forgetful map

$$\mathcal{CM}_R \longrightarrow \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}).$$

We call $\mathcal{CM}_R$ the (naive) fat big CM cycle, or simply CM cycle.

Remark 7.8. Strictly speaking our moduli space does not cover the big CM cycle in [19] where R is a maximal order and hence may not be monogenic of the desired type. Note that we do not impose any Kottwitz signature condition in our Definition 7.6, while [19] does. Unlike the moduli space for $Z^{\mathbb{Q}}$, our moduli space $\mathcal{M}_0'$ for F'/F_0' is alway non-empty (cf. Remark 6.1). Note that the emptiness was caused by non-existence of certain hermitian space. In our case, we start from the one-dimensional hermitian space W. The price we pay is that we have lost control on the order R and the moduli space $\mathcal{CM}_R$ could have very bad fibers, e.g., a large dimension in positive characteristic. In this sense, the fat big CM cycle seems not very practically useful. However, the AFL type identity allows us to understand some basic property of such moduli spaces, at the bad fibers (with a natural derived structure on them) where the ambient $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ has good reduction.

We define a twisted variant of $\mathcal{CM}_R$. Let $R = O_F[1/\mathfrak{d}, g_0]$ be as before.

Definition 7.9. Let $g \in \prod_{v|\mathfrak{d}} G(F_v)$. The functor $\mathcal{CM}_R(g)$ associates to each $O_E[1/\mathfrak{d}]$-scheme S the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}, \varphi)$ where $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}) \in \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ and $\varphi \in \mathrm{End}_{O_F}(A) \otimes \mathbb{Z}[1/\mathfrak{d}]$ such that

- the characteristic polynomial $\mathrm{char}_F(g_0)$ annihilates the endomorphism φ;
- φ is compatible with λ, i.e., $\varphi^*\lambda = \lambda$, or equivalently, the Rosati involution sends φ to φ^{-1}; and
- We have a commutative diagram

$$\begin{tikzcd} \mathrm{V}_{\mathfrak{d}}(A_0, A) \arrow[r, "\eta_1"] \arrow[d, "\varphi"] & V(F_{0,\mathfrak{d}}) \arrow[d, "g"] \\ \mathrm{V}_{\mathfrak{d}}(A_0, A) \arrow[r, "\eta_2"] & V(F_{0,\mathfrak{d}}), \end{tikzcd}$$

where $\eta_i \in \overline{\eta}$.

Morphisms are defined in the obvious way.

Remark 7.10. When K_G is a normal subgroup of K_G°, and $g \in K_G^\circ$, the last condition is simplified as $\varphi = \eta^{-1}(g)$ for some $\eta \in \overline{\eta}$ in the sense that we have a commutative diagram

$$\begin{tikzcd} \mathrm{V}_{\mathfrak{d}}(A_0, A) \arrow[r, "\eta"] \arrow[d, "\varphi"] & V(F_{0,\mathfrak{d}}) \arrow[d, "g"] \\ \mathrm{V}_{\mathfrak{d}}(A_0, A) \arrow[r, "\eta"] & V(F_{0,\mathfrak{d}}). \end{tikzcd}$$

Remark 7.11. We may think of $\mathcal{CM}_{R=O_F[1/\mathfrak{d},g_0]}$ as the (naive) "fixed point locus of g_0" even though there is no group action of F'^1 on the entire moduli space $\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$. Then the twisted variant may be viewed as the (naive) "fixed point locus of g_0 composed with a Hecke correspondence corresponding to the double coset $K_G\, g\, K_G$".

Proposition 7.12. *Let $g \in \prod_{v|\mathfrak{d}} G(F_v)$.*

(a) The morphism $\mathcal{CM}_R(g) \to \mathcal{M}$ is representable, finite and unramified.

(b) The morphism $\mathcal{CM}_R(g) \to \operatorname{Spec} O_E[1/\mathfrak{d}]$ is proper. Its restriction to the open sub-scheme $\operatorname{Spec} O_E[1/\mathfrak{d}] \setminus \operatorname{Ram}(R)$ is finite étale.

Proof. The first part follows similarly to Proposition 7.3 (it is representable of locally finite type by the theory of Hilbert scheme; it is unramified by the rigidity of quasi-isogeny; it is quasi-finite because there are only finitely many ways to endow an action of R to a given (A, ι, λ) over an arbitrary field; it satisfies the valuative criterion by the Néron property of abelian scheme, therefore it is proper, and hence finite).

The properness of $\mathcal{CM}_R(g) \to \operatorname{Spec} O_E[1/\mathfrak{d}]$ follows by the valuative criterion (the toric part of a semi-abelian scheme will have too small dimension to have an action of R). Finally, the argument of [19, Prop. 3.12] still holds to show the finiteness and étaleness over $\operatorname{Spec} O_E[1/\mathfrak{d}] \setminus \operatorname{Ram}(R)$: at every place above $v \notin \operatorname{Ram}(R)$, the local order $R \otimes_{O_F} O_{F,v}$ is maximal and hence the p-divisible group has formal multiplication by a local maximal order. $\square$

7.5. **Hecke correspondences and their fixed point loci.** We first define the Hecke correspondence.

Definition 7.13. Let $g \in \prod_{v|\mathfrak{d}} G(F_v)$. The functor $\mathrm{Hk}_{[K_G\, g\, K_G]}$ associates to each $O_E[1/\mathfrak{d}]$-scheme S the groupoid of tuples

$$(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}, A', \iota', \lambda', \overline{\eta}', \varphi)$$

where $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}), (A_0, \iota_0, \lambda_0, A', \iota', \lambda', \overline{\eta}') \in \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})(S)$, and a quasi-isogeny $\varphi \in \operatorname{Hom}_{O_F}(A, A') \otimes \mathbb{Z}[1/\mathfrak{d}]$ such that

- φ is compatible with λ, λ'.
- There exist $\eta \in \overline{\eta}$ and $\eta' \in \overline{\eta}'$ such that the diagram

$$\begin{tikzcd} \mathrm{V}_{\mathfrak{d}}(A_0, A) \arrow[r, "\eta"] \arrow[d, "\varphi"] & V(F_{0,\mathfrak{d}}) \arrow[d, "g"] \\ \mathrm{V}_{\mathfrak{d}}(A_0, A') \arrow[r, "\eta'"] & V(F_{0,\mathfrak{d}}) \end{tikzcd}$$

  commutes. Here the left vertical map on rational Tate modules is induced by φ. Note that this is to be understood similarly to the definition of level structure (cf. Definition 6.3).

Morphisms are defined in the obvious way.

We have a natural morphism

$$\mathrm{Hk}_{[K_G\, g\, K_G]} \longrightarrow \mathcal{M} \times_{O_E[1/\mathfrak{d}]} \mathcal{M}.$$

This morphism is finite, and the projection to any one factor is a finite étale morphism.

Now consider the fiber product, called the "fixed point locus of the Hecke correspondence $\mathrm{Hk}_{[K_G\, g\, K_G]}$"

$$\begin{tikzcd} \mathcal{M}_{[K_G\, g\, K_G]} := \mathrm{Hk}_{[K_G\, g\, K_G]} \times_{\mathcal{M}\times\mathcal{M}} \Delta_{\mathcal{M}} \arrow[r] \arrow[d] & \mathrm{Hk}_{[K_G\, g\, K_G]} \arrow[d] \\ \mathcal{M} \arrow[r, "\Delta"] & \mathcal{M} \times_{O_E[1/\mathfrak{d}]} \mathcal{M}. \end{tikzcd}$$

Let S be a connected scheme. To any object in $\mathcal{M}_{[K_G\, g\, K_G]}(S)$, we may associate a characteristic polynomial with coefficients in F as follows. First of all we may assume that such an element in $\mathcal{M}_{[K_G\,g\,K_G]}(S)$ is of the form $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}, A, \iota, \lambda, \overline{\eta}, \varphi)$. Fix any geometric point $\overline{s}$ of S, and a representative η in the $\prod_{v|\mathfrak{d}} K_{G,v}$-orbit $\overline{\eta}$ of isometries of hermitian modules

$$\eta\colon\ \mathrm{V}_{\mathfrak{d}}(A_{0,\overline{s}}, A_{\overline{s}}) \xrightarrow{\sim} V(F_{0,\mathfrak{d}})\ .$$

Then we have an endomorphism

$$\eta(\varphi_{\overline{s}}) := \eta \circ \varphi_{\overline{s}} \circ \eta^{-1} \in \mathrm{End}_{F_{\mathfrak{d}}}(V(F_{0,\mathfrak{d}}))\ .$$

Then we define the characteristic polynomial (over F) of φ, denoted by $\mathrm{char}_F(\varphi)$, to be the characteristic polynomial of $\eta(\varphi_{\overline{s}})$, an element in the polynomial ring $F_{\mathfrak{d}}[T]$ (where T is the indeterminant) of degree n. This is independent of the choice of the geometric point $\overline{s}$ on S, and the choice of the representative $\eta \in \overline{\eta}$.

Here is an alternative definition of the characteristic polynomial $\mathrm{char}_F(\varphi)$ of φ. First of all, any $\varphi_{\overline{s}} \in \mathrm{End}^\circ(A_{\overline{s}})$ (not necessarily commuting with the O_F-action) induces an endomorphism of the $\mathbb{Q}_\ell$-vector space $\mathrm{V}_\ell(A_{\overline{s}})$ where ℓ is any prime different from the characteristics of $\overline{s}$. The characteristic polynomial, a priori with $\mathbb{Q}_\ell$-coefficients, has coefficients in $\mathbb{Q}$ because they can be computed as intersection numbers between algebraic cycles (with $\mathbb{Q}$-coefficients) on powers of A. We denote it by $\mathrm{char}_{\mathbb{Q}}(\varphi) \in \mathbb{Q}[T]_{\deg=n[F:\mathbb{Q}]}$, where $\mathbb{Q}[T]_{\deg=n[F:\mathbb{Q}]}$ denotes the set of monic polynomials of degree $n[F:\mathbb{Q}]$ with coefficients in $\mathbb{Q}$. Similarly we can define the trace $\mathrm{tr}_{\mathbb{Q}}(\varphi) \in \mathbb{Q}$, and knowing $\mathrm{tr}_{\mathbb{Q}}(\varphi^i)$ for all $i \geq 0$ is equivalent to knowing $\mathrm{char}_{\mathbb{Q}}(\varphi)$. If φ commutes with the O_F-action on A, we can define $\mathrm{tr}_F(\varphi) \in F$, characterized by

$$\mathrm{tr}_{F/\mathbb{Q}}(a\,\mathrm{tr}_F(\varphi)) = \mathrm{tr}_{\mathbb{Q}}(\iota(a)\varphi), \quad \text{for all } a \in O_F.$$

From $\mathrm{tr}_F(\varphi^i)$ for all $i \geq 0$, there exists a unique $\mathrm{char}_F(\varphi) \in F[T]_{\deg=n}$ with expected properties.

It is clear that the two definitions of $\mathrm{char}_F(\varphi)$ coincide (via the natural embedding $F \hookrightarrow F_{\mathfrak{d}}$). Therefore we obtain a locally constant map (for the Zariski topology on the source)

$$\mathrm{char}_F\colon \mathcal{M}_{[K_G\,g\,K_G]} \longrightarrow F[T]_{\deg=n}. \tag{7.13}$$

The image is a finite set because the source is of finite type and hence has only finitely many connected components. It follows that the fixed point locus $\mathcal{M}_{[K_G\,g\,K_G]}$ is a disjoint union of open and closed substacks, indexed by the image under the map (7.13):

$$\mathcal{M}_{[K_G\,g\,K_G]} = \coprod_{a\in \mathrm{Im}(\mathrm{char}_F)} \mathrm{char}_F^{-1}(a). \tag{7.14}$$

Finally, back to Definition 7.9, let R be a monogenic order $R = O_F[1/\mathfrak{d}, g_0]$ for some element $g_0 \in F'^1$ with (irreducible) characteristic polynomial $\mathrm{char}(g_0) \in F[T]_{\deg=n}$.

Lemma 7.14. *The fiber of the map* char_F (7.13) *above the polynomial* $\mathrm{char}_F(g_0)$ *is canonically isomorphic to the twisted CM cycle* $\mathcal{CM}_R(g)$ *as in Definition 7.9.*

Proof. By Definition (7.13), the fiber of $\mathrm{char}_F\, P$ above $\mathrm{char}_F(g_0)$ is the functor whose S-points are the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \overline{\eta}, \varphi)$ satisfying the same conditions as in Definition 7.9, except the first one, i.e., $\mathrm{char}_F(g_0)(\varphi) = 0$. This condition is equivalent to the condition on the characteristic polynomial of φ by Cayley–Hamilton theorem and the assumption that $\mathrm{char}_F(g_0)$ is irreducible.

□

7.6. **Derived CM cycle** $^{\mathbb{L}}\mathcal{CM}_R(g)$**.** In §7.5, the twisted variant $\mathcal{CM}_R(g)$ is recognized as a union of some connected components of the fixed point locus $\mathcal{M}_{[K_G\,g\,K_G]}$, cf. (7.13):

$$\begin{tikzcd}
\mathcal{CM}_R(g) \arrow[r] & \mathcal{M}_{[K_G\,g\,K_G]} \arrow[r] \arrow[d] & \mathrm{Hk}_{[K_G\,g\,K_G]} \arrow[d] \\
& \mathcal{M} \arrow[r, "\Delta"] & \mathcal{M} \times_{O_E[1/\mathfrak{d}]} \mathcal{M}.
\end{tikzcd}$$

This allows us to endow $\mathcal{CM}_R(g)$ with a derived structure, by taking the restriction of the derived tensor product

$$^{\mathbb{L}}\mathcal{CM}_R(g) := [\mathcal{O}_{\mathrm{Hk}_{[K_G\, g\, K_G]}} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{M}}] \,|_{\mathcal{CM}_R(g)} \in K_0'(\mathcal{CM}_R(g)). \tag{7.15}$$

Moreover, since Δ is a regular immersion, by the dimension calculation, this element lies in the filtration

$$^{\mathbb{L}}\mathcal{CM}_R(g) \in F_1\, K_0'(\mathcal{CM}_R(g)). \tag{7.16}$$

We extend the derived CM cycle to a weighted version. Let $\mathcal{S}\left(\prod_{v|\mathfrak{d}} G(F_v), K_{G,\mathfrak{d}}\right)$ be the space of bi-$K_{G,\mathfrak{d}}$-invariant Schwartz functions. For $\phi_{\mathfrak{d}} \in \mathcal{S}\left(\prod_{v|\mathfrak{d}} G(F_v), K_{G,\mathfrak{d}}\right)$, we denote $\phi_0 = \mathbf{1}_{K_G^{\mathfrak{d}}} \otimes \phi_{\mathfrak{d}} \in \mathcal{S}(G(\mathbb{A}_{0,f}))$ (here $K_G^{\mathfrak{d}} = \prod_{v\nmid\mathfrak{d}} K_{G,v}^{\circ}$). We then define $^{\mathbb{L}}\mathcal{CM}_R(\phi_0)$ as a formal sum of above twisted variants

$$^{\mathbb{L}}\mathcal{CM}_R(\phi_0) = \sum_{g\in K_G\backslash G(\mathbb{A}_{0,f})/K_G} \phi_0(g)\; ^{\mathbb{L}}\mathcal{CM}_R(g), \tag{7.17}$$

where we now view the summands as elements in $\bigoplus_{g\in K_G\backslash G(\mathbb{A}_{0,f})/K_G} K_0'(\mathcal{CM}_R(g))$. Moreover, these elements lie in the filtration, cf. (7.16),

$$^{\mathbb{L}}\mathcal{CM}_R(\phi_0) \in \bigoplus_{g\in K_G\backslash G(\mathbb{A}_{0,f})/K_G} F_1\, K_0'(\mathcal{CM}_R(g)). \tag{7.18}$$

7.7. **Hecke correspondences in the formal neighborhood of the basic locus.** We now consider the restriction of the Hecke correspondence $\mathrm{Hk}_{K_G\, g\, K_G}$ to the formal neighborhood of the basic locus at a non-archimedean place $v_0 \nmid \mathfrak{d}$ *inert* in F, via the RZ uniformization (7.4). We resume the notation there.

We consider the fiber product (in the category of locally noetherian formal schemes)

$$\begin{tikzcd}
\widehat{\mathrm{Hk}}_{[K_G\, g\, K_G]} \arrow[r] \arrow[d] & \mathrm{Hk}_{[K_G\, g\, K_G]} \arrow[d] \\
\widehat{\mathcal{M}}_{O_{\breve{E}_\nu}} \times_{\mathrm{Spf}\, O_{\breve{E}_\nu}} \widehat{\mathcal{M}}_{O_{\breve{E}_\nu}} \arrow[r] & \mathcal{M} \times_{O_E[1/\mathfrak{d}]} \mathcal{M}.
\end{tikzcd}$$

The commutative diagram in fact lives over the base $Z^{\mathbb{Q}}(\mathbb{A})\backslash\big(Z^{\mathbb{Q}}(\mathbb{A}_f)/K_{Z^{\mathbb{Q}}}\big)$, cf. (7.5). Therefore it suffices to consider the fiber (cf. (7.6)) over any fixed element of $Z^{\mathbb{Q}}(\mathbb{A})\backslash\big(Z^{\mathbb{Q}}(\mathbb{A}_f)/K_{Z^{\mathbb{Q}}}\big)$.

Then it follows immediately that

Proposition 7.15. *Let*

$$\mathrm{Hk}^{(v_0)}_{[K_G\, g\, K_G]} := \{(g_1, g_2) \in G(\mathbb{A}_f^{v_0})/K_G^{v_0} \times G(\mathbb{A}_f^{v_0})/K_G^{v_0} \mid g_1^{-1} g_2 \in K_G g K_G\}$$

with the two obvious projection maps, and the diagonal action by $G'(\mathbb{Q})$ from the left multiplication. Then the Hecke correspondence $\widehat{\mathrm{Hk}}_{[K_G\, g\, K_G],0}$ can be identified with

$$\begin{tikzcd}
\widehat{\mathrm{Hk}}_{[K_G\, g\, K_G],0} \arrow[r, "\sim"] \arrow[d] & G'(\mathbb{Q})\Big\backslash\Big[\mathcal{N}_{O_{\breve{E}_\nu}} \times \mathrm{Hk}^{(v_0)}_{[K_G\, g\, K_G]}\Big] \arrow[d] \\
\widehat{\mathcal{M}}_{O_{\breve{E}_\nu},0} \times \widehat{\mathcal{M}}_{O_{\breve{E}_\nu},0} \arrow[r, "\sim"] & G'(\mathbb{Q})\Big\backslash\Big[\mathcal{N}_{O_{\breve{E}_\nu}} \times G(\mathbb{A}_f^{v_0})/K_G^{v_0}\Big] \times G'(\mathbb{Q})\Big\backslash\Big[\mathcal{N}_{O_{\breve{E}_\nu}} \times G(\mathbb{A}_f^{v_0})/K_G^{v_0}\Big]
\end{tikzcd}$$

where the right vertical map is induced by the diagonal $\mathcal{N}_{O_{\breve{E}_\nu}} \to \mathcal{N}_{O_{\breve{E}_\nu}} \times \mathcal{N}_{O_{\breve{E}_\nu}}$, and the two projection maps from $\mathrm{Hk}_g^{(v_0)}$.

7.8. **CM cycles in the formal neighborhood of the basic locus.** We now consider the restriction of the fat big CM cycle and its derived version to the formal neighborhood of the basic locus at a non-archimedean place $v_0 \nmid \mathfrak{d}$ *inert* in F, via the RZ uniformization (7.4). We resume the notation there.

Fix an embedding $F'^1 \to G'$ (viewed as $\mathbb{Q}$-algebraic groups). Such embedding exists if the cycle $\mathcal{CM}_R$ has non-empty restriction to the basic locus.

Lemma 7.16. *Let $G'(g_0)$ be the variety of G'-orbit of g_0 (i.e., elements with the same characteristic polynomial $\mathbf{P}_{g_0}$). Then there is only one $G'(\mathbb{Q})$-orbit in $G'(g_0)(\mathbb{Q})$.*

Proof. The $G'(\mathbb{Q})$-orbits in $G'(g_0)(\mathbb{Q})$ is parameterized by the kernel of the natural map between Galois cohomology (as pointed sets)

$$H^1(\mathbb{Q}, F'^1) \longrightarrow H^1(\mathbb{Q}, G').$$

This map is injective when F'^1 is not a product of tori of lower ranks (i.e., "no nontrivial endoscopy" of G' involves the torus F'^1). $\square$

For $g_{v_0} \in G'(F_{0,v_0})$, let $\mathcal{N}^{g_{v_0}}$ be the fixed point locus of g on the RZ space $\mathcal{N}$ for $F_{w_0}/F_{0,v_0}$(cf. §3.1), and its base change $\mathcal{N}^{g_{v_0}}_{O_{\breve{E}_\nu}}$. For $(\gamma, h) \in G'(\mathbb{Q}) \times G(\mathbb{A}_f^{v_0})/K_G^{v_0}$, we define a closed formal subscheme of $\mathcal{N}_{O_{\breve{E}_\nu}} \times G(\mathbb{A}_f^{v_0})/K_G^{v_0}$:

$$\mathcal{CM}(\gamma, h)_{K_G^{v_0}} = \mathcal{N}^{\gamma}_{O_{\breve{E}_\nu}} \times \mathbf{1}_{hK_G^{v_0}}, \tag{7.19}$$

and its image in the quotient formal scheme (7.6)

$$\big[\mathcal{CM}(\gamma, h)\big]_{K_G^{v_0}} = \sum \mathcal{CM}(\gamma', h'), \tag{7.20}$$

where the sum runs over (γ', h') in the $G'(\mathbb{Q})$-orbit of $(\gamma, h) \in G'(\mathbb{Q}) \times G(\mathbb{A}_f^{v_0})/K_G^{v_0}$. Here $G'(\mathbb{Q})$ acts diagonally on $G'(\mathbb{Q}) \times G(\mathbb{A}_f^{v_0})/K_G^{v_0}$ by $g \cdot (\gamma, h) = (g\gamma g^{-1}, gh)$.

Furthermore, we have a derived version of (7.19) and (7.20) by replacing the naive fixed point locus $\mathcal{N}^{g_{v_0}}_{O_{\breve{E}_\nu}}$ in (7.19) by the derived fixed point locus ${}^{\mathbb{L}}\mathcal{N}^{g_{v_0}}_{O_{\breve{E}_\nu}}$ defined by (3.8).

We then have an analog of Proposition 7.4.

Proposition 7.17. *Let $R = O_F[1/\mathfrak{d}, g_0]$ be monogenic for some element $g_0 \in F'^1$.*
(i) The restriction of the CM cycle $\mathcal{CM}_R$ to each fiber of the projection (7.5) *is the disjoint union*

$$\coprod_{(\gamma,h)} \big[\mathcal{CM}(\gamma, h)\big]_{K_G^{v_0}},$$

where the index runs over the set

$$\Big\{(\gamma, h) \in G'(\mathbb{Q})\backslash\big(G'(g_0)(\mathbb{Q}) \times G(\mathbb{A}_f^{v_0})/K_G^{v_0}\big) \mid h^{-1}\gamma h \in K_G^{v_0}\Big\}.$$

(ii) Let $\phi_0 = \mathbf{1}_{K_G^{\mathfrak{d}}} \otimes \phi_{\mathfrak{d}} \in \mathcal{S}(G(\mathbb{A}_{0,f}))$ where $\phi_{\mathfrak{d}} \in \mathcal{S}\Big(\prod_{v|\mathfrak{d}} G(F_v), K_{G,\mathfrak{d}}\Big)$. The restriction of the twisted (derived) CM cycle ${}^{\mathbb{L}}\mathcal{CM}_R(g)$ (7.17) *to each fiber of the projection* (7.5) *is the sum*

$$\sum_{(\gamma,h)\in G'(\mathbb{Q})\backslash\big(G'(g_0)(\mathbb{Q})\times G(\mathbb{A}_f^{v_0})/K_G^{v_0}\big)} \phi_0^{v_0}(h^{-1}\gamma h) \cdot \big[{}^{\mathbb{L}}\mathcal{CM}(\gamma, h)\big]_{K_G^{v_0}},$$

as an element in the group (7.18).

Remark 7.18. One can define an analog of the cycle ${}^{\mathbb{L}}\mathcal{CM}_R(\phi_0)$ on a semiglobal integral model (i.e., over the localization $O_{E,(\nu)}$ of O_E at a place ν above v_0, cf. [40, §4]) where one allows more general level structure $K_G^{v_0}$ away v_0, and therefore allows $\phi_0 = \mathbf{1}_{K_{G,v_0}} \otimes \phi^{v_0} \in \mathcal{S}(G(\mathbb{A}_{0,f}))$ where $\phi^{v_0} \in \mathcal{S}\big(G(\mathbb{A}_f^{v_0}), K_G^{v_0})\big)$.

Proof. We prove part (i), and the other assertion concerning the derived version follows along the same line.

Over the formal scheme (7.6), $\mathcal{CM}_R$ consists of $G'(\mathbb{Q})$-cosets of $(X, hK_G^{v_0}) \in \mathcal{N}_{O_{\breve{E}_\nu}} \times G(\mathbb{A}_f^{v_0})/K_G^{v_0}$ together with a quasi-isogeny $\varphi_{v_0} : X \to X$ and $g \in G(\mathbb{A}_f^{v_0})$, satisfying the following conditions: there exists $\gamma \in G'(\mathbb{Q})$ such that the endomorphism of the framing object $\mathbb{X}_n$ induced by φ_{v_0} is γ, and both g and γ fix $hK_G^{v_0}$ and they induce the same automorphism of $hK_G^{v_0}$; the characteristic polynomial $\mathbf{P}_{g_0}$ annihilates g and φ_{v_0} (or equivalently γ by the rigidity of quasi-isogeny). In particular, $\gamma \in G'(g_0)(\mathbb{Q})$.

Here we view $G(\mathbb{A}_f^{v_0})/K_G^{v_0}$ as a groupoid in which the automorphism group of $hK_G^{v_0}$ is isomorphic to $hK_G^{v_0}h^{-1}$. If both γ and g fix $hK_G^{v_0}$ and induce the same automorphism of $hK_G^{v_0}$, then $g = \gamma$ ("rigidity away from v_0"). It follows that the condition that g fixes $hK_G^{v_0}$ is equivalent to $\gamma hK_G^{v_0} = hK_G^{v_0}$, i.e., $h^{-1}\gamma h \in K_G^{v_0}$.

The condition on the existence of a quasi-isogeny φ_{v_0} lifting γ amounts to $X \in \mathcal{N}^{\gamma}_{O_{\breve{E}_\nu}}$.

Therefore for a fixed $\gamma \in G'(g_0)(\mathbb{Q})$, we obtain that the desired pairs $(X, hK_G^{v_0})$ are exactly those lying on $\mathcal{N}^{\gamma}_{O_{\breve{E}_\nu}} \times \mathbf{1}_{hK_G^{v_0}}$ subject to the condition $h^{-1}\gamma h \in K_G^{v_0}$. Then we just need to sum over all $\gamma \in G'(g_0)(\mathbb{Q})$ to complete the proof of Part (i). $\square$

8. Modular generating functions of special divisors

In this section we collect a few modularity results for the generating functions of special divisors with valued in Chow groups, and in a reduced version of arithmetic Chow groups.

8.1. **Generating functions of special divisors on $M_{\widetilde{K}_G}(\widetilde{G})$.** We first define the generating functions of special divisors on the canonical model $M_{\widetilde{K}_G}(\widetilde{G})$ over $\operatorname{Spec} E$. The moduli functor is introduced at the end of §6, for an arbitrary compact open subgroup $K_{\widetilde{G}}^\circ$ of the form $K_{\widetilde{G}} = K^\circ_{Z^{\mathbb{Q}}} \times K_G$. For $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))^{K_G}$, and $\xi \in F_{0,+}$, we have defined the divisor $Z(\xi,\phi) \in \mathrm{Ch}^1(M_{K_{\widetilde{G}}}(\widetilde{G}))$ by (7.1). When $\xi = 0$, we define

$$Z(0,\phi) = -\phi(0)\, c_1(\omega) \in \mathrm{Ch}^1(M_{\widetilde{K}_G}(\widetilde{G})), \tag{8.1}$$

where ω is the automorphic line bundle [23], and c_1 denotes the first Chern class.

In §11.1 we will recall the Weil representation ω of $\mathrm{SL}_2(\mathbb{A}_{0,f})$ on $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))^{K_G}$. We define the generating function on $\mathrm{SL}_2(\mathbb{A}_0)$ by

$$Z(h,\phi) = Z(0,\omega(h_f)\phi)W_0^{(n)}(h_\infty) + \sum_{\xi \in F_{0,+}} Z(\xi,\omega(h_f)\phi)\, W_\xi^{(n)}(h_\infty), \tag{8.2}$$

where $h = (h_\infty, h_f) \in \mathrm{SL}_2(\mathbb{A}_0)$, $h_\infty = (h_v)_{v|\infty} \in \prod_{v|\infty} \mathrm{SL}_2(F_v)$, and

$$W_\xi^{(n)}(h_\infty) = \prod_{v|\infty} W_\xi^{(n)}(h_v),$$

cf. (1.8) for the weight-n Whittaker function $W_\xi^{(n)}$ on $\mathrm{SL}_2(\mathbb{R})$.

Theorem 8.1. *The generating function $Z(h,\phi)$ lies in $\mathcal{A}_{\mathrm{hol}}(\mathrm{SL}_2(\mathbb{A}_0), K, n)_{\mathbb{Q}} \bigotimes_{\mathbb{Q}} \mathrm{Ch}^1(M_{\widetilde{K}_G}(\widetilde{G}))$, where $K \subset \mathrm{SL}_2(\mathbb{A}_{0,f})$ is a compact open subgroup which fixes $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))$ under the Weil representation.*

We refer to (1.6) (and (1.5)) for the definition of the vector space in the statement.

The result has an analog for orthogonal Shimura varieties, which is due to Borcherds when $F_0 = \mathbb{Q}$ (generalizing Gross–Kohnen–Zagier theorem), and [45] for totally real fields F_0; Bruinier also gave a proof in [5] where he also constructed the automorphic Green function. By the embedding trick [29, §3.2, Lemma 3.6], this result implies the analogous modularity for Shimura varieties $\mathrm{Sh}_{K_G}\big(\mathrm{Res}_{F/F_0} G, \{h_G\}\big)$.[4] Then the assertion in the theorem above follows from the fact that, after base change to $\mathbb{C}$, $M_{\widetilde{K}_G}(\widetilde{G})$ is a disjoint union of copies of $\mathrm{Sh}_{K_G}\big(\mathrm{Res}_{F/F_0} G, \{h_G\}\big)$, cf. (6.7).

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[4]In the unitary case, one expect to obtain a $\mathrm{U}(1,1)$-automorphic form. However, the SL_2-automorphic form suffices for our purpose, and in fact the extra information in $\mathrm{U}(1,1)$-automorphy is not useful for us at all because the analytic side only has SL_2-automorphy.

8.2. **Complex uniformization of special divisors.** We now study the analog over the complex numbers of the uniformization of the special divisors in the formal neighborhood of the basic locus, cf. §7.3.

We start with the complex uniformization of our Shimura varieties. This is very much similar to the (7.4), cf. [40, Remark 3.2, Prop. 3.5]. Let $\nu : E \hookrightarrow \mathbb{C}$ be a complex place of the reflex field E. Its restriction to F (F_0, resp.) is a place denoted by w_0 (v_0, resp.). Let $M_{\nu,\mathbb{C}} = M_{K_{\widetilde{G}}}(\widetilde{G}) \otimes_{E,\nu} \mathbb{C}$ be the complex orbifold via ν. Let V' be the "nearby" hermitian space, i.e., the unique one that is positive definite at all archimedean places except v_0 where the signature is $(n-1,1)$, and isomorphic to V locally at all non-archimedean places. Then let G' be the unitary group (viewed as a $\mathbb{Q}$-algebraic group) associated to V'. Let $\mathcal{D}_{v_0}$ be the Grassmannian of negative definite $\mathbb{C}$-lines in $V' \otimes_{F,w_0} \mathbb{C}$. Then we have a complex uniformization

$$M_{\nu,\mathbb{C}} = \widetilde{G}'(\mathbb{Q}) \Big\backslash \Big[\mathcal{D}_{v_0} \times \widetilde{G}(\mathbb{A}_f)/K_{\widetilde{G}}\Big]. \tag{8.3}$$

Analogous to (7.5), we have a partition by the projection

$$M_{\nu,\mathbb{C}} \longrightarrow Z^{\mathbb{Q}}(\mathbb{Q})\backslash\big(Z^{\mathbb{Q}}(\mathbb{A}_f)/K_{Z^{\mathbb{Q}}}\big), \tag{8.4}$$

where each fiber is naturally isomorphic to

$$M_{\nu,\mathbb{C},0} :\, = G'(\mathbb{Q}) \Big\backslash \Big[\mathcal{D}_{v_0} \times G(\mathbb{A}_f)/K_G\Big]. \tag{8.5}$$

Here we fix an isomorphism $G'(\mathbb{A}_f) \simeq G(\mathbb{A}_f)$.

Now we return to describe the complex uniformization of the special divisors. For each $u \in V'(F_0)$ with totally positive norm, let $\mathcal{D}_{v_0,u} \subset \mathcal{D}_{v_0}$ be the space of negative definite $\mathbb{C}$-lines perpendicular to u.[5] For a pair $(u,g) \in V'(F_0) \times G(\mathbb{A}_f)/K_G$, we define

$$Z(u,g)_{K_G} = \mathcal{D}_{v_0,u} \times \mathbf{1}_{g\,K_G} \tag{8.6}$$

and its image in the quotient (8.5):

$$[Z(u,g)]_{K_G} = \sum Z(u',g')_{K_G}, \tag{8.7}$$

where the sum is over (u',g') in the $G'(\mathbb{Q})$-orbit of the pair (u,g) (for the diagonal action of $G'(\mathbb{Q})$ on $V'(F_0) \times G(\mathbb{A}_f)/K_G$.

Then, we have an archimedean analog of Proposition 7.4 for the special divisor $Z(\xi,\phi)$ defined by (7.1).

Proposition 8.2. *Let $\xi \in F_{0,+}$. Then the restriction of the special divisor $Z(\xi,\phi) \otimes_{E,\nu} \mathbb{C}$ to each fiber of the projection* (7.5) *is the formal sum*

$$\sum_{(u,g)\in G'(\mathbb{Q})\backslash(V'(F_0)_\xi \times G(\mathbb{A}_f)/K_G)} \phi(g^{-1}u) \cdot [Z(u,g)]_{K_G}. \tag{8.8}$$

Remark 8.3. We may rewrite the above result into a form that has appeared in the formula of special divisors in [45, §1]. Let $G'_u \subset G'$ the stabilizer of u under the action of G' on V', viewed as an algebraic group over $\mathbb{Q}$. Instead of (8.6), we define

$$\widetilde{Z}(u,g)_{K_G} := \mathcal{D}_{v_0,u} \times \mathbf{1}_{G'_u(\mathbb{A}_f)\,g\,K_G}.$$

Similarly we denote its image in the quotient (8.5) by $[\widetilde{Z}(u,g)]_{K_G}$. Then we may rewrite the sum as (8.8)

$$\sum_{u\in G'(\mathbb{Q})\backslash V'(F_0)_\xi} \ \sum_{g\in G'_u(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K_G} \phi(g^{-1}u) \cdot [\widetilde{Z}(u,g)]_{K_G}.$$

This is exactly the formula in *loc. cit.*.

[5] The codimension one analytic space $\mathcal{D}_{v_0,u}$ on $\mathcal{D}_{v_0}$ is the archimedean analog of the local KR divisor $\mathcal{Z}(u)$ on $\mathcal{N}$ in §7.3.

8.3. **Green's functions.** We recall the Green functions of Kudla [24], and the automorphic Green functions of Bruinier [5]. The former is more convenient when comparing with the analytic side, while the latter is more suitable for proving (holomorphic) modularity of generating series. The difference between them is studied by Ehlen–Sankaran in [9] when $F_0 = \mathbb{Q}$.

We first recall Kudla's Green functions, defined for the orthogonal case in [24] which can be carried over easily to the unitary case (cf. [29, §4B]). Let $u \in V'(F_0)$ be as in the previous subsection. Let $z \in \mathcal{D}_{v_0}$. Let u_z be the orthogonal projection to the negative definite $\mathbb{C}$-line z of $V' \otimes_{F,w_0} \mathbb{C}$. Define

$$R(u,z) = \langle u_z, u_z\rangle = \frac{\langle u, \widetilde{z}\rangle^2}{\langle \widetilde{z}, \widetilde{z}\rangle}, \tag{8.9}$$

where $\widetilde{z}$ is any $\mathbb{C}$-basis of the line z.

We will need the exponential integral defined by

$$\mathrm{Ei}(-r) = -\int_r^\infty \frac{e^{-t}}{t} dt, \quad r > 0. \tag{8.10}$$

This function has a logarithmic singularity around 0, more precisely, when $r \to 0^+$,

$$\mathrm{Ei}(-r) = \gamma + \log r + \sum_{n=1}^{\infty} \frac{(-r)^n}{n \cdot n!}.$$

Here γ is the Euler constant.

Let $h_\infty = (h_v)_{v|\infty} \in \mathrm{SL}_2(F_0 \otimes_{\mathbb{Q}} \mathbb{R})$ and $h_v = \begin{pmatrix} 1 & b_v \\ & 1 \end{pmatrix} \begin{pmatrix} \sqrt{a_v} & \\ & 1/\sqrt{a_v} \end{pmatrix} \kappa_v$ in Iwasawa decomposition, cf. (1.9). For each *non-zero* vector $u \in V(F_0)$, Kudla defined a Green function on $\mathcal{D}_{v_0}$, parameterized by h_∞

$$\mathcal{G}^{\mathbf{K}}(u, h_\infty)(z) = -\mathrm{Ei}(2\pi a_{v_0} R(u,z)), \quad z \in \mathcal{D}_{v_0} \setminus \mathcal{D}_{v_0,u}. \tag{8.11}$$

It has logarithmic singularity along the divisor $\mathcal{D}_{v_0,u}$. Note that this is defined for *every* non-zero vector $u \in V'(F_0)$ (in particular, u may have null-norm). If $\mathcal{D}_{v_0,u}$ is empty, the function is then smooth on $\mathcal{D}_{v_0}$. When $u = 0$, we set

$$\mathcal{G}^{\mathbf{K}}(0, h_\infty) = -\log |a_{v_0}|. \tag{8.12}$$

Now we descend the Green function on $\mathcal{D}_{v_0}$ to the quotient (8.5): for all $\xi \in F_0$, define

$$\mathcal{G}^{\mathbf{K}}(\xi, h_\infty, \phi) = \sum \phi(g^{-1}u) \cdot \left(\mathcal{G}^{\mathbf{K}}(u, h_\infty) \times \mathbf{1}_{G_u(\mathbb{A}_f)\, g\, K_G}\right) \tag{8.13}$$

where the sum is over the double coset $(u,g) \in G'(F_0)\backslash (V'(F_0)_\xi \times G(\mathbb{A}_f)/K_G)$. This defines a Green's function for the divisor $Z(\xi, \phi)$, cf. [29, Prop. 4.9].

We now recall the automorphic Green functions of Bruinier [5, 6]. Since the role of them are indirect to this paper, we just say that there is a Green's function $\mathcal{G}^{\mathbf{B}}(\xi, \phi)$ for each $\xi \in F_{0,+}$, and $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))$, cf. [6, §7.3].

We define the generating function of the difference of the two Green functions

$$\mathcal{Z}_{v_0,\mathrm{corr}}(h, \phi) \colon = \sum_{\xi \in F_0} \left(\mathcal{G}^{\mathbf{K}}(\xi, h_\infty, \omega(h_f)\phi) - \mathcal{G}^{\mathbf{B}}(\xi, \omega(h_f)\phi)\right) W_\xi^{(n)}(h_\infty), \tag{8.14}$$

where the notation is the same as (8.2). We note that this definition depends on the archimedean place v_0 of F_0, though it is omitted in the right hand side of the equality.

The following theorem is due to Ehlen–Sankaran [9].

Theorem 8.4. *Assume that $F_0 = \mathbb{Q}$. The generating function $\mathcal{Z}_{\infty,\mathrm{corr}}(h, \phi)$ lies in the space $\mathcal{A}_{\exp}(\mathrm{SL}_2(\mathbb{A}_0), K, n)$, in the sense that, for each $[z, g] \in M_{\nu,\mathbb{C}}$, the value of the generating functions at $[z, g]$ lies in $\mathcal{A}_{\exp}(\mathrm{SL}_2(\mathbb{A}_0), K, n)$. Here $K \subset \mathrm{SL}_2(\mathbb{A}_{0,f})$ is a compact open subgroup which fixes $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))$ under the Weil representation.*

Proof. In [9, Thm. 3.6], the authors proved the assertion for orthogonal groups, from which the case of unitary groups follows (e.g, by the embedding trick [29, §3.2]). □

8.4. **Modularity in the reduced arithmetic Chow group $\widehat{\mathrm{Ch}}^1_\circ(\mathcal{M})$.** We will use the Gillet–Soulé arithmetic intersection theory cf. [13, 11] (in the non-proper case, cf. [4]). First we recall the arithmetic Chow group $\widehat{\mathrm{Ch}}^1(\mathcal{M})$ (with $\mathbb{Q}$-coefficient) for a regular flat DM stack (possibly non-proper) $\mathcal{M} \to \mathrm{Spec}\, O_E$. Elements are represented by arithmetic divisors, i.e., $\mathbb{Q}$-linear combinations of tuples $\Big(Z, (g_{Z,w})_{w\in \mathrm{Hom}_\mathbb{Q}(E,\overline{\mathbb{Q}})}\Big)$, where Z is a divisor on $\mathcal{M}$ and $g_{Z,w}$ is a Green's function of $Z_w(\mathbb{C})$ on the orbifold $\mathcal{M}_w(\mathbb{C})$ via the embedding $w : E \hookrightarrow \overline{\mathbb{Q}} \subset \mathbb{C}$ (cf. [13, §3.3]). Principal arithmetic divisors are tuples associated to rational functions $f \in E(\mathcal{M})^\times$:

$$\Big(\mathrm{div}(f), (-\log|f|_w^2)_{w\in \mathrm{Hom}_\mathbb{Q}(E,\overline{\mathbb{Q}})}\Big).$$

(e.g., when $E = \mathbb{Q}$, we have $\mathbf{V}_\mathfrak{p} = (0, 2\log|p|)$ in $\widehat{\mathrm{Ch}}^1(\mathcal{M})$, where $\mathbf{V}_p$ is the fiber of $\mathcal{M}$ over a prime p.)

Fix a finite set S of *non-archimedean* places, including all ν where $\mathcal{M}_\nu$ has bad reduction over $O_{E,\nu}$. Denote by $\mathrm{Ch}^1_{|S|}(\mathcal{M})$ the subgroup of $\widehat{\mathrm{Ch}}^1(\mathcal{M})$ consisting of elements supported at the fibers above $\nu \in S$. This is a finite dimensional vector space. Define the reduced arithmetic Chow group (w.r.t. the fixed S) as the the quotient of $\widehat{\mathrm{Ch}}^1(\mathcal{M})$

$$\widehat{\mathrm{Ch}}^1_\circ(\mathcal{M}):=\widehat{\mathrm{Ch}}^1(\mathcal{M})/\mathrm{Ch}^1_{|S|}(\mathcal{M}).$$

From the definition, the reduced arithmetic Chow group depends only on the integral model over the Zariski open $\mathrm{Spec}\, O_E \setminus S$. In fact, it is clear that the definition only requires a regular flat DM stack $\mathcal{M} \to \mathrm{Spec}\, O_E \setminus S$ (rather than the restriction of some regular flat DM stack over $\mathrm{Spec}\, O_E$).

Now we specialize to our interest, the moduli stack $\mathcal{M} = \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ introduced in Definition 6.3. Let S be a finite set of *non-archimedean* places, including all ν where $\mathcal{M}_\nu$ has bad reduction over $O_{E,\nu}$ (in particular all places above $\mathfrak{d}$).

Let $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))^{K_G}$ be of the form $\phi = \mathbf{1}_{\Lambda^\mathfrak{d}} \otimes \phi_\mathfrak{d}$ (cf. (7.2)). For $\xi \in F_{0,+}$, we endow the special divisor $\mathcal{Z}(\xi, \phi)$ (cf. (7.2)) with the automorphic Green function $\mathcal{G}^\mathbf{B}(\xi, \phi)$. Denote by $\widehat{\mathcal{Z}}^\mathbf{B}(\xi, \phi)$ the resulting element in $\widehat{\mathrm{Ch}}^1_\circ(\mathcal{M})$. When $\xi = 0$, we define

$$\mathcal{Z}(0, \phi) = -\phi(0)\, c_1(\widehat{\omega}) \in \widehat{\mathrm{Ch}}^1_\circ(\mathcal{M}), \tag{8.15}$$

where $\widehat{\omega} = (\omega, \|\cdot\|_{\mathrm{Pet}})$ is the extension of the automorphic line bundle ω to the integral model $\mathcal{M}$, endowed with its Petersson metric [6, §7.2].

We define the generating series with coefficients in the reduced arithmetic Chow group $\widehat{\mathrm{Ch}}^1_\circ(\mathcal{M})$

$$\widehat{\mathcal{Z}}^\mathbf{B}(\tau, \phi) = \sum_{\xi\in F_0,\ \xi\geq 0} \widehat{\mathcal{Z}}^\mathbf{B}(\xi, \phi)\, q^\xi, \tag{8.16}$$

where

$$\tau = (\tau_v)_{v|\infty} \in \prod_{v|\infty} \mathcal{H}, \quad q^\xi := e^{2\pi i\, \mathrm{tr}_{F_0/\mathbb{Q}}(\tau\xi)}. \tag{8.17}$$

The following theorem can be deduced from by [6].

Theorem 8.5. *Let $F_0 = \mathbb{Q}$. The generating series $\widehat{\mathcal{Z}}^\mathbf{B}(\cdot, \phi)$ lies in $\mathcal{A}_{\mathrm{hol}}(\Gamma(N), n)_\mathbb{Q} \bigotimes_\mathbb{Q} \widehat{\mathrm{Ch}}^1_\circ(\mathcal{M})$, where N depends only on ϕ and all prime factors of N are contained in S.*

Proof. In [6] the authors proved a stronger version in the maximal level case (principle polarized) over the full ring of integers of E. Since the reduced arithmetic Chow group omits a finite set of bad places S (including primes ramified in F), the computation of divisors of the regularized theta lifts and Borcherds product on the integral models over $\mathrm{Spec}\, O_E[1/\mathfrak{d}]$ of *loc. cit.* still applies to our (simpler) situation.

□

Remark 8.6. Note that the statement of the modularity in the (reduced) arithmetic Chow group is weaker than the analog on the generic fiber (the special case $F_0 = \mathbb{Q}$ of Theorem 8.1). The main difference is that we have not defined the special divisors on the arithmetic Chow group for an arbitrary function $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))^{K_G}$, cf. (7.1) and (7.2). This is the reason we define a generating function (8.2) in $h \in \mathrm{SL}_2(\mathbb{A}_0)$ for special divisors on the generic fiber, while only a generating function (8.16) in $\tau \in \mathcal{H}^{[F_0:\mathbb{Q}]}$ for the integral model.

9. Local intersection: non-archimedean places

9.1. **Arithmetic intersection theory for the reduced arithmetic Chow groups.** Now let $\mathcal{M} \to \mathcal{B} = \operatorname{Spec} O_E$ be a pure dimensional flat (not necessarily proper) morphism of regular schemes with smooth generic fiber. Let $\widetilde{\mathcal{Z}}_{1,c}(\mathcal{M})$ be the group of proper (*over the base* $\mathcal{B}$) 1-cycles on $\mathcal{M}$ (with $\mathbb{Q}$-coefficient). Then there is an arithmetic intersection pairing between two $\mathbb{Q}$-vector spaces (cf. [3, §2.3] when the ambient scheme is proper)

$$(\cdot,\cdot)\colon \quad \widehat{\mathrm{Ch}}^1(\mathcal{M}) \times \widetilde{\mathcal{Z}}_{1,c}(\mathcal{M}) \longrightarrow \mathbb{R}. \tag{9.1}$$

Let S be a finite set of places of E, and let S_p be the subset of places above p. Let $\widehat{\mathrm{Ch}}^1_\circ(\mathcal{M})$ be the reduced arithmetic Chow group. Consider the quotient of $\mathbb{R}$ by a finite dimensional $\mathbb{Q}$-vector space:

$$\mathbb{R}_S := \mathbb{R}/\mathrm{span}_{\mathbb{Q}}\{\log p : \#S_p \neq 0\} \tag{9.2}$$

which is an (infinite dimensional) $\mathbb{Q}$-vector space. Then the above pairing descends to the quotient $\widehat{\mathrm{Ch}}^1_\circ(\mathcal{M})$ with values in $\mathbb{R}_S$

$$(\cdot,\cdot)\colon \quad \widehat{\mathrm{Ch}}^1_\circ(\mathcal{M}) \times \widetilde{\mathcal{Z}}_{1,c}(\mathcal{M}) \longrightarrow \mathbb{R}_S. \tag{9.3}$$

Moreover, the definition of [3] works directly if we replace the base $\mathcal{B} = \operatorname{Spec} O_E$ by $\mathcal{B} = \operatorname{Spec} O_{E,S}$ (i.e., without an integral model over the full ring of integers O_E), and yields a pairing with valued in $\mathbb{R}_S$. Note that the cycles in $\widetilde{\mathcal{Z}}_{1,c}(\mathcal{M})$ are assumed to be *proper* over $\mathcal{B}$. This will be the pairing we will apply to our integral models (away from a finite set of primes) of Shimura varieties.

Here we remark that, by [3, Prop. 2.3.1 (ii)], for cycles in $\widetilde{\mathcal{Z}}_{1,c}(\mathcal{M})$ supported on special fibers, the pairing only depends on their rational equivalence classes. This motivates us to define a quotient group $\mathcal{Z}_{1,c}(\mathcal{M})$ of $\widetilde{\mathcal{Z}}_{1,c}(\mathcal{M})$ by the subgroup generated by 1-cycles that are supported on *proper* substacks Y of the special fibers and are rationally equivalent to zero on Y. We have the resulting pairing

$$(\cdot,\cdot)\colon \quad \widehat{\mathrm{Ch}}^1_\circ(\mathcal{M}) \times \mathcal{Z}_{1,c}(\mathcal{M}) \longrightarrow \mathbb{R}_S. \tag{9.4}$$

We apply the above remark to $\mathcal{M} = \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G}) \to \mathcal{B} = \operatorname{Spec} O_{E,S}$. For the moment, S can be any finite set containing all places $\nu \mid \mathfrak{d}$. We define an element in $\mathcal{Z}_{1,c}(\mathcal{M})$ starting from the derived CM cycle ${}^{\mathbb{L}}\mathcal{CM}_R(g)$ (7.15), which is an element in $F_1 K_0'(\mathcal{CM}_R(g))$, (7.16). The finite morphism $\mathcal{CM}_R(g) \to \mathcal{M}$ induces a homomorphism

$$K_0'(\mathcal{CM}_R(g)) \longrightarrow K'_{0,\mathcal{CM}_R(g)}(\mathcal{M})$$

preserving the respective filtrations, where $K'_{0,\mathcal{CM}_R(g)}(\mathcal{M})$ denotes the K-group of coherent sheaves with support on the image of $\mathcal{CM}_R(g)$. Since $\mathcal{CM}_R(g) \to \mathcal{B}$ is proper and the generic fiber of $\mathcal{CM}_R(g)$ is zero dimensional (cf., Prop. 7.12 (b)), there is a natural homomorphism $\mathrm{Ch}_{1,\mathcal{CM}_R(g)}(\mathcal{M}) \to \mathcal{Z}_{1,c}(\mathcal{M})$. We now consider the composition

$$F_1 K_0'(\mathcal{CM}_R(g)) \longrightarrow \mathrm{Gr}_1 K'_{0,\mathcal{CM}_R(g)}(\mathcal{M}) \xrightarrow{\sim} \mathrm{Ch}_{1,\mathcal{CM}_R(g)}(\mathcal{M}) \longrightarrow \mathcal{Z}_{1,c}(\mathcal{M}),$$

where the isomorphism in the middle is [12, Theorem 8.2], and Gr_1 denotes the grading F_1/F_0. By abuse of notation, we still denote by ${}^{\mathbb{L}}\mathcal{CM}_R(g)$ the image in $\mathcal{Z}_{1,c}(\mathcal{M})$ of the element ${}^{\mathbb{L}}\mathcal{CM}_R(g) \in F_1 K_0'(\mathcal{CM}_R(g))$ (cf. (7.16)) under the above composition.

9.2. **Intersection of special divisors and CM cycles.** We now let $\mathcal{M} = \mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ be the moduli stack introduced in Definition 6.3. Let $\Phi = \otimes_{v_0} \Phi_{v_0} \in \mathcal{S}((G \times V)(\mathbb{A}_{0,f}))$ be of the form $\phi_0 \otimes \phi$, where

- $\phi_0 = \mathbf{1}_{K_G^{\mathfrak{d}}} \otimes \phi_{0,\mathfrak{d}}$ and $\phi_{0,\mathfrak{d}} \in \mathcal{S}\left(\prod_{v|\mathfrak{d}} G(F_v), K_{G,\mathfrak{d}}\right)$ (cf. (7.17)), and
- $\phi = \mathbf{1}_{\Lambda^{\mathfrak{d}}} \otimes \phi_{\mathfrak{d}}$ and $\phi_{\mathfrak{d}} \in \mathcal{S}(V(F_{0,\mathfrak{d}}))^{K_{G,\mathfrak{d}}}$ (cf. (7.2)).

Let $R = O_F[1/\mathfrak{d}, g_0]$ be our fixed monogenic order for some $g_0 \in F'^1$. We define

$$\mathrm{Int}(\tau, \Phi) \colon = \frac{1}{\tau(Z^{\mathbb{Q}}) \cdot [E : F]} \left(\widehat{\mathcal{Z}}^{\mathbf{B}}(\tau, \phi), \quad {}^{\mathbb{L}}\mathcal{CM}_R(\phi_0) \right), \tag{9.5}$$

where $\mathcal{Z}^{\mathbf{B}}(\tau, \phi)$ is (8.16), and

$$\tau(Z^{\mathbb{Q}}) := \# Z^{\mathbb{Q}}(\mathbb{A}) \backslash \big(Z^{\mathbb{Q}}(\mathbb{A}_f) / K_{Z^{\mathbb{Q}}} \big). \tag{9.6}$$

Remark 9.1. By Theorem 8.5, when $F_0 = \mathbb{Q}$, this is a holomorphic modular form (of weight n, and level depending only on ϕ) valued in $\mathbb{R}_S$, i.e.,

$$\mathrm{Int}(\cdot, \Phi) \in \mathcal{A}_{\mathrm{hol}}(\Gamma(N), n)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}_S. \tag{9.7}$$

Our results in this and the next section are still valid for general totally real fields F_0 since they do not use the modularity.

Similarly we define for each $\xi \in F_0$

$$\mathrm{Int}(\xi, \Phi) \colon = \frac{1}{\tau(Z^{\mathbb{Q}}) \cdot [E : F]} \left(\widehat{\mathcal{Z}}(\xi, \phi), \quad {}^{\mathbb{L}}\mathcal{CM}_R(\phi_0) \right). \tag{9.8}$$

When $\xi = 0$, this is by definition

$$\mathrm{Int}(0, \Phi) = \frac{1}{\tau(Z^{\mathbb{Q}}) \cdot [E : F]} \left(\widehat{\omega}, \quad {}^{\mathbb{L}}\mathcal{CM}_R(\phi_0) \right) \phi(0). \tag{9.9}$$

Then by (8.16),

$$\mathrm{Int}(\tau, \Phi) = \sum_{\xi \in F_0,\, \xi \geq 0} \mathrm{Int}(\xi, \Phi)\, q^{\xi}. \tag{9.10}$$

Now let $\xi \neq 0$. We will express the arithmetic intersection number (9.8) in terms of the local intersection numbers from the AFL over good places and the archimedean local intersection.

9.3. **The support of the intersection.** We first study the intersection of the special divisor $\mathcal{Z}(\xi, \phi)$ and the CM cycle ${}^{\mathbb{L}}\mathcal{CM}_R(\phi_0)$. First we have the following analog to [40, Thm. 8.5].

Theorem 9.2. *Let $\xi \neq 0$ and $\Phi = \otimes_{v_0} \Phi_{v_0} \in \mathcal{S}((G \times V)(\mathbb{A}_{0,f}))^{K_G}$. Let S be a finite set of places containing all places above $\mathfrak{d}$, and such that at $v_0 \notin S$, $\Phi_{v_0} = \mathbf{1}_{K^{\circ}_{G,v_0}} \otimes \mathbf{1}_{\Lambda^{\circ}_{v_0}}$.*

Then the following statements on the support of the intersection of the special divisor $\mathcal{Z}(\xi, \phi)$ and the CM cycle $\mathcal{CM}_R(\phi_0)$ on $\mathcal{M}$ hold.

(i) The support does not meet the generic fiber.

(ii) Let $\nu \notin S$ be a place of E lying over a place of F_0 which splits in F. Then the support does not meet the special fiber $\mathcal{M} \otimes_{O_E} \kappa_\nu$.

(iii) Let $\nu \notin S$ be a place of E lying over a place of F_0 which does not split in F. Then the support meets the special fiber $\mathcal{M} \otimes_{O_E} \kappa_\nu$ only in its basic locus.

Proof. Note that $R = O_F[1/\mathfrak{d}, g_0]$ is monogenic with $g_0 \in F'^1$. The proof of [40, Thm. 8.5] goes through verbatim (since g_0 generates a field F' over F, hence the pair (g, u) is regular semisimple for any non-zero vector u in $V(F_0)$). □

Since their generic fibers do not intersect by Theorem 9.2, the intersection pairing $\mathrm{Int}(R, \xi, \phi)$ localizes to a sum over all places of E. We define

$$\mathrm{Int}^{\natural}_{\nu}(\xi, \Phi) := \big\langle \widehat{\mathcal{Z}}(\xi, \phi), \quad {}^{\mathbb{L}}\mathcal{CM}_R(\phi_0) \rangle_{\nu} \log q_{\nu}, \tag{9.11}$$

where q_ν is the cardinality of the residue field of $O_{E,(\nu)}$. Here we recall that the local intersection number $\langle \cdot, \cdot \rangle_\nu$ is defined for a non-archimedean place ν through the Euler–Poincaré characteristic of a derived tensor product on $\mathcal{M} \otimes_{O_E} O_{E,(\nu)}$, comp. [13, 4.3.8(iv)]. For an archimedean place

ν, the local intersection number is the value of the Green's function at the complex point of the CM cycle:

$$\mathrm{Int}^{\natural}_{\nu}(\xi,\Phi) := \langle \mathcal{G}^{\mathbf{B}}_{\nu}(\xi,\phi), \quad {}^{\mathbb{L}}\mathcal{CM}_R(\phi_0)_{\nu,\mathbb{C}} \rangle \log q_{\nu} \tag{9.12}$$

where by definition $\log q_\nu = 2$ for complex places ν (and 1 if ν were a real place).

For a place v_0 of F_0, we set

$$\mathrm{Int}_{v_0}(\xi,\Phi) := \frac{1}{\tau(Z^{\mathbb{Q}})\cdot[E:F]} \sum_{\nu|v_0} \mathrm{Int}^{\natural}_{\nu}(\xi,\Phi). \tag{9.13}$$

Then we have a decomposition into a sum over places v_0 of F_0

$$\mathrm{Int}(\xi,\Phi) = \sum_{v_0} \mathrm{Int}_{v_0}(\xi,\Phi). \tag{9.14}$$

Combining (9.10), we obtain a decomposition of the generating function of arithmetic intersection numbers

$$\mathrm{Int}(\tau,\Phi) = \mathrm{Int}(0,\Phi) + \sum_{v_0} \mathrm{Int}_{v_0}(\tau,\Phi), \tag{9.15}$$

where

$$\mathrm{Int}_v(\tau,\Phi) :\, = \sum_{\xi\in F_{0,+}} \mathrm{Int}_{v_0}(\xi,\Phi)\, q^{\xi}, \tag{9.16}$$

Corollary 9.3. *(to Theorem 9.2) If v_0 is split in F/F_0, we have*

$$\mathrm{Int}_{v_0}(\xi,\phi) = 0. \tag{9.17}$$

9.4. **Local intersection: inert non-archimedean places.** Now let v_0 be a place of F_0 inert in F, and w_0 the unique place of F above v_0. The notation here follows §7.3.

Theorem 9.4. *Assume that $v_0 \nmid \mathfrak{d}$ and $\Phi = \Phi_{v_0} \otimes \Phi^{v_0}$ where*

$$\Phi_{v_0} = \mathbf{1}_{K^{\circ}_{G,v_0}} \otimes \mathbf{1}_{\Lambda^{\circ}_{v_0}}.$$

Then

$$\mathrm{Int}_{v_0}(\xi,\Phi) = 2\log q_{v_0} \sum_{(g,u)} \mathrm{Int}_{v_0}(g,u)\cdot \mathrm{Orb}\left((g,u),\Phi^{v_0}\right), \tag{9.18}$$

where the sum runs over the $G'(\mathbb{Q})$-orbits (g,u) in the product

$$G'(g_0)(\mathbb{Q}) \times V'(F_0)_{\xi}.$$

Here $\mathrm{Int}_{v_0}(g,u)$ is the quantity defined in the AFL conjecture (semi-Lie algebra version) for the unramified quadratic extension $F_{w_0}/F_{0,v_0}$, cf. (3.9), *and the orbital integral is the product of the local orbital integral defined by* (2.13) *with Haar measures on $G(F_{0,v})$ such that* $\mathrm{vol}(K_{G,v}) = 1$.

Proof. The proof goes along a similar line to [47, Thm. 3.11] and [40, Thm. 8.15].

First, by Theorem 9.2 (iii), the intersection only takes place in the basic locus. Hence it suffices to consider the question in the formal completion along the basic locus. We now fix a place ν of E above v_0. Now by Proposition 7.4, and Proposition 7.17, it suffices to consider the intersection number for each fiber of the projection (7.5), and multiply the result by the factor $\tau(Z^{\mathbb{Q}})$ (hence canceling the factor $\tau(Z^{\mathbb{Q}})$ in the denominator of (9.13)). Therefore we consider only the fiber $\mathcal{M}_{\widehat{O}_{\breve{E}_\nu},0}$ by (7.6).

Recall that by Proposition 7.4, the restriction to $\mathcal{M}_{\widehat{O}_{\breve{E}_\nu},0}$ of the special divisor $\mathcal{Z}(\xi,\phi)$ is

$$\sum_{(u,g')\in G'(\mathbb{Q})\backslash(V'(F_0)_{\xi}\times G(\mathbb{A}_f^{v_0})/K_G^{v_0})} \phi^{v_0}(g'^{-1}u)\cdot \left[\mathcal{Z}(u,g')\right]_{K_G^{v_0}},$$

and by Proposition 7.17 the restriction of the derived CM cycle ${}^{\mathbb{L}}\mathcal{CM}_R(\phi_0)$ is the sum

$$\sum_{(\gamma,h)\in G'(\mathbb{Q})\backslash(G'(g_0)(\mathbb{Q})\times G(\mathbb{A}_f^{v_0})/K_G^{v_0})} \phi_0^{v_0}(h^{-1}\gamma h)\cdot \left[{}^{\mathbb{L}}\mathcal{CM}(\gamma,h)\right]_{K_G^{v_0}}.$$

We may compute the intersection number by pulling-back to the covering formal scheme $\mathcal{N}_{O_{\breve{E}_\nu}} \times G(\mathbb{A}_f^{v_0})/K_G^{v_0}$ in the uniformization (7.6). The intersection number ${}^{\mathbb{L}}\mathcal{CM}_R(\phi_0) \cap^{\mathbb{L}} \mathcal{Z}(\xi,\phi)\log q_\nu$ (restricted to $\mathcal{M}_{\widehat{O}_{\breve{E}_\nu},0}$) is equal to a sum of

$$\phi_0^{v_0}(h^{-1}\gamma h)\phi^{v_0}(g'^{-1}u)\cdot {}^{\mathbb{L}}\mathcal{CM}(\gamma,h)_{K_G^{v_0}} \cap^{\mathbb{L}} \mathcal{Z}(u,g')_{K_G^{v_0}}\cdot \log q_\nu,$$

over $G'(\mathbb{Q})$-orbits (via diagonal action) of tuples (γ,h,u,g'):

$$(\gamma,h)\in G'(g_0)(\mathbb{Q})\times G(\mathbb{A}_f^{v_0})/K_G^{v_0},\qquad \text{and}\qquad (u,g')\in V'(F_0)_\xi\times G(\mathbb{A}_f^{v_0})/K_G^{v_0}.$$

Here, we are abusing the notation $\cap^{\mathbb{L}}$ to denote the Euler–Poincare characteristics of the corresponding derived tensor product.

By (7.7) and (7.19), we obtain

$${}^{\mathbb{L}}\mathcal{CM}(\gamma,h)_{K_G^{v_0}} \cap^{\mathbb{L}} \mathcal{Z}(u,g')_{K_G^{v_0}}\cdot\log q_\nu = {}^{\mathbb{L}}\mathcal{N}^{\gamma}_{O_{\breve{E}_\nu}} \cap^{\mathbb{L}} \mathcal{Z}(u)_{O_{\breve{E}_\nu}}\log q_\nu\cdot \mathbf{1}_{K_G^{v_0}}(g'^{-1}h).$$

The first term is equal to

$$\begin{aligned}{}^{\mathbb{L}}\mathcal{N}^{\gamma}_{O_{\breve{E}_\nu}} \cap^{\mathbb{L}} \mathcal{Z}(u)_{O_{\breve{E}_\nu}}\log q_\nu &= [E_\nu:F_{w_0}]\cdot\left({}^{\mathbb{L}}\mathcal{N}^{\gamma}\cap^{\mathbb{L}}\mathcal{Z}(u)\right)\log q_{w_0}\\ &= 2[E_\nu:F_{w_0}]\cdot \mathrm{Int}_{v_0}(\gamma,u)\log q_{v_0}.\end{aligned}$$

Here the factor 2 is due to $q_{w_0}=q_{v_0}^2$. In particular, it is invariant under the (diagonal) action of $G'(\mathbb{Q})$ on the product $G'(g_0)(\mathbb{Q})\times V'(F_0)_\xi$.

The second term $(g',h)\in (G'(\mathbb{A}_f^{v_0})/K_G^{v_0})^2\mapsto \mathbf{1}_{K_G^{v_0}}(g'^{-1}h)$ is also invariant under the (diagonal) $G'(\mathbb{Q})$-action. For a fixed pair (γ,u), we obtain

$$\begin{aligned}\sum_{(g',h)\in(G'(\mathbb{A}_f^{v_0})/K_G^{v_0})^2}&\phi_0^{v_0}(h^{-1}\gamma h)\phi^{v_0}(g'^{-1}u)\cdot\mathbf{1}_{K_G^{v_0}}(g'^{-1}h)\\ &=\sum_{h\in G'(\mathbb{A}_f^{v_0})/K_G^{v_0}}\phi_0^{v_0}(h^{-1}\gamma h)\phi^{v_0}(h^{-1}\cdot u)\\ &=\int_{G'(\mathbb{A}_f^{v_0})}\phi_0^{v_0}(h^{-1}\gamma\, h)\phi^{v_0}(h^{-1}\cdot u)\,dh\\ &=\mathrm{Orb}\left((\gamma,u),\Phi^{v_0}\right),\end{aligned}$$

where we note that the Haar measure on $G'(\mathbb{A}_f^{v_0})$ is normalized such that $\mathrm{vol}(K_G^{v_0})=1$.

To summarize, the intersection number ${}^{\mathbb{L}}\mathcal{CM}_R(\phi_0)\cap^{\mathbb{L}}\mathcal{Z}(\xi,\phi)\log q_\nu$ (restricted to $\mathcal{M}_{\widehat{O}_{\breve{E}_\nu},0}$) is equal to

$$2[E_\nu:F_{w_0}]\sum_{(\gamma,u)}\mathrm{Orb}\left((\gamma,u),\Phi^{v_0}\right)\cdot\mathrm{Int}_{v_0}(\gamma,u)\log q_{v_0},$$

where the sum is over $G'(\mathbb{Q})$-orbits of pairs $(\gamma,u)\in G'(g_0)(\mathbb{Q})\times V'(F_0)_\xi$.

Finally the sum over all places $\nu\mid v_0$ will cancel the factor $[E:F]$ in (9.13), by

$$\sum_{\nu|w_0}e_{\nu/w_0}f_{\nu/w_0}=\sum_{\nu|w_0}d_{\nu/w_0}=[E:F],$$

where e_{ν/w_0} (resp., $f_{\nu/w_0},d_{\nu/w_0}$) denotes the ramification degree (resp., inert degree, degree) of the extension E_ν/F_{w_0}. This completes the proof. □

10. Local intersection: archimedean places

The goal of this section is to compute the local intersection at ν of E above an archimedean place v_0 of F_0. In fact we will replace the automorphic Green function by Kudla's Green function, i.e., we consider the analog of (9.12):

$$\mathrm{Int}_\nu^{\natural,\mathbf{K}}(\xi,\Phi):=\left\langle\mathcal{G}_\nu^{\mathbf{K}}(\xi,\phi),\quad {}^{\mathbb{L}}\mathcal{CM}_R(\phi_0)_{\nu,\mathbb{C}}\right\rangle\log q_\nu.\tag{10.1}$$

When $F_0=\mathbb{Q}$ the difference is addressed by Theorem 8.4. Similar to (9.13), we set for $\xi\in F_0$,

$$\mathrm{Int}_{v_0}^{\mathbf{K}}(\xi,\Phi):=\frac{1}{\tau(Z^{\mathbb{Q}})\cdot[E:F]}\sum_{\nu|v_0}\mathrm{Int}_\nu^{\natural,\mathbf{K}}(\xi,\Phi).\tag{10.2}$$

We note that by (8.11) and (8.12), there is a parameter $h_\infty \in \mathrm{SL}_2(F_0 \otimes_{\mathbb{Q}} \mathbb{R})$ implicitly in the above expression.

The strategy is analogous to Theorem 9.4. We follow the notation in §8.2 and §8.3.

Theorem 10.1. *Let $\Phi \in \mathcal{S}((G \times V)(\mathbb{A}_f))$. Let $\xi \neq 0$. Then we have*

$$\mathrm{Int}^{\mathbf{K}}_{v_0}(\xi, \Phi) = \sum_{(g,u)} \mathrm{Int}_{v_0}(g,u) \cdot \mathrm{Orb}\left((g,u), \Phi\right), \tag{10.3}$$

where the sum runs over the $G'(\mathbb{Q})$-orbits (g,u) on the product

$$G'(g_0)(\mathbb{Q}) \times V'(F_0)_\xi.$$

Here $\mathrm{Int}_{v_0}(g,u)$ is defined as the special value of the function

$$\mathrm{Int}_{v_0}(g,u) = \mathcal{G}^{\mathbf{K}}(u, h_\infty)(z_g), \tag{10.4}$$

where z_g is the unique fixed point of g on $\mathcal{D}_{v_0}$. Moreover, the point z_g does not lie on any $\mathcal{D}_{v_0,u}$ for non-zero vector $u \in V'(F_0)$.

Proof. The proof goes along the same line as that of Theorem 9.4, so we will not repeat the detail, except to prove the claim on the point z_g. By the embedding $F'^1 \hookrightarrow G'$ twisted by g, the hermitian space V' is endowed with a one-dimensional F'/F'_0-hermitian structure, cf. (7.10). Then the $\mathbb{C}$-algebra $F'_{w_0} := F' \otimes_{F,w_0} \mathbb{C}$ acts on the n-dimensional $\mathbb{C}$-vector space $V' \otimes_{F,w_0} \mathbb{C}$. If a negative definite $\mathbb{C}$-line is fixed by g, then it is also fixed by the algebra F'_{w_0}. Therefore it must be an eigen-line for F'_{w_0}, which must be unique by the signature $(n-1,1)$ condition.

If z_g lies on a divisor $\mathcal{D}_{v_0,u}$ for non-zero vector $u \in V'(F_0)$, it also lies on $\mathcal{D}_{v_0, g^i\cdot u}$, the translation of g^i, for all $i \in \mathbb{Z}$. Equivalently, the line z_g is perpendicular to all $g^i \cdot u$. Since u is non-zero vector, and $\{g^i \mid 0 \le i \le n-1\}$ generate F' over F, the vectors $g^i \cdot u$ span $V' \otimes_{F,w_0} \mathbb{C}$ over F_{w_0}. Contradiction! □

It remains to compute (10.4), or equivalently $R(u, z_g)$ defined by (8.9). By the invariance of $R(u, z_g)$ under the action (by conjugation) of $G'(F_{0,v_0})$, we may assume that $g = g_0$. The F-vector space V' then carries the structure of a one-dimensional F'/F'_0-hermitian space,

$$\langle \cdot, \cdot \rangle_{F'_0} \colon\ V \times V \longrightarrow F', \tag{10.5}$$

cf. (7.10), with signatures $(1,0)$ for all but one archimedean place v'_0 of F'_0 over v_0. We define a refined invariant

$$\xi' = \mathfrak{q}'(u) \in F'_0, \tag{10.6}$$

where $\mathfrak{q}'$ is the quadratic form to the hermitian form, cf. 1.4.

According to the action of F'_0, we have an orthogonal direct sum decomposition

$$V' \otimes_{F,w_0} \mathbb{C} = \bigoplus_{v' \in \mathrm{Hom}(F'_0, \mathbb{R}), v'|_{F_0} = v_0} \mathbb{C}_{v'},$$

where F'_0 acts on the line $\mathbb{C}_{v'}$ through $v' : F'_0 \hookrightarrow \mathbb{R}$. Then there is a unique negative-definite summand, say $\mathbb{C}_{v'_0}$ for a place v'_0 above v_0. It follows that

$$R(u, z_{g_0}) = v'_0(\mathfrak{q}'(u)) = -|\xi'|_{v'_0}, \tag{10.7}$$

where the last equality is due to the fact $v'_0(\mathfrak{q}'(u)) < 0$.

Corollary 10.2. *Under the same assumptions as Theorem 10.1, we have*

$$\mathrm{Int}^{\mathbf{K}}_{v_0}(\xi, \Phi) = -\sum \mathrm{Ei}(-2\pi|\xi'|_{v'_0}) \cdot \mathrm{Orb}\left((g_0,u), \Phi\right), \tag{10.8}$$

where (g_0, u) is the unique orbit with the refined invariant $\mathfrak{q}'(u) = \xi' \in F'_0$, and the sum runs over all $\xi' \in F'_0$ such that

- $\mathrm{tr}_{F'_0/F_0}(\xi') = \xi$, *and*
- *there exists exactly one archimedean place v'_0 of F'_0 where ξ' is negative, and this place v'_0 is above v_0.*

Finally, we address the difference between the two Green functions. Define, for any place $v \mid \infty$ of F_0, and $h \in \mathrm{SL}_2(\mathbb{A}_0)$,

$$\mathrm{Int}_v^{\mathbf{K-B}}(h, \Phi) = \frac{1}{\tau(Z^{\mathbb{Q}}) \cdot [E : F]} \left(\mathcal{Z}_{v,\mathrm{corr}}(h, \phi), \quad {}^{\mathbb{L}}\mathcal{CM}_R(\phi_0) \right), \tag{10.9}$$

cf. (8.14), and

$$\mathrm{Int}^{\mathbf{K-B}}(h, \Phi) = \sum_{v \mid \infty} \mathrm{Int}_v^{\mathbf{K-B}}(h, \Phi). \tag{10.10}$$

We note that the definition works without any reference to the integral models $\mathcal{M}$, hence makes sense for all $\phi_0 \in \mathcal{S}\left(G(\mathbb{A}_{0,f}), K_G\right)$ and $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))^{K_G}$.

Corollary 10.3 (to Theorem 8.4)**.** *Let $F_0 = \mathbb{Q}$. Then the function $h \in \mathrm{SL}_2(\mathbb{A}_0) \mapsto \mathrm{Int}^{\mathbf{K-B}}(h, \Phi)$ belongs to $\mathcal{A}_{\exp}(\mathrm{SL}_2(\mathbb{A}_0), K, n)$, where $K \subset \mathrm{SL}_2(\mathbb{A}_{0,f})$ is a compact open subgroup which fixes $\phi \in \mathcal{S}(V(\mathbb{A}_{0,f}))$ under the Weil representation.*

11. Weil representation and RTF

Starting from this section, we study a partially linearized version of the Jacquet–Rallis relative trace formula, and the "action" on the RTF by $\mathrm{SL}_2(\mathbb{A})$ under Weil representation (by changing testing functions on the linear factor of the RTF).

11.1. **Weil representation and theta functions.** For now we let F be a global field. Let $(V, \mathfrak{q})$ be a (non-degenerate) quadratic space over F of *even* dimension d, where $\mathfrak{q} : V \to F$ is the quadratic form with the associated symmetric bilinear pairing $\langle \cdot, \cdot \rangle : V \times V \to F$ by (1.2). Let $O(V) = O(V, \mathfrak{q})$ be the isometry group, viewed as an algebraic group over F.

Let $\mathcal{S}(V(\mathbb{A}))$ be the space of Schwarz functions. The product group $O(V)(\mathbb{A}) \times \mathrm{SL}_2(\mathbb{A})$ acts on $\mathcal{S}(V(\mathbb{A}))$ via Weil representation denoted by ω: for $\phi \in \mathcal{S}(V(\mathbb{A}))$, the function $\omega(g,h)\phi$ is defined by

$$(\omega(g,h)\phi)(x) = (\omega(h))\phi(g^{-1}x), \quad (g,h) \in O(V)(\mathbb{A}) \times \mathrm{SL}_2(\mathbb{A}),$$

where the action of $\mathrm{SL}_2(\mathbb{A})$ is defined as follows. Let $\chi_V = \prod_v \chi_{V_v}$ be the quadratic character of $F^\times \backslash \mathbb{A}_F^\times$ defined by

$$\chi_V(a) = (a, (-1)^{d/2} \det(V))_F,$$

where $(\cdot, \cdot)$ is the Hilbert symbol over F, and $\det(V) \in F^\times/(F^\times)^2$ is the determinant of the moment matrix $\frac{1}{2}(\langle x_i, x_j \rangle)_{1 \le i,j \le d}$ of any F-basis $x_1, \cdots, x_d$ of V. For a place v of F, and $\phi_v \in \mathcal{S}(V(F_v))$, the action of $\mathrm{SL}_2(F_v)$ is determined by

$$\begin{aligned} \omega_v \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \phi_v(x) &= \chi_{V_v}(a)|a|_v^{d/2}\phi_v(ax), \\ \omega_v \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \phi_v(x) &= \psi_v(b\mathfrak{q}(x))\phi_v(x), \\ \omega_v \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \phi_v(x) &= \gamma_{V_v} \widehat{\phi}_v(x), \end{aligned} \tag{11.1}$$

where γ_{V_v} is the Weil constant (an eighth root of unity), and the Fourier transform is defined by

$$\widehat{\phi}_v(x) = \int_{V(F_v)} \phi_v(y)\psi_v\left(\langle x, y \rangle\right) dy.$$

Here dy is a self-dual Haar measure on $V(F_v)$.

For $\phi \in \mathcal{S}(V(\mathbb{A}))$, we define the theta function by the absolute convergent sum

$$\theta_\phi(g,h) = \sum_{\xi \in V} \omega(g,h)\phi(\xi), \quad (g,h) \in O(V)(\mathbb{A}) \times \mathrm{SL}_2(\mathbb{A}).$$

This is left invariant under $O(V)(F) \times \mathrm{SL}_2(F)$.

11.2. **Automorphic kernel functions.** In this subsection we work with a fairly general setting. It serves to explain the idea behind the more explicit setting in later sections.

Let G be a connected reductive algebraic group over F, acting on V and preserving the quadratic form $\mathfrak{q}$ (i.e., the homomorphism $G \to \mathrm{GL}(V)$ factors through $O(V, \mathfrak{q})$). Let X be an affine variety over F with an action of G. Consider the diagonal action r of G on $X \times V$. Then $G(\mathbb{A})$ acts on $\mathcal{S}((X \times V)(\mathbb{A}))$.

The group $\mathrm{SL}_2(\mathbb{A})$ acts on $\mathcal{S}((X\times V)(\mathbb{A}))$ through the second factor V via Weil representation. Note that now the formula for the action of $\mathrm{SL}_2(\mathbb{A})$ is only applied to the second coordinate, e.g., locally at v, the element $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ acts on $\mathcal{S}((X \times V)(F_v))$ by (up to the Weil constant γ_{V_v}) the partial Fourier transform w.r.t. the V-component.

Let $\phi_0 \in \mathcal{S}(X(\mathbb{A}))$, and let $x_0 \in X(F)$ be a fixed semisimple element (w.r.t. the G-action). Let $\phi \in \mathcal{S}(V(\mathbb{A}))$. We define the automorphic kernel function associated to $\Phi = \phi_0 \otimes \phi \in \mathcal{S}((X \times V)(\mathbb{A}))$,

$$\begin{aligned}\mathcal{K}_{\Phi,x_0}(g,h) &:= \sum_{u\in V, x\in G(F)x_0} \phi_0(g^{-1}\cdot x)\omega(h)\phi(g^{-1}\cdot u) \\ &= \sum_{u\in V, x\in G(F)x_0} \omega(h)\Phi(g^{-1}\cdot(x,u)),\end{aligned} \tag{11.2}$$

where $g \in G(\mathbb{A}), h \in \mathrm{SL}_2(\mathbb{A})$. This is again left invariant under $G(F) \times \mathrm{SL}_2(F)$. It follows that

$$h \in \mathrm{SL}_2(\mathbb{A}) \longmapsto \mathbb{J}(h,\Phi) = \int_{[G]} \mathcal{K}_{\Phi,x_0}(g,h)dg,$$

when absolutely convergent, is invariant under $\mathrm{SL}_2(F)$. The same applies if we replace the pure tensor $\phi_0 \otimes \phi$ by more general function Φ in $\mathcal{S}((X \times V)(\mathbb{A}))$ (this does not make any essential difference at non-archimedean places, but does at archimedean places).

Let T_0 be the stabilizer of x_0 (for the G-action on X). The above kernel function may be rewritten as a sum over the $G(F)$-orbits in $G(F)x_0 \times V(F)$

$$\begin{aligned}\sum_{u\in V, x\in G(F)x_0} r(g)\Phi(x,u) &= \sum_{\substack{u\in V,\\ \gamma\in T_0(F)\backslash G(F)}} r(g)\Phi(\gamma^{-1}\cdot x_0, u) \\ (\text{changing } u \text{ to } \gamma^{-1}\cdot u) &= \sum_{\gamma\in T_0(F)\backslash G(F)} \sum_{u\in V} r(g)\Phi(\gamma^{-1}\cdot(x_0,u)) \\ &= \sum_{u\in V(F)/T_0(F)} \sum_{\substack{\gamma\in T_0(F)\backslash G(F),\\ t\in T_{0,u}(F)\backslash T_0(F)}} r(g)\Phi(\gamma^{-1}\cdot(x_0, t\cdot u)) \\ &= \sum_{u\in V(F)/T_0(F)} \sum_{\gamma\in T_{0,u}(F)\backslash G(F)} r(g)\Phi(\gamma^{-1}\cdot(x_0,u)),\end{aligned}$$

where $T_{0,u}$ is the stabilizer of $u \in V$. Note that $T_{0,u}$ is equal to $G_{(x_0,u)}$, the stabilizer of (x_0, u) under the diagonal G-action on $X \times V$. Then the inner sum as a function of $G(\mathbb{A})$ is left invariant under $G(F)$. We obtain

$$\mathbb{J}(h,\Phi) = \int_{[G]} \sum_{u\in V,\, x\in G(F)x_0} r(g)\omega(h)\Phi(x,u)\, dg \tag{11.3}$$

$$= \sum_{u\in V(F)/T_0(F)} \int_{G_{(x_0,u)}(F)\backslash G(\mathbb{A})} \omega(h)\Phi(g^{-1}\cdot(x_0,u))\, dg. \tag{11.4}$$

So far we have not justified the convergence, but we will do so later in the cases of our interest. Luckily in this paper we will consider actions where all but very few orbits in $V(F)/T_0(F)$ are regular semisimple. In fact, below we apply the set up to group actions arising from the Jacquet–Rallis relative trace formulas. In these cases, the stabilizers of (x_0, u) will turn out to be either trivial or T_0.

Now we return to our earlier convention. Let F_0 be a totally real field, and F/F_0 a CM field extension. Let

$$\eta = \eta_{F/F_0} : F^\times \backslash \mathbb{A}_0^\times \longrightarrow \{\pm 1\}$$

be the quadratic character by class field theory. Note that now F_0 plays the role of the base field F in above discussion. From now on we denote

$$\mathbf{H} = \mathrm{SL}_2$$

as an algebraic group over F_0.

11.3. **The case of unitary groups.** Now we consider the Jacquet–Rallis RTF for unitary groups. Let V be a F/F_0-hermitian space of dimension n. Let $G = \mathrm{U}(V)$ be the unitary group. Viewing V as an F_0-vector space $\mathrm{R}_{F/F_0}V$ of dimension $2n$, the hermitian form defines a quadratic form on $\mathrm{R}_{F/F_0}V$ (cf. 1.2 (1.4) for the convention on hermitian forms and quadratic forms). We consider the adjoint action of $G = \mathrm{U}(V)$ on G, and on its Lie algebra $\mathfrak{g} = \mathfrak{u}(V)$, respectively.

Let $x_0 \in X = G$ or $\mathfrak{g}$ be a *regular elliptic* element, i.e., the stabilizer (for the G-action on X) is an anisotropic maximal torus T_0 of G. Then $V(F_0)$ breaks into $T_0(F_0)$-orbits among which, except $u = 0$, all the others are regular semisimple (w.r.t. T_0-action) and have trivial stabilizer. Then we rewrite (11.3)

$$\begin{aligned}
\mathbb{J}(h, \Phi) &= \int_{[G]} \sum_{u \in V, x \in G(F_0)x_0} r(g)\omega(h)\Phi(x, u)\, dg \\
&= \mathrm{vol}([T_0]) \int_{T_0(\mathbb{A}_0)\backslash G(\mathbb{A}_0)} \omega(h)\Phi(g^{-1}\cdot x_0, 0)\, dg \\
&\quad + \sum_{u \in V(F_0)/T_0(F_0), u \neq 0} \int_{G(\mathbb{A}_0)} \omega(h)\Phi(g^{-1}\cdot (x_0, u))\, dg.
\end{aligned} \tag{11.5}$$

Lemma 11.1. *For any $\Phi \in \mathcal{S}((X \times V)(\mathbb{A}_0))$, the sum (11.5) over nonzero u is absolutely convergent,*

$$\sum_{u \in V(F_0)/T_0(F_0), u \neq 0} \int_{G(\mathbb{A}_0)} |\Phi(g^{-1}\cdot (x_0, u))|\, dg < \infty.$$

In particular, the function $h \in \mathbf{H}(\mathbb{A}_0) = \mathrm{SL}_2(\mathbb{A}_0) \mapsto \mathbb{J}(h, \Phi)$ is left invariant under $\mathbf{H}(F_0)$.

Proof. The convergence follows from [1, Prop. A.2.1]. □

The summands in (11.5) are related to the global Jacquet–Rallis (relative) orbital integral (for the G-action on $X \times V$) of $\omega(h)\Phi$. For $\Phi \in \mathcal{S}((X \times V)(\mathbb{A}_0))$, and a regular semisimple $(x_0, u) \in (X \times V)(F_0)$, we define

$$\mathrm{Orb}((x_0, u), \Phi) :\, = \int_{G(\mathbb{A}_0)} \Phi(g^{-1}\cdot (x_0, u))\, dg. \tag{11.6}$$

When $u = 0$, we define

$$\mathrm{Orb}((x_0, 0), \Phi) := \mathrm{vol}([T_0]) \int_{T_0(\mathbb{A}_0)\backslash G(\mathbb{A}_0)} \Phi(g^{-1}\cdot x_0, 0). \tag{11.7}$$

It follows easily that, for $\xi \in F_0^\times$, the ξ-th Fourier coefficient of $\mathbb{J}(\cdot, \Phi)$ is equal to

$$\sum_{u \in V(F_0)/T_0(F_0),\, \mathfrak{q}(u) = \xi} \mathrm{Orb}((x_0, u), \omega(h)\Phi). \tag{11.8}$$

Here, for a left $N(F_0)$-invariant continuous function φ on $\mathbf{H}(\mathbb{A}_0)$, its ξ-th Fourier coefficient for $\xi \in F_0$ is defined as the function

$$h \in \mathbf{H}(\mathbb{A}_0) \longmapsto \int_{F_0\backslash \mathbb{A}_0} \varphi\left[\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} h\right] \psi(-\xi b) db. \tag{11.9}$$

Note that the orbit of $u \in V(F_0)/T_0(F_0)$ depends only on the refined invariant $\xi' = \mathfrak{q}'(u) \in F_0'$, cf. (10.6), and $\xi = \mathrm{tr}_{F_0'/F_0}\,\xi'$. We may rewrite (11.8) as

$$\sum_{\xi' \in F_0',\, \mathrm{tr}_{F_0'/F_0}\,\xi' = \xi} \mathrm{Orb}((x_0, u), \Phi), \tag{11.10}$$

where for a fixed ξ', $u \in V(F_0)/T_0(F_0)$ is the unique orbit such that $\mathfrak{q}'(u) = \xi'$.

Now assume that V are *positive definite* at all archimedean places $v \mid \infty$ of F. In particular, the group $G(\mathbb{R})$ is compact.

Note that, since x_0 is semisimple, the map

$$\begin{aligned} G(\mathbb{A}_0)/T_0(\mathbb{A}_0) &\longrightarrow X(\mathbb{A}_0) \\ g &\longmapsto g \cdot x_0 \end{aligned}$$

is a closed embedding. It follows that the image has compact intersection with the support of ϕ_0 in both the group and the Lie algebra case; this is due to the compactness of $G(\mathbb{R})$ and of the support of ϕ_0 away from archimedean places. Therefore we may fix a large compact subset Ω of $G(\mathbb{A}_0)$ such that $\Phi(g \cdot x_0, u) = 0$ unless $g \in \Omega\, T_0(\mathbb{A}_0)$. We introduce a Schwartz function on $V(\mathbb{A}_0)$

$$\phi(u) := \int_\Omega \Phi(g^{-1} \cdot (x_0, u))\, dg. \tag{11.11}$$

We may normalize the measure such that the orbital integral (11.6) is simplified as

$$\mathrm{Orb}((x_0, u), \Phi) = \int_{T_0(\mathbb{A}_0)} \phi(t^{-1} \cdot u)\, dt \tag{11.12}$$

and we denote the right hand by $\mathrm{Orb}(u, \phi)$. Similarly (11.7) becomes

$$\mathrm{Orb}(0, \phi) = \mathrm{vol}([T_0])\phi(0).$$

We simplify (11.5)

$$\mathbb{J}(h, \Phi) = \sum_{u \in V(F_0)/T_0(F_0)} \mathrm{Orb}(u, \omega(h)\phi), \quad h \in \mathrm{SL}_2(\mathbb{A}_0). \tag{11.13}$$

11.4. **The case of general linear groups.** We resume from the end of §11.2. Now we consider the Jacquet–Rallis RTF for general linear groups. Now let $V_0 = F_0^n$ be the n-dimensional vector space of column vectors over F_0. We identify the dual vector space $V_0^* = \mathrm{Hom}_{F_0}(V_0, F_0)$ with the space of row vectors. Consider the natural quadratic form on $V' = V_0 \times V_0^*$:

$$\begin{aligned} \mathfrak{q}\colon \quad V_0 \times V_0^* &\longrightarrow F_0 \quad . \\ u' = (u_1, u_2) &\longmapsto u_2(u_1) \end{aligned} \tag{11.14}$$

Let

$$\langle \cdot, \cdot \rangle \colon V' \times V' \longrightarrow F_0\ .$$

be the the associated symmetric bilinear pairing (so that $\langle u', u' \rangle = 2\mathfrak{q}(u')$). Let $G' = \mathrm{GL}(V_0)$ act on V' by $(\mathrm{std}, \mathrm{std}^\vee)$. Then $G' \simeq \mathrm{GL}_{n,F_0}$ via the given identification $V_0 = F_0^n$.

Consider the diagonal action of G' on $X' \times V'$ where X' is either the symmetric space S_n, or its Lie algebra $\mathfrak{s}_n$, cf. (2.2) and (2.3). Let $x_0' \in X'(F_0)$ be a regular elliptic element in the sense that $F' = F[x_0']$ is a field extension of F of degree n. Note that the condition implies and is stronger than that the stabilizer x_0' (for the G'-action on X') is a maximal torus T_0' that is anisotropic modulo the center of G'.

Let F_0' be the subfield fixed by the involution on F induced by that of F/F_0 and $x_0' \mapsto x_0'^{-1}$ (resp., $x_0' \mapsto -x_0'$) when $X' = S_n$ (resp., $\mathfrak{s}_n$). Then we have a natural isomorphism

$$T_0' \simeq \mathrm{Res}_{F_0'/F_0}\, \mathbb{G}_m,$$

viewed as F_0-algebraic groups. Note that $F' = FF_0'$ and the character $\eta \circ \det$ (of $G'(\mathbb{A}_0)$) is nontrivial on $T_0'(\mathbb{A}_0)$. Via the action of F_0', the vector space V_0 (hence V_0^*) carries a structure of a one-dimensional F_0'-vector space. Furthermore, we can identify

$$\mathrm{Hom}_{F_0'}(V_0, F_0') \simeq V_0^*$$

as one-dimensional F_0'-vector spaces. There is a unique bi-F_0'-linear symmetric pairing

$$\langle\cdot,\cdot\rangle_{F_0'}\colon V'\times V'\longrightarrow F_0' \tag{11.15}$$

such that

$$\langle u_1,u_2\rangle=\mathrm{tr}_{F_0'/F_0}\,\langle u_1,u_2\rangle_{F_0'}.$$

Let

$$\mathfrak{q}'\colon\ V_0\times V_0^*\longrightarrow F_0' \tag{11.16}$$

be the associated quadratic form over F_0'.

Then $V'(F_0)$ breaks into $T_0'(F_0)$-orbits among which, except the (relative) x_0-nilpotent cone, all the others are regular semisimple (w.r.t. T_0'-action) and have trivial stabilizer. The nilpotent cone breaks into three orbits

$$\begin{cases}\{(0,0)\},\\ 0_+:=\{(u_1,0): u_1\in V_0(F_0)\setminus\{0\}\},\\ 0_-:=\{(0,u_2): u_2\in V_0^*(F_0)\setminus\{0\}\}.\end{cases} \tag{11.17}$$

The last two are regular (i.e., with trivial stabilizers).

For $\Phi'\in\mathcal{S}((X'\times V')(\mathbb{A}_0))$, we consider, for $s\in\mathbb{C}$

$$\mathbb{J}(h,\Phi',s)=\int_{[G']}\left(\sum_{u'\in V',x'\in G'(F_0)x_0'} r(g)\omega(h)\Phi'(x',u')\right)|\det(g)|_{F_0}^s\eta(g)\,dg. \tag{11.18}$$

Here and thereafter we will simply denote by η the character $\eta\circ\det$ of $G'(\mathbb{A}_0)$. The definition depends on the fixed x_0', which will be suppressed in the notation. Similar to the unitary case, we write it as a sum over orbits:

$$\mathbb{J}(h,\Phi',s)=\mathbb{J}(h,\Phi',s)_0+\sum_{u'\in V'(F_0)_{\mathrm{rs}}/T_0'(F_0)}\int_{G'(\mathbb{A}_0)}\omega(h)\Phi'(g^{-1}\cdot(x_0',u'))|\det(g)|_{F_0}^s\eta(g)\,dg, \tag{11.19}$$

where the sum is over the regular semisimple orbits in $u'\in V'(F_0)_{\mathrm{rs}}/T_0'(F_0)$, and the term $\mathbb{J}(h,\Phi',s)_0$ is the sum over the regular x_0'-nilpotent orbits $0_\pm$ in (11.17), which will be defined in §12.6 by an analytic continuation. We have discarded the orbit $\{(0,0)\}$ since η is a non-trivial character on the stabilizer $T_0(\mathbb{A}_0)$.

Lemma 11.2. *For any $\Phi\in\mathcal{S}((X'\times V')(\mathbb{A}_0))$, the sum over regular semisimple orbits in $u'\in V'(F_0)/T_0'(F_0)$ is absolutely convergent*

$$\sum_{u'\in V'(F_0)_{\mathrm{rs}}/T_0'(F_0)}\int_{G'(\mathbb{A}_0)}|\Phi'(g^{-1}\cdot(x_0',u'))|\det(g)|_{F_0}^s\,dg<\infty,$$

and uniformly for s in any compact subset in $\mathbb{C}$.

Proof. This also follows the same argument as in [1, Prop. A.2.1]. □

We will defer the $\mathbf{H}(F_0)$-invariance to the next section, cf. Lemma 12.15. For now, we define the (global) Jacquet–Rallis (relative) orbital integral (for the G'-action on $X'\times V'$). For $\Phi'\in\mathcal{S}((X'\times V')(\mathbb{A}_0))$, and a regular semisimple $(x_0',u')\in(X'\times V')(F_0)$, we define

$$\mathrm{Orb}((x_0',u'),\Phi',s):=\int_{G'(\mathbb{A}_0)}\Phi'(g^{-1}\cdot(x_0',u'))\,|\det(g)|_{F_0}^s\eta(g)\,dg. \tag{11.20}$$

Choose the product Haar measure on $G'(\mathbb{A}_0)$. Then the global orbital integral is a product of local orbital integrals

$$\mathrm{Orb}((x_0',u'),\Phi_v',s):=\int_{G'(F_{0,v})}\Phi_v'(g^{-1}\cdot(x_0',u'))\,|\det(g)|_v^s\eta(g)\,dg. \tag{11.21}$$

12. RTF with Gaussian test functions

It is difficult to explicate the sum (11.5) (resp., (11.19)) in its full generality (this is essentially the same difficult as the geometric side of Jacquet–Rallis relative trace formula). Instead, we will simplify the sum for fixed x_0 (resp., x_0') by plugging in a Gaussian test function at every archimedean place.

12.1. **Gaussian test functions: the compact unitary group case.** Now let F/F_0 be the the archimedean local field extension $\mathbb{C}/\mathbb{R}$. Let V be an n-dimensional *positive definite* hermitian space with the unitary group $G = \mathrm{U}(V)$ and its Lie algebra $\mathfrak{u}(V)$. We define a special test function, called the Gaussian test function (cf. [40, §7]), in the Lie algebra setting

$$\Phi(x,u) = e^{-|x|^2} \otimes e^{-\pi\langle u,u\rangle} \in \mathcal{S}((\mathfrak{u}(V)\times V)(\mathbb{R})), \tag{12.1}$$

and in the semi-Lie algebra setting,

$$\Phi(x,u) = \mathbf{1}_{G(\mathbb{R})}(x) \otimes e^{-\pi\langle u,u\rangle} \in \mathcal{S}((G\times V)(\mathbb{R})). \tag{12.2}$$

Since they are invariant under $G(\mathbb{R})$, their orbital integrals (2.13) take a very simple form. We will normalize the Haar measure on $G(\mathbb{R})$ such that $\mathrm{vol}(G(\mathbb{R})) = 1$.

We explicate the action of $\mathrm{SL}_2(\mathbb{R})$ by the Weil representation (w.r.t. the fixed additive character $\psi : x \in \mathbb{R} \mapsto e^{2\pi i x}$). Write $h \in \mathrm{SL}_2(\mathbb{R})$ according to the Iwasawa decomposition

$$h = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} a^{1/2} & \\ & a^{-1/2} \end{pmatrix} \kappa_\theta, \quad a \in \mathbb{R}_+, \quad b \in \mathbb{R},$$

where $\kappa(\theta)$ is as in (1.10). First of all, the Gaussian test functions above are eigen-vectors of weight $k = n$ under the action of the maximal compact $\mathrm{SO}(2,\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$, i.e.,

$$\omega(\kappa_\theta)\Phi = \chi_n(\kappa_\theta)\Phi, \tag{12.3}$$

where χ_n is the character (1.11). In general, for h of the form (1.9),

$$\omega(h)\Phi(x,u) = \chi_n(\kappa_\theta)\mathbf{1}_{G(\mathbb{R})}(x) \otimes |a|^{1/2} e^{\pi i(b+ia)\langle u,u\rangle},$$

on $\mathrm{U}(V)\times V$ (12.2); a similar formula holds for the Gaussian function Φ (12.1) on $\mathfrak{u}(V)\times V$.

12.2. **Gaussian test functions: the general linear group case.** On the general linear group side, we define Gaussian test functions as the smooth transfer of the Gaussian test functions on the unitary side (cf. [40, §7]). We recall the bijection of regular semisimple orbits (2.7) and (2.9). Note that in the disjoint union, one component is from the positive definite hermitian space V. We then defined the notion of transfer at the end of §2.3.

Definition 12.1. We call $\Phi' \in \mathcal{S}((S_n\times V_n')(\mathbb{R}))$ (resp., $\mathcal{S}((\mathfrak{s}_n\times V_n')(\mathbb{R}))$) a Gaussian test function (relative to Ω_0) if it is a transer of the tuple $\{\Phi_V\}_V$ where Φ_V is the Gaussian test functions (12.1) (resp., (12.2)) for the positive definite hermitian space V, and $\Phi_V = 0$ for all the other (isometric classes of) hermitian spaces V.

Theorem 12.2. *Gaussian test functions on $(S_n\times V_n')(\mathbb{R})$ and $(\mathfrak{s}_n\times V_n')(\mathbb{R})$ exist.*

Proof. Since the group $G(\mathbb{R})$ is compact, the dual uncertainty principle in [48] holds for $(\mathfrak{u}(V)\times V)(\mathbb{R})$. Therefore, the existence of smooth transfer of the Gaussian test function on $(\mathfrak{u}(V)\times V)(\mathbb{R})$ follows from [44] and the procedure in [48] for non-archimedean local fields. Then the Lie algebra case implies the group case by the localization method in [48]. □

However, it seems very difficult to explicate the Gaussian test functions on $(S_n\times V')(\mathbb{R})$ or $(\mathfrak{s}_n\times V')(\mathbb{R})$ (with one exception: the case $n = 1$). Fortunately a weaker version suffices for our purpose. We only need a partial matching, i.e., only Schwartz functions that have matching orbital integrals for elements with a fixed component on $\mathfrak{s}_n$ or S_n; we will name them "partial Gaussian test functions".

For $\mathfrak{s}_n$ or S_n, we call the subset $\mathfrak{t}_n, T_n$ of diagonal elements their compact Cartan subspaces. We have

$$\mathfrak{t}_n \simeq \mathfrak{u}(1)(\mathbb{R})^n, \quad T_n \simeq \mathrm{U}(1)(\mathbb{R})^n.$$

Let $\mathfrak{t}_n^{\mathrm{rs}}$ and T_n^{rs} denote the open sets of the regular semisimple elements in the Cartan subspaces (i.e., those with distinct diagonal entries).

Definition 12.3. Let Ω_0 be a compact subset of $\mathfrak{t}_n^{\mathrm{rs}}$ or T_n^{rs}. We call $\Phi' \in \mathcal{S}((S_n \times V_n')(\mathbb{R}))$ or $\mathcal{S}((\mathfrak{s}_n \times V_n')(\mathbb{R}))$ a partial Gaussian test function (relative to Ω_0) if, for all regular semisimple $(x_0', u') \in \Omega_0 \times V'$ matching $(x, u) \in (\mathrm{U}(V) \times V)(\mathbb{R})$ (resp., $(\mathfrak{u}(V) \times V)(\mathbb{R})$) for the positive definite hermitian space V, we have

$$\mathrm{Orb}((x_0', u'), \Phi') = \mathrm{Orb}((x_0, u), \Phi), \tag{12.4}$$

where in the right hand side Φ is the Gaussian test functions (12.1) (resp., (12.2)), and $\mathrm{Orb}((x_0', u'), \Phi') = 0$ whenever a regular semisimple (x_0', u') matches an orbit from non-positive-definite hermitian spaces in (2.7) (resp., (2.9)) .

Now we construct "partial Gaussian test functions" explicitly, for any compact subset Ω_0 of $\mathfrak{t}_n^{\mathrm{rs}}$ or T_n^{rs}. We first consider the case $n = 1$, and then reduce the general case to $n = 1$.

12.3. **Gaussian test functions when** $n = 1$. Assume $n = \dim V = 1$. Then $G'(\mathbb{R}) \simeq \mathbb{R}^\times$, and the symmetric space $S_1(\mathbb{R})$ is compact. The orbital integrals have been defined in §2.3, cf. (2.12). Since the G'-action on $\mathfrak{s}_1$ and S_1 is trivial, we simply work with the vector space component and suppress the fixed $x_0' \in \mathfrak{s}_1$ or S_1 in the orbital integrals.

Let $V = \mathbb{C}$ be 1-dimensional hermitian space (with the standard norm), and let

$$\phi(z) = e^{-\pi\, z\bar{z}} \in \mathcal{S}(V).$$

Then we have $\widehat{\phi} = \phi$.

Let $V' \simeq \mathbb{R} \times \mathbb{R}$, with $\mathbb{R}^\times$-action

$$t \cdot (x, y) = (t^{-1}x, ty)$$

Recall from (11.14) that the quadratic form on V' is $\mathfrak{q}(x, y) = xy$. We consider the following Schwartz function in the Fock model,

$$\phi'(x, y) = 2^{-3/2}(x + y)e^{-\frac{1}{2}\pi(x^2+y^2)} \in \mathcal{S}(\mathbb{R} \times \mathbb{R}). \tag{12.5}$$

It has the symmetry

$$\phi'(x, y) = \phi'(y, x), \quad \phi'(-x, -y) = -\phi'(x, y).$$

Recall that the K-Bessel function is defined as

$$K_s(c) = \frac{1}{2}\int_{\mathbb{R}_+} e^{-\frac{1}{2}c(u+1/u)} u^s \frac{du}{u}, \quad c > 0, s \in \mathbb{C}.$$

Lemma 12.4. *Let* $\xi \in \mathbb{R}^\times$. *Then*

$$\mathrm{Orb}((1, \xi), \phi', s) = 2^{-1/2}|\xi|^{(-s+1)/2}\left(K_{(s+1)/2}(\pi|\xi|) + \eta(\xi)K_{(s-1)/2}(\pi|\xi|)\right).$$

In particular,

$$\mathrm{Orb}((1, \xi), \phi') = \begin{cases} e^{-\pi\xi}, & \xi > 0, \\ 0, & \xi < 0, \end{cases}$$

and when $\xi < 0$,

$$\partial\mathrm{Orb}((1, \xi), \phi') = \frac{1}{2}e^{-\pi\xi}\,\mathrm{Ei}(-2\pi|\xi|).$$

Here Ei *is the exponential integral* (8.10).

Remark 12.5. Here the special value at $s = 0$ has taken into account of the transfer factors, cf. §2.3.

Proof. By definition of orbital integrals (2.12) (except we have suppressed the $\mathfrak{s}_1$ and S_1 component), we have

$$\begin{aligned}\mathrm{Orb}((1,\xi),\phi',s) &= 2^{-1/2}\int_{\mathbb{R}_+}(t+\eta(\xi)|\xi|/t)e^{-\frac{1}{2}\pi(t^2+\xi^2/t^2)}t^{-s}\frac{dt}{t}\\ &= 2^{-1/2}|\xi|^{(-s+1)/2}\int_{\mathbb{R}_+}(t+\eta(\xi)/t)e^{-\frac{1}{2}\pi|\xi|(t^2+1/t^2)}t^{-s}\frac{dt}{t}\\ &= 2^{-1/2}|\xi|^{(-s+1)/2}\int_{\mathbb{R}_+}e^{-\frac{1}{2}\pi|\xi|(t^2+1/t^2)}(t^{-s+1}+\eta(\xi)t^{-s-1})\frac{dt}{t}\\ &= 2^{-3/2}|\xi|^{(-s+1)/2}\int_{\mathbb{R}_+}e^{-\frac{1}{2}\pi|\xi|(u+1/u)}(u^{(-s+1)/2}+\eta(\xi)u^{(-s-1)/2})\frac{du}{u}\\ &= 2^{-1/2}|\xi|^{(-s+1)/2}\left(K_{(-s+1)/2}(\pi|\xi|)+\eta(\xi)K_{(-s-1)/2}(\pi|\xi|)\right).\end{aligned}$$

To evaluate at $s=0$, we note

$$K_{1/2}(\xi)=\sqrt{\frac{\pi}{2}}\frac{e^{-\xi}}{\xi^{1/2}}.$$

Also we note that the transfer factor (2.15) takes value one at elements of the form $(1,\xi)$, applied to $F/F_0=\mathbb{C}/\mathbb{R}$.

The assertion for the first derivative follows from the following identity [37],

$$\frac{d}{ds}\Big|_{s=1/2}K_s(y)=-\sqrt{\frac{\pi}{2}}\frac{e^{y}}{y^{1/2}}\,\mathrm{Ei}(-2y),\quad y>0.$$

□

We now explicates the action of $\mathrm{SL}_2(\mathbb{R})$ by the Weil representation ω. Similar to the unitary case, the Gaussian test functions above are eigen-vectors of weight $k=n=1$ under the action of the maximal compact $\mathrm{SO}(2,\mathbb{R})$, cf. (12.3), (1.11). Write $h\in\mathrm{SL}_2(\mathbb{R})$ according to the Iwasawa decomposition

$$h=\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\begin{pmatrix}a^{1/2} & \\ & a^{-1/2}\end{pmatrix}\kappa_\theta,\quad a\in\mathbb{R}_+,\quad b\in\mathbb{R},$$

where $\kappa_\theta\in\mathrm{SO}(2,\mathbb{R})$ is as in (1.10).

Lemma 12.6. *Let* $\xi\in\mathbb{R}^\times$. *Then*

$$\mathrm{Orb}((1,\xi),\omega(h)\phi',s)=2^{-1/2}\chi_1(\kappa_\theta)a|\xi|^{(-s+1)/2}\left(K_{(-s+1)/2}(\pi a|\xi|)+\eta(\xi)K_{(-s-1)/2}(\pi a|\xi|)\right).$$

In particular,

$$\mathrm{Orb}((1,\xi),\omega(h)\phi')=\begin{cases}a^{1/2}e^{\pi i\xi(b+ia)}, & \xi>0,\\ 0, & \xi<0,\end{cases}$$

and when $\xi<0$,

$$\partial\mathrm{Orb}((1,\xi),\omega(h)\phi')=\frac{1}{2}\chi_1(\kappa_\theta)a^{1/2}\,e^{\pi i|\xi|(b-ia)}\,\mathrm{Ei}(2\pi a|\xi|).$$

Here Ei *is the exponential integral* (8.10).

Proof. This follows by straightforward computation using Lemma 12.4, and the formulas (11.1) defining Weil representation in §11.1. □

12.4. **Partial Gaussian test functions: general** n**.** We will use the Iwasawa decomposition of the group $G'(\mathbb{R})=\mathrm{GL}_n(\mathbb{R})$,

$$G'(\mathbb{R})=ANK, \tag{12.6}$$

where $K=\mathrm{SO}(n,\mathbb{R})$, N the group of unipotent upper triangular matrices, and $A\simeq(\mathbb{R}^\times)^n$ the diagonal torus. We have a homeomorphism

$$G'(\mathbb{R})\simeq AN\times_{\mu_2^{n-1}}K \tag{12.7}$$

as real manifolds, where the fiber product is over the intersection $AN \cap K$, which is equal to

$$K \cap A = \ker(\mu_2^n \longrightarrow \mu_2) \simeq \mu_2^{n-1}.$$

We will take the natural Haar measure on each factor (e.g., the measure $\frac{dt}{|t|}$ on $\mathbb{R}^\times$ and the product measure on $(\mathbb{R}^\times)^n \simeq A$) and take the induced measure on $G'(\mathbb{R})$ by the above product (12.7).

Note that the torus A is the stabilizer of a regular semisimple element in the Cartan subspace $\mathfrak{t}_n$ or T_n. Then $NK \cdot T_n^{\mathrm{rs}}$ (the conjugation action) defines an open subset $S_n^{c,\mathrm{rs}}$ ("c" is for "compact") in S_n:

$$\begin{aligned} NK \times T_n^{\mathrm{rs}} &\xrightarrow{\sim} S_n^{c,\mathrm{rs}} \subset S_n \\ (h,t) &\longmapsto h^{-1}th. \end{aligned}$$

The map is a $K \cap A$-torsor, and induces a $K \cap A$-torsor:

$$\begin{aligned} NK \times T_n^{\mathrm{rs}} \times (V_0 \times V_0^*) &\longrightarrow S_n^{c,\mathrm{rs}} \times (V_0 \times V_0^*) \\ (h,t,u') &\longmapsto (h^{-1}th, h \cdot u'). \end{aligned} \tag{12.8}$$

Now let $\Omega_0 \subset T_n^{\mathrm{rs}}$ be any compact subset. We consider functions on $NK \times T_n^{\mathrm{rs}} \times (V_0 \times V_0^*)$ of the form $\Psi = \phi_0 \otimes \phi'$, with $\phi' \in \mathcal{S}(V_0 \times V_0^*)$ and

$$\phi_0 = \varphi_N \otimes \varphi_K \otimes \varphi_{T_n}, \tag{12.9}$$

where

(1) the function $\varphi_{T_n} \in C_c^\infty(T_n^{\mathrm{rs}})$ satisfies $\varphi_{T_n}|_{\Omega_0} = \mathbf{1}_{\Omega_0}$,
(2) the function $\varphi_N \in C_c^\infty(N)$ satisfies $\int_N \varphi_N(n)dn = 1$,
(3) the function φ_K is a constant multiple of $\mathbf{1}_K$ such that $\int_K \varphi_K(k)dk = 1$,
(4) the function ϕ' is invariant under the finite group $K \cap A$.

By the $K \cap A$-invariance of ϕ' and of ϕ', the function $\Psi = \phi_0 \otimes \phi'$ descends along the map (12.8) to a Schwartz function Φ'^c on $S_n^{c,\mathrm{rs}} \times (V_0 \times V_0^*)$. Then the extension-by-zero of Φ'^c, denoted by Φ', is a Schwartz function on $S_n \times (V_0 \times V_0^*)$.

For $\Omega_0 \subset \mathfrak{s}_n$, we note that the Cartan subspace $\mathfrak{t}_n$ is also naturally embedded into the Lie algebra $\mathfrak{u}(V)$ of the compact unitary group $\mathrm{U}(V)$ (by choosing an orthogonal basis of V). Then we replace $\mathbf{1}_{\Omega_0}$ by the restriction of the first component of the Gaussian test function (12.1) to the Cartan subspace $\mathfrak{t}_n$, viewed as a subspace of $\mathfrak{u}(V)$.

Finally we specify ϕ' on $V_0 \times V_0^*$. Identify $V_0 \times V_0^*$ with $\mathbb{R}^n \times \mathbb{R}^n \simeq (\mathbb{R} \times \mathbb{R})^n$ and we define

$$\phi' = 2^{-3n/2} \prod_{1 \le i \le n} (x_i + y_i) e^{-\frac{1}{2}\pi(x_i^2 + y_i^2)}, \tag{12.10}$$

cf. (12.5) for the case $n = 1$. It is obviously invariant under $K \cap A$. Therefore by our recipe this function ϕ' (with any ϕ_0 above) gives us a Schwartz function Φ' on $S_n \times (V_0 \times V_0^*)$.

Now we define the orbital integral $\mathrm{Orb}(u', \phi', s)$ for $u' \in V_0 \times V_0^*$, relative to the A-action on $V_0 \times V_0^*$, in the obvious way generalizing the case $n = 1$, cf. (2.12).

Lemma 12.7. *Let $x_0' \in \Omega_0 \subset T_n^{\mathrm{rs}}$. Then for regular semisimple (x_0', u'), the local orbital integral* (11.21) *is equal to*

$$\mathrm{Orb}((x_0', u'), \Phi', s) = \mathrm{Orb}(u', \phi', s).$$

In particular, by Lemma 12.4, the function Φ' is a partial Gaussian test function (relative to the compact subset Ω_0).

Proof. By the Iwasawa decomposition (12.6), the local orbital integral (11.21) is equal to

$$\int_A \int_{NK} \Phi'((nk)^{-1} \cdot (x_0', a^{-1} \cdot u')) \, |\det(a)|^s \eta(a) \, dn \, dk \, da.$$

By our choice of Φ', we obtain

$$\begin{aligned}&\int_{NK}\Phi'((nk)^{-1}\cdot(x'_0,u'))\,dn\,dk\\ =&\left(\int_N\varphi_N(n)dn\int_K\varphi_K(k)dk\right)\varphi_{T_n}(x'_0)\phi'(u')\\ =&\phi'(u').\end{aligned}$$

Therefore

$$\mathrm{Orb}((x'_0,u'),\Phi',s)=\int_A\phi'(a^{-1}\cdot u')|\det(a)|^s\eta(a)da=\mathrm{Orb}(u',\phi',s).$$

This completes the proof.

□

Remark 12.8. This result holds even if u' is a regular nilpotent orbit and the orbital integral is regularized by (12.17) and (12.23) below.

Similar result holds for $\mathfrak{s}_n\times(V_0\times V_0^*)$, and we omit the detail.

12.5. **Modular analytic generating functions when** $n=1$**.** Now we return to the global situation §11.4. Assume that $n=1$. Then we may identify $V'=F_0\times F_0$ and the special orthogonal group $\mathrm{SO}(V',\mathfrak{q})$ can be identified with the F_0-group $G':=\mathrm{GL}_{1,F_0}$, via the action on the V' by $g\cdot(u_1,u_2)=(g^{-1}u_1,gu_2)$. The map $u'=(u_1,u_2)\mapsto\xi=\mathfrak{q}(u')=u_1u_2$ identifies the categorical quotient $V'_{/\!\!/G'}$ with the affine line. Note that regular semisimple orbits (w.r.t. the G'-action) are exactly the fibers over $\xi\neq 0$, and each fiber has exactly one G'-orbit and has trivial stabilizer.

Let $\phi'\in\mathcal{S}(V'(\mathbb{A}_0))$. Consider the integral,

$$\mathbb{J}(\phi',s)=\int_{[G']}\left(\sum_{u'\in V'(F_0)}\phi'(g^{-1}\cdot u')\right)|g|^s\eta(g)\,dg. \tag{12.11}$$

The integral is not necessarily convergent, and we define it by a regularization procedure.

As before (cf. (11.19)), we write the integrand as a sum over the $G'(F_0)$-orbits in $V'(F_0)$. For each $\xi=u_1u_2\in F_0\simeq V'_{/\!\!/G'}(F_0)$, denote its fiber by $V'(F_0)_\xi$. Then the sum over $\xi\in F_0^\times$ yields

$$\int_{[G']}\left(\sum_{\xi\in F_0^\times}\sum_{u'\in V'(F_0)_\xi}\phi'(g^{-1}\cdot u'))\right)|g|^s\eta(g)\,dg=\sum_{\xi\in F_0^\times}\int_{G'(\mathbb{A}_0)}\phi'(g^{-1}\cdot u')|g|^s\eta(g)\,dg \tag{12.12}$$

where each integral and the sum are absolutely convergent, and uniformly for s a compact set in $\mathbb{C}$ (cf. Lemma 11.2). We denote the term for (u_1,u_2) with $\xi=u_1u_2\in F^\times$ by

$$\mathrm{Orb}((u_1,u_2),\phi',s):=\int_{G'(\mathbb{A}_0)}\phi'(g^{-1}\cdot u')|g|^s\eta(g)\,dg. \tag{12.13}$$

The fiber over $\xi=0$ breaks into three orbits

$$\begin{cases}\{(0,0)\},\\ 0_+=\{(u_1,0):u_1\in F_0^\times\},\\ 0_-=\{(0,u_2):u_2\in F_0^\times\}.\end{cases}$$

The stabilizer of the first one is G', and the other two have trivial stabilizer. Note that η is non-trivial on $G'(\mathbb{A}_0)$, and hence we *define* the integral for the first orbit to be zero. For the other two orbits, we define

$$\mathrm{Orb}(0_+,\phi',s):=\int_{\mathbb{A}_0^\times}\phi'(g,0)|g|^s\eta(g)\,dg, \tag{12.14}$$

and

$$\mathrm{Orb}(0_-,\phi',s)=\int_{\mathbb{A}_0^\times}\phi'(0,g^{-1})|g|^s\eta(g)\,dg=\int_{\mathbb{A}_0^\times}\phi'(0,g)|g|^{-s}\eta(g)\,dg. \tag{12.15}$$

Both will be understood as Tate's global zeta integrals. More precisely,

$$\mathrm{Orb}(0_+,\phi',s)=L(s,\eta)\prod_v \mathrm{Orb}(0_+,\phi'_v,s), \tag{12.16}$$

where the local orbital integral for the regular nilpotent 0_+ is defined as (the analytic continuation of)

$$\mathrm{Orb}(0_+,\phi'_v,s):=\frac{\int_{F_{0,v}^\times}\phi'_v(g,0)|g|_v^s\eta_v(g)\,dg}{L(s,\eta_v)}. \tag{12.17}$$

Note that the local Tate integral (12.17) is absolutely convergent when $\mathrm{Re}(s)>0$, extends to an entire function of s (a polynomial in $q_v^{\pm s}$ when v is non-archimedean) and equal to one for unramified data. Here $L(s,\eta)$ is the *complete* L-function of the Hecke character η. Similarly for 0_-, we have

$$\mathrm{Orb}(0_-,\phi',s)=L(-s,\eta)\prod_v \mathrm{Orb}(0_-,\phi'_v,-s), \tag{12.18}$$

where

$$\mathrm{Orb}(0_-,\phi'_v,-s)=\frac{\int_{F_{0,v}^\times}\phi'_v(0,g)|g|_v^{-s}\eta_v(g)\,dg}{L(-s,\eta_v)}.$$

To summarize, we define (12.11) as the sum of (12.12), (12.14), and (12.15) (or rather, their analytic continuation to $s\in\mathbb{C}$).

Define the analytic generating function on $\mathbf{H}(\mathbb{A}_0)$ (for $\mathbf{H}=\mathrm{SL}_2$ over F_0),

$$\mathbb{J}(h,\phi',s)=\mathbb{J}(\omega(h)\phi',s),\quad h\in\mathbf{H}(\mathbb{A}_0).$$

Remark 12.9. The function $\mathbb{J}(\cdot,\phi',s)$ may be viewed the generating function of the above relative orbital integrals (12.13), (12.14), and (12.15), parameterized by $\xi\in F_0$. This is the analytic counterpart of the modular generating function of special divisors in §8.

Theorem 12.10. *The function $\mathbb{J}(h,\phi',s)$ is entire for $s\in\mathbb{C}$. As a function in $h\in\mathbf{H}(\mathbb{A}_0)$, it is left invariant under $\mathbf{H}(F_0)$.*

Proof. The entireness follows from the same property for each of (12.12), (12.14), and (12.15). To show the $\mathbf{H}(F_0)$-invariance, we first note that the invariance under the upper triangular elements follow from the definition of Weil representation and that of the function $\mathbb{J}(h,\phi',s)$. It remains to show the invariance under $w=\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$, i.e., the functional equation

$$\mathbb{J}(h,\phi',s)=\mathbb{J}(h,\widehat{\phi}',s). \tag{12.19}$$

By Poisson summation formula (note that the group G'-action commutes with Weil representation)

$$\sum_{u'\in V'}\phi'(g^{-1}\cdot u')=\sum_{u'\in V'}\widehat{\phi}'(g^{-1}\cdot u'),\quad g'\in G'(\mathbb{A}),$$

or equivalently,

$$\begin{aligned}\sum_{u'\in V',\xi\neq 0}\phi'(g^{-1}\cdot u')-&\sum_{u'\in V',\xi\neq 0}\widehat{\phi}'(g^{-1}\cdot u')\\ &=-\sum_{u'\in V'_{\xi=0}}\phi'(g^{-1}\cdot u')+\sum_{u'\in V'_{\xi=0}}\widehat{\phi}'(g^{-1}\cdot u'),\quad g'\in G'(\mathbb{A}).\end{aligned} \tag{12.20}$$

Denote $[G']^1=G'(F_0)\backslash G'(\mathbb{A}_0)^1$, where

$$G'(\mathbb{A}_0)^1:=\ker(G'(\mathbb{A}_0)\longrightarrow\mathbb{R}_+),\quad g\longmapsto|\det(g)|.$$

We embed $\mathbb{R}_+$ into $G'(\mathbb{A}_0)=\mathbb{A}_0^\times$ by sending $t\in\mathbb{R}_+$ to (t_v) where

$$t_v=\begin{cases}t^{1/[F_0:\mathbb{Q}]}, & v\mid\infty,\\ 1, & v\nmid\infty.\end{cases}$$

Then we have a direct product

$$\begin{aligned} G'(\mathbb{A}_0) &\xrightarrow{\sim} G'(\mathbb{A}_0)^1 \times \mathbb{R}_+ \\ g &\longmapsto (g_1, t) \end{aligned} \tag{12.21}$$

Since the quotient $[G']^1$ is compact, we may integrate (12.20) over $[G']^1$ first, and this kills the zero orbits (due to the non-triviality of $\eta|_{[G']^1}$). Then we integrate over $\mathbb{R}_+$, this is where the Tate integrals are needed to define the regular nilpotent orbit integrals (12.14) and (12.15).

We introduce a partial Fourier transform $\phi' \mapsto \mathcal{F}_i(\phi')$ for each of the two variables[6], e.g.,

$$\mathcal{F}_1(\phi')(u_1, u_2) = \int_{\mathbb{A}} \phi'(u_2, w_2)\psi(-u_1 w_2)\, dw_2.$$

Then Tate's global functional equation gives (noting that $\eta^{-1} = \eta$)

$$\begin{aligned} \mathrm{Orb}(0_+, \mathcal{F}_1(\phi'), 1-s) &= \mathrm{Orb}(0_-, \phi', s), \\ \mathrm{Orb}(0_-, \mathcal{F}_1(\phi'), s) &= \mathrm{Orb}(0_+, \mathcal{F}_1(\phi'), s), \end{aligned}$$

and similarly for $\mathcal{F}_2$

$$\begin{aligned} \mathrm{Orb}(0_+, \mathcal{F}_2(\phi'), s) &= \mathrm{Orb}(0_-, \mathcal{F}_2(\phi'), s), \\ \mathrm{Orb}(0_-, \mathcal{F}_2(\phi'), 1-s) &= \mathrm{Orb}(0_+, \phi', s). \end{aligned}$$

It follows that the full Fourier transform satisfies

$$\begin{aligned} \mathrm{Orb}(0_+, \widehat{\phi}', s) &= \mathrm{Orb}(0_+, \mathcal{F}_1(\mathcal{F}_2\phi'), s) \\ &= \mathrm{Orb}(0_-, \mathcal{F}_2(\phi'), 1-s) = \mathrm{Orb}(0_+, \phi', s). \end{aligned}$$

Similarly we have

$$\mathrm{Orb}(0_-, \widehat{\phi}', s) = \mathrm{Orb}(0_-, \phi', s).$$

This completes the proof of (12.19). □

Remark 12.11. The integral (12.11) can be viewed as the theta lifting for the pair

$$(\mathrm{SO}(V', \mathfrak{q}), \quad \mathrm{SL}_2),$$

from the automorphic representation $\eta|\cdot|^s$ of $\mathrm{SO}(V') \simeq \mathrm{GL}_1$ to SL_2. Therefore, the representation space spanned by $h \mapsto \mathbb{J}(h, \phi', s)$ is the space of degenerate Eisenstein series for the induced representation $\mathrm{Ind}_{B(\mathbb{A}_0)}^{\mathbf{H}(\mathbb{A}_0)}(\eta|\cdot|^s)$ (B the Borel subgroup of upper triangular matrices). In this way, the two nilpotent orbital integrals become the constant terms of the associated Eisenstein series.

Lemma 12.12. *Let $v \mid \infty$, and ϕ'_v the Gaussian test function* (12.5). *Then the local nilpotent orbital integral* (12.17) *is equal to*

$$\mathrm{Orb}(0_+, \phi'_v, s) = 2^{\frac{s}{2}-1}.$$

The action of the group $\mathrm{SL}_2(\mathbb{R})$ *is given as follows, for* $h \in \mathrm{SL}_2(\mathbb{R})$ *in the form* (1.9),

$$\mathrm{Orb}(0_+, \omega(h)\phi', s) = \chi_1(\kappa_\theta) a^{(-s+1)/2} 2^{\frac{s}{2}-1}.$$

Proof. By (12.5), we obtain $\phi'_v(x, 0) = 2^{-3/2} x e^{-\frac{1}{2}\pi x^2}$. Then $\mathrm{Orb}(0_+, \phi'_v, s)$ is the Tate's local zeta integral at an archimedean place:

$$\begin{aligned} 2\int_{\mathbb{R}_+} e^{-\frac{1}{2}\pi x^2} |x|^{s+1} \frac{dx}{x} &= \int_{\mathbb{R}_+} e^{-\frac{1}{2}\pi x} |x|^{(s+1)/2} \frac{dx}{x} \\ &= (\pi/2)^{-(s+1)/2} \Gamma((s+1)/2). \end{aligned}$$

Note the local L-factor in (12.17) is by definition

$$L(s, \eta) = \pi^{-(s+1)/2} \Gamma\left(\frac{s+1}{2}\right).$$

[6] $\mathcal{F}_1$ partial Fourier transform is the composition of the full one with $\mathcal{F}_2$ (up to changing the sign of the variable, which is harmless in the global setting).

We obtain

$$\mathrm{Orb}(0_+, \phi'_v, s) = 2^{\frac{s}{2}-1}.$$

The action of $\mathrm{SL}_2(\mathbb{R})$ is determined in the way similar to Lemma 12.6.

□

12.6. **Modular analytic generating functions for general** n**.** We now return to the setting of §11.4 for general n.

Definition 12.13. An element $x_0 \in S_n(F_0)$ (resp., $\mathfrak{s}_n(F_0)$) is compact, if locally at all places $v \mid \infty$, it lies in the compact Cartan subspace $T_n(F_{0,v})$ (resp., $\mathfrak{t}_n(F_{0,v})$).

From now on, let $X'_n = S_n$ or $\mathfrak{s}_n$, and we fix the following data.

(a) Let $x'_0 \in X'_n(F_0)$ be a regular elliptic compact element. Hence, $F' = F[x'_0]$ is a quadratic field extension of F'_0 such that $T'_0 \simeq \mathrm{Res}_{F'_0/F_0} \mathbb{G}_m$, cf. §11.4. Denote by

$$\eta' = \eta_{F'/F'_0} : \mathbb{A}^\times_{F'_0} \longrightarrow \{\pm 1\}$$

the quadratic character associated to F'/F'_0.

(b) For every $v \mid \infty$, we fix the archimedean $\Phi'_v \in \mathcal{S}((X'_n \times V'_n)(F_{0,v}))$ to be the partial Gaussian test function constructed in §12.4 (relative to a fixed compact neighborhood of x'_0). Let ϕ'_v be the function associated to Φ'_v (cf. (12.10)).

Under the condition (a) above, there are two regular nilpotent orbits for the T'_0-action on $V'(F_0)$, denoted by $0_\pm$ in (11.17). We now define the constant term $\mathbb{J}(h, \Phi', s)_0$ in (11.19) as the sum of the two regular x'_0-nilpotent orbital integrals $\mathrm{Orb}((x'_0, 0_\pm), \Phi', s)$ in a similar way to (12.14). More precisely, we define

$$\mathrm{Orb}((x'_0, 0_+), \Phi', s) := L(s, \eta') \prod_v \mathrm{Orb}((x'_0, 0_+), \Phi'_v, s), \tag{12.22}$$

where the local orbital integral is defined as

$$\mathrm{Orb}((x'_0, 0_+), \Phi'_v, s) = \frac{\int_{G'(F_{0,v})} \Phi'_v(g^{-1} \cdot (x'_0, 0_+)) \, |\det(g)|_v^s \eta(g) \, dg}{L(s, \eta'_v)}. \tag{12.23}$$

Here the denominator is defined as

$$L(s, \eta'_v) = \prod_{v' \mid v} L(s, \eta'_{v'})$$

where v' runs over all places of F'_0 above v. Note that $L(s, \eta') = L(s, \mathrm{Ind}_{F'_0}^{F_0} \eta')$. We define $\mathrm{Orb}((x'_0, 0_-), \Phi', s)$ similarly. Here we normalize the measure on $G'(F_{0,v})$ such that $\mathrm{vol}(G'(O_{F_{0,v}})) = 1$ for all but finitely non-archimedean places v.

Lemma 12.14. *The integral* (12.23) *is absolutely convergent when* $\mathrm{Re}(s) > 0$, *extends to an entire function of* s *(a polynomial of* $q_v^{\pm s}$ *for non-archimedean* v*).*

Moreover, for a fixed x'_0 *and a pure tensor* $\Phi = \otimes_v \Phi_v$ *where* $\Phi'_v = \mathbf{1}_{(X' \times V')(O_{F_{0,v}})}$ *for all but finitely many* v, *the integral* (12.23) *is equal to one for all but finitely many places* v *(depending on* x'_0 *and* Φ*).*

Proof. When $v \mid \infty$, by Lemma 12.7, the desired claim follows from (the product of n copies of) the same claim for $n = 1$.

Now let v be non-archimedean, and Φ'_v as in §12.4. We fix a large compact subset Ω_v of $G'(F_{0,v})$ such that $\Phi'_v(g^{-1} \cdot x'_0, u') = 0$ unless $g \in \Omega_v \cdot T'_0(F_{0,v})$. We introduce a Schwartz function (with a parameter $s \in \mathbb{C}$) on $V'(F_{0,v})$

$$\phi'_{v,s}(u') := \int_{\Omega_v} \Phi'_v(g^{-1} \cdot (x'_0, u')) \, |\det(g)|_v^s \eta_v(g) \, dg. \tag{12.24}$$

It is easy to see that it is of the form

$$\phi'_{v,s} = \sum_{1 \le i \le m} a_i \, \lambda_i^s \, \phi_i, \tag{12.25}$$

where

$$a_i \in \mathbb{Q}, \quad \lambda_i \in \mathbb{Q}_+^\times, \quad \phi_i \in \mathcal{S}(V'(F_{0,v})).$$

Then, for a suitable choice of measure dg on Ω_v in the integral (12.24)

$$\mathrm{Orb}((x_0', 0_+), \Phi'_{v_0}, s) = \mathrm{Orb}(0_+, \phi'_{v_0,s}, s). \tag{12.26}$$

Here we view V_0 as a one-dimensional F_0'-vector space, and V_0^* as its F_0'-dual vector space, and the right hand side is (12.18) relative to the quadratic extension $F_v'/F_{0,v}'$ at v (i.e., F_v' is the product of $F_{v'}' = F' \otimes_{F_0'} F_{0,v'}'$ over all places v' of F_0' over v). This shows that the local orbital integral for 0_+ is a polynomial of $q_{v'}^{\pm s}, v'|v$, particularly, an entire function in s.

Finally, let's assume that v is unramified in F' and $\Phi_v' = \mathbf{1}_{(X' \times V')(O_{F_0,v})}$ (here we implicitly identified $V_0 = F_0^n$ and endow it with the natural integral structure). For all but finitely many places v, the element x_0 belongs to $X_n'(O_{F_0,v})$ and generates the maximal order $O_{F_v'}$ in F_v'. Then it is easy to see that $\phi_{v,s}' = \mathbf{1}_{V'(O_{F_0,v})}$, and hence the integral is equal to one by the standard computation of Tate's local zeta integral for unramified data.

□

Lemma 12.15. *The function $(h, s) \in \mathbf{H}(\mathbb{A}_0) \times \mathbb{C} \mapsto \mathbb{J}(h, \Phi', s)$ is entire in $s \in \mathbb{C}$, and left invariant under $\mathbf{H}(F_0)$.*

Proof. By the proof of Lemma 12.14, (12.25) and (12.26), we see that

$$\mathbb{J}(h, \Phi', s) = \sum_{1 \le i \le m} a_i\, \lambda_i^s\, \mathbb{J}(h, \phi_i', s).$$

The desired claims follow now from Theorem 12.10 for $n = 1$, applied to the new quadratic extension F'/F_0'. □

For simplicity, we combine the two nilpotent orbital integrals into one

$$\mathrm{Orb}((x_0', 0_\pm), \Phi', s) \colon = \mathrm{Orb}((x_0', 0_+), \Phi', s) + \mathrm{Orb}((x_0', 0_-, \Phi', s). \tag{12.27}$$

Now we obtain an expansion as a sum of orbital integrals

$$\begin{aligned}\mathbb{J}(h, \Phi', s) = &\,\mathrm{Orb}((x_0', 0_\pm), \omega(h)\Phi', s) \\ &+ \sum_{u' \in V'(F_0)/T_0'(F_0), \mathfrak{q}(u') \neq 0} \mathrm{Orb}((x_0', u'), \omega(h)\Phi', s).\end{aligned} \tag{12.28}$$

Moreover, for $\xi \in F_0^\times$, the ξ-th Fourier coefficient of $\mathbb{J}(\cdot, \Phi', s)$ is equal to

$$\sum_{u' \in V'(F_0)/T_0'(F_0), \mathfrak{q}(u') = \xi} \mathrm{Orb}((x_0', u'), \omega(h)\Phi', s). \tag{12.29}$$

This is the analog of (11.8) on the unitary side. Analogous to (11.10), we may rewrite (12.29) as

$$\sum_{\xi' \in F_0', \mathrm{tr}_{F_0'/F_0} \xi' = \xi} \mathrm{Orb}((x_0', u'), \Phi', s), \tag{12.30}$$

where for a fixed ξ', $u' \in V'(F_0)/T_0'(F_0)$ is the unique orbit such that $\mathfrak{q}'(u') = \xi'$.

12.7. **The decomposition of the special value at $s = 0$.** We set

$$\mathbb{J}(h, \Phi') := \mathbb{J}(h, \Phi', 0). \tag{12.31}$$

Then the decomposition (12.28) specializes to

$$\mathbb{J}(h, \Phi') = \sum_{u' \in V'(F_0)/T_0'(F_0)} \mathrm{Orb}((x_0', u'), \omega(h)\Phi'), \tag{12.32}$$

where

$$\mathrm{Orb}((x_0', u'), \omega(h)\Phi') := \mathrm{Orb}((x_0', u'), \omega(h)\Phi', 0).$$

We set

$$\begin{aligned}\partial\mathbb{J}(h,\Phi') &:= \frac{d}{ds}\Big|_{s=0}\mathbb{J}(h,\Phi',s),\\ \partial\mathrm{Orb}((x_0',u'),\Phi_v') &:= \frac{d}{ds}\Big|_{s=0}\mathrm{Orb}((x_0',u'),\Phi_v',s).\end{aligned} \tag{12.33}$$

Now we introduce

$$\begin{aligned}\partial\mathbb{J}_v(h,\Phi') &:= \partial\mathbb{J}_v(\omega(h)\Phi'), \quad \text{and}\\ \partial\mathbb{J}_v(\Phi') &:= \sum_{u'\in V'(F_0)/T_0'(F_0),u'\neq 0}\partial\mathrm{Orb}((x_0',u'),\Phi_v')\cdot\mathrm{Orb}((x_0',u'),\Phi'^{v}).\end{aligned} \tag{12.34}$$

Then by Leibniz's rule, we have a decomposition, with an extra term from the nilpotent orbital integrals (12.22)

$$\partial\mathbb{J}(h,\Phi') = \partial\mathrm{Orb}(0_\pm,\omega(h)\Phi') + \sum_v \partial\mathbb{J}_v(\omega(h)\Phi'), \tag{12.35}$$

where we define

$$\partial\mathrm{Orb}(0_\pm,\omega(h)\Phi') = \frac{d}{ds}\Big|_{s=0}\mathrm{Orb}((x_0',0_\pm),\omega(h)\Phi',s). \tag{12.36}$$

We call it the nilpotent term; it is part of the constant term (i.e., the 0-th Fourier coefficient).

Part 3. Proof of the main theorems

13. The proof of FL

13.1. **Smooth transfer: the global situation.** In §2.3, we have defined the local transfer factor, cf. (2.15). The definition depends on a choice of an extension $\widetilde{\eta}$ of the quadratic character η attached to local quadratic extension. In the global case, we fix an extension of the quadratic character η_{F/F_0} of $F_0^\times\backslash\mathbb{A}_0^\times$ to a character $\widetilde{\eta}$ of $F^\times\backslash\mathbb{A}^\times$ (not necessarily of order 2). The transfer factor for a global element then satisfies a product formula, and transforms according to the desired rule, cf. [40, §7.3].

We are now in the setting of §12.6. For simplicity we only consider S_n, though it will be clear how to extend the result to $\mathfrak{s}_n$. Let $\Phi' = \otimes_v\Phi_v' \in \mathcal{S}((S_n\times V_n')(\mathbb{A}_0))$ be a pure tensor such that

- for every $v\mid\infty$, Φ_v' is the partial Gaussian test function, cf. 12.6, and
- for every non-archimedean v, Φ_v' is pure in the sense of [47, Def. 3.5], i.e., Φ_v' transfers to a non-zero function on $(\mathrm{U}(V_v)\times V_v)(F_{0,v})$ for (at most) one hermitian space V_v (cf. Definition 2.2, where there are at most two isometric classes of hermitian spaces at every non-archimedean place).

We define a weaker notion of smooth transfer.

Definition 13.1. For fixed $g_0\in\mathrm{U}(V_v)(F_{0,v})$ and $x_0'\in S_n(F_{0,v})$ with the same characteristic polynomial, we say that Φ_v' partially (w.r.t. g_0 and x_0') transfers to $\Phi_v\in\mathcal{S}(\mathrm{U}(V_v)\times V_v)(F_{0,v})$, if we only require the equality (2.16) in Definition 2.2 to hold for regular semisimple orbits of the form $(x_0',u')\in(S_n\times V_n')(F_{0,v})_{\mathrm{rs}}$ and $(g_0,u)\in(\mathrm{U}(V_v)\times V_v)(F_{0,v})_{\mathrm{rs}}$.

For such $\Phi'^{\infty} = \otimes_{v\nmid\infty}\Phi_v' \in \mathcal{S}((S_n\times V_n')(\mathbb{A}_{0,f}))$, we say that it partially transfers to (or matches) $\Phi^\infty = \otimes_{v\nmid\infty}\Phi_v\in\mathcal{S}((\mathrm{U}(V)\times V)(\mathbb{A}_{0,f}))$ if Φ_v' partially transfers to Φ_v for every non-archimedean place v.

Remark 13.2. At those places of F_0 split in F, we will further demand Φ_v and Φ_v' to match in an elementary way analogous to [48].

13.2. **Comparison.** In this subsection, we compare $\mathbb{J}(h,\Phi')$ with $\mathbb{J}(h,\Phi)$ in the "coherent case", i.e., $\Phi=\otimes_v\Phi_v\in\mathcal{S}((\mathrm{U}(V)\times V)(\mathbb{A}_0))$ for an n-dimensional F/F_0-hermitian space V. We further assume that V is totally positively definite and Φ_v is the Gaussian test function for every $v\mid\infty$, cf. (12.2) in §12.1.

Proposition 13.3. *The function*

$$h \in \mathrm{SL}_2(\mathbb{A}_0) \longmapsto \mathbb{J}(h, \Phi'), \ \textit{respectively}\ \mathbb{J}(h, \Phi),$$

lies in $\mathcal{A}_{\mathrm{hol}}(\mathrm{SL}_2(\mathbb{A}_0), K(N), n)$, *where* $K(N)$ *is a compact open subgroup of* $\mathrm{SL}_2(\mathbb{A}_{0,f})$ *that acts trivially on both* Φ *and* Φ'.

Proof. The $K(N)$-invariance follows immediately from the definition of $\mathbb{J}(h, \Phi')$ and $\mathbb{J}(h, \Phi)$.

By Lemma 12.15 (resp., Lemma 11.1) the functions $\mathbb{J}(\cdot, \Phi')$ (resp., $\mathbb{J}(\cdot, \Phi)$) are invariant under $\mathrm{SL}_2(F_0)$. The weight n-condition follows from the action under $\mathrm{SO}(2, \mathbb{R})$, by (12.3) for Φ, and by Lemma 12.6 for Φ'.

Finally we need to show the holomorphy on the complex upper half plane $\mathcal{H}$ and at all cusps. Equivalently, for any $h_f \in \mathrm{SL}_2(\mathbb{A}_{0,f})$, the function

$$b + ai \in \mathcal{H} \longmapsto |a|^{-n/2}\mathbb{J}((h_\infty, h_f), \Phi'), \ \text{respectively}\ |a|^{-n/2}\mathbb{J}((h_\infty, h_f), \Phi),$$

where $h_\infty = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} a^{1/2} & \\ & a^{-1/2} \end{pmatrix}$, is holomorphic, and holomorphic at the cusp $i\infty$.

By (12.32), Lemma 12.4, and Lemma 12.7, the ξ-th Fourier coefficient vanishes unless $\xi \in F_0$ and $\xi \geq 0$ (i.e., totally semi-positive), and hence the Fourier expansion takes the form

$$\sum_{\xi \in F_0,\, \xi \geq 0} A_\xi\, q^\xi, \quad A_\xi \in \mathbb{C},$$

where $A_\xi = 0$ unless ξ lies in a (fractional) ideal of F_0 depending on Φ'_f and h_f. This shows that $\mathbb{J}(\cdot, \Phi') \in \mathcal{A}_{\mathrm{hol}}(\mathrm{SL}_2(\mathbb{A}_0), K(N), n)$. The assertion for Φ is proved similarly. $\square$

For the rest of this section, we assume that $F_0 = \mathbb{Q}$. Now let's fix a regular elliptic compact element $x'_0 \in S_n(F_0)$ (cf. §12.6), and fix $g_0 \in \mathrm{U}(V)(F_0)$ with the same characteristic polynomial as x'_0.

Let S be a finite set of non-archimedean places of F_0 such that

- S contains all places with residue characteristics 2,
- for all $v \in S$, Φ'_v partially (w.r.t. g_0 and x'_0) transfers to $\Phi_v \in \mathcal{S}(\mathrm{U}(V_v) \times V_v)(F_{0,v})$, [7] and
- for every non-archimedean $v \notin S$, the hermitian space V_v is split, $\Phi_v = \mathbf{1}_{(\mathrm{U}(V)\times V)(O_{F_0,v})}$ (w.r.t. to a self-dual lattice in V_v), and $\Phi'_v = \mathbf{1}_{(S_n \times V')(O_{F_0,v})}$.

Then in Proposition 13.3, we can assume that the compact open subgroup $K(N) \subset \mathrm{SL}_2(\mathbb{A}_{0,f})$ is a principle congruence subgroup of level N for some integer N whose prime factors are all contained in S.

We consider the difference

$$\mathscr{E}(h) = \mathbb{J}(h, \Phi') - \mathbb{J}(h, \Phi), \quad h \in \mathrm{SL}_2(\mathbb{A}_0).$$

By Proposition 13.3, we obtain a classical holomorphic modular form $\mathscr{E}^\flat \in \mathcal{A}_{\mathrm{hol}}(\Gamma(N), n)$, with its Fourier expansion (at the cusp ∞)

$$\mathscr{E}^\flat(\tau) = \sum_{\xi \geq 0,\, \xi \in N^{-1}\mathbb{Z}} A_\xi\, q^\xi, \quad q = e^{2\pi i \tau}, \tau \in \mathcal{H}.$$

Here, we refer to §1.2 (1.7) for the association $\mathscr{E} \mapsto \mathscr{E}^\flat$.

Theorem 13.4. *Assume that Conjecture 2.4 part* (a) *holds for all p-adic fields $\mathbb{Q}_p$ with $p \notin S$ and for S_n. Then $\mathscr{E}^\flat = 0$. (Note that we are in the case* $\dim V = n$.)

Proof. Let B (for "bad") be the (finite) set of non-archimedean places $v \notin S$ of F_0 where $O_{F,v}[g_0]$ is not a maximal order in $F_v[g_0]$.

By the vanishing criterion Lemma 13.6 below, it suffices to show that the Fourier coefficient

$$A_\xi = 0$$

whenever $(\xi, B) = 1$ (here $\xi \in N^{-1}\mathbb{Z}$).

[7] Transfers exist by the result of [48]. Here we only need the weaker result of the existence of partial transfers for fixed x'_0 and g_0, which can be deduced easily from the $n = 1$ case.

Now let ξ be coprime to B, in particular, $\xi \neq 0$. By (12.29), the ξ-th Fourier coefficient of $\mathbb{J}^\flat(\tau, \Phi')$ is

$$\sum_{u' \in V'(F_0)/T_0'(F_0), \mathfrak{q}(u')=\xi} \mathrm{Orb}((x_0', u'), \Phi'^{\infty}).$$

Similarly, by (11.8), the ξ-th Fourier coefficient of $\mathbb{J}^\flat(\tau, \Phi)$ is

$$\sum_{u \in V(F_0)/T_0(F_0),\, \mathfrak{q}(u)=\xi} \mathrm{Orb}((g_0, u), \Phi^{\infty}).$$

We now *claim* that the equality

$$\mathrm{Orb}((x_0', u'), \Phi_v') = \mathrm{Orb}((g_0, u), \Phi_v), \tag{13.1}$$

holds for every place v and every u' with $\mathfrak{q}(u) = \xi$. From the claim it follows that $A_\xi = 0$ whenever $(\xi, B) = 1$.

To show the claim, first let $v \in B$. Since $(\xi, B) = 1$, ξ is a unit at v. Therefore by Proposition 2.7 (ii), Conjecture 2.4 part (a) implies (13.1).

If $v \notin S \cup B$, then $O_{F,v}[g_0]$ is a maximal order in $F_v[g_0]$, and (13.1) follows from Proposition 2.6.

If $v \in S$, by our assumption on Φ_v' and Φ_v, they partially match (w.r.t. the fixed g_0, x_0), and hence (13.1) holds.

If v is archimedean, then (13.1) follows from our choices of partial Gaussian test functions. This proves the claim, and therefore completes the proof. □

Corollary 13.5. *Under the assumption of Theorem 13.4, we have for all $\xi \in F_0^\times$,*

$$\sum_{\xi' \in F'_{0,+},\, \mathrm{tr}_{F_0'/F_0} \xi' = \xi} \mathrm{Orb}((x_0', u'), \Phi') = \sum_{\xi' \in F'_{0,+},\, \mathrm{tr}_{F_0'/F_0} \xi' = \xi} \mathrm{Orb}((g_0, u), \Phi),$$

where for each ξ', on the left (resp., right) hand side, $u' \in V'(F_0)/T_0'(F_0)$ (resp., $u \in V(F)/T(F)$) is the unique orbit with the refined invariant $\mathfrak{q}'(u') = \xi'$ (resp., $\mathfrak{q}'(u) = \xi'$).

Proof. This follows from Theorem 13.4, the alternative expression of the ξ-th Fourier coefficient in terms of the refined invariants ξ', cf. (11.10) and (12.30), and the vanishing of archimedean local orbital integrals unless ξ' is totally positive by Lemma 12.4. □

13.3. **A lemma on Fourier coefficients of modular forms.** Let $f \in \mathcal{A}_{\mathrm{hol}}(\Gamma(N), k)$ be a holomorphic modular form of level $\Gamma(N)$ and weight $k \in \mathbb{Z}_{>0}$. It has a Fourier expansion (at the cusp ∞):

$$f(\tau) = \sum_{n \geq 0,\, n \in N^{-1}\mathbb{Z}} a_n(f) q^n, \quad q = e^{2\pi i \tau}, \tau \in \mathcal{H}.$$

Similarly, $f \in \mathcal{A}_{\mathrm{hol}}(\Gamma_1(N), k)$ has a Fourier expansion (at the cusp ∞):

$$f(\tau) = \sum_{n \geq 0,\, n \in \mathbb{Z}} a_n(f) q^n.$$

We have the following vanishing criterion.

Lemma 13.6. *(a) Let $f \in \mathcal{A}_{\mathrm{hol}}(\Gamma(N), k)$. Let $B \in \mathbb{N}$ be a positive integer coprime to N. If $a_n(f) = 0$ for all $n \in N^{-1}\mathbb{Z}$ such that $(n, B) = 1$ (i.e., $(nN, B) = 1$), then $f = 0$.*

(b) Let $f \in \mathcal{A}_{\mathrm{hol}}(\Gamma_1(N), k)$. Let $B \in \mathbb{N}$ be a positive integer coprime to N. If $a_n(f) = 0$ for all $n \in \mathbb{Z}$ such that $(n, B) = 1$, then $f = 0$.

Proof. This is essentially a part of the newform theory. For reader's convenience we will give a proof.

Note

$$\Gamma(N) \simeq \Gamma_1(N) \cap \Gamma_0(N^2) \supset \Gamma_1(N^2).$$

We have an injective linear map

$$\begin{aligned}\iota_N : \mathcal{A}_{\mathrm{hol}}(\Gamma(N), k) &\longrightarrow \mathcal{A}_{\mathrm{hol}}(\Gamma_1(N^2), k)\\ f &\longmapsto \mathrm{diag}(N,1)f(\tau) :\, = f(N\tau).\end{aligned}$$

Under the map $\iota_N(f)$ has a Fourier expansion (at the cusp ∞):

$$\iota_N f(\tau) = \sum_{n\geq 0,\, n\in N^{-1}\mathbb{Z}} a_n(f) q^{nN}.$$

Therefore part (b) implies part (a).

To prove part (b), now let $\Gamma = \Gamma_1(N)$. We first consider the special case $B = p$ where $p \nmid N$ is a prime. Recall that "Ihara's lemma" over $\mathbb{C}$ (this can be proved directly) asserts that the "level-raising" map

$$\begin{aligned}\mathrm{LR}_p : \mathcal{A}_{\mathrm{hol}}(\Gamma, k)^{\oplus 2} &\longrightarrow \mathcal{A}_{\mathrm{hol}}(\Gamma \cap \Gamma_0(p), k)\\ (f_1, f_2) &\longmapsto f_1 + \iota_p(f_2),\end{aligned}$$

is *injective*, where $\iota_p(f_2)$ is given by $\mathrm{diag}(p,1)f_2(\tau) = f_2(p\tau)$. Now let $f \in \mathcal{A}_{\mathrm{hol}}(\Gamma, k)$. Then $g(\tau) := f(\tau/p)$ is an element in $\Gamma\cap\Gamma^0(p)$ where $\Gamma^0(p)$ is the transpose of $\Gamma_0(p)$. If $a_n(f) = 0$ for all $n \in \mathbb{Z}$ such that $p \nmid N$, then g is invariant under $\tau \mapsto \tau+1$. Since $\Gamma\cap\Gamma^0(p)$ and $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ generate Γ, the function $g(\tau) = f(\tau/p)$ lies in $\mathcal{A}_{\mathrm{hol}}(\Gamma, k)$. Now we check $\mathrm{LR}_p(-f, g) = -f + \iota_p(g) = 0$. It follows that $f = 0$ by the injectivity of the map LR_p.

In general, we may assume that $B = pB'$ is square-free with a prime factor p. Consider the map sending $f = \sum_{n\geq 0,\, n\in\mathbb{Z}} a_n(f)q^n \in \mathcal{A}_{\mathrm{hol}}(\Gamma_1(N), k)$ to the

$$\widetilde{f} := \sum_{n\geq 0,\, n\in\mathbb{Z}, (n,B')=1} a_n(f) q^n.$$

Then $\widetilde{f} \in \mathcal{A}_{\mathrm{hol}}(\Gamma_1(NB'^2), k)$. Now apply the special case above to conclude that $\widetilde{f} = 0$. By induction on the number of prime factors of B, we complete the proof.

□

13.4. **A refinement of Corollary 13.5.** We denote by $\mathcal{B}$ the common categorical quotient (cf. [48] for the analogous case $\mathrm{U}(V^\sharp)_{/\!/\mathrm{U}(V)} \simeq S_{n+1/\!/\mathrm{GL}_n}$)

$$\mathcal{B} :\, = (\mathrm{U}(V)\times V)_{/\!/\mathrm{U}(V)} \simeq (S_n \times V_n')_{/\!/\mathrm{GL}_n}.$$

This is an affine variety over F_0.

Lemma 13.7. *Fix $b_0 \in \mathcal{B}(F_0)$. Fix a non-archimedean place v_1 of F_0, split in F. For every place $v \neq v_1$ of F_0, we fix a compact neighborhood $\Omega_v \subset \mathcal{B}(F_{0,v})$ of b_0, such that, for all but finitely many non-archimedean places v, Ω_v is the image of $(S_n \times V_n')(O_{F_v})$. Then there exists a neighborhood $\Omega_{v_1} \subset \mathcal{B}(F_{0,v_1})$ of b_0 such that*

$$\mathcal{B}(F_0) \cap \prod_v \Omega_v = \{b_0\},$$

where the intersection is taken inside $\mathcal{B}(\mathbb{A}_0)$.

Proof. Embed $\mathcal{B}$ as a closed sub-variety of some affine space $Y = \mathbf{A}^m$ over F_0. By the compactness of Ω_v for $v \neq v_1$, we may choose compact subset $\widetilde{\Omega}_v \subset Y(F_{0,v})$ such that $\Omega_v = \widetilde{\Omega}_v \cap \mathcal{B}(F_{0,v})$. Then by the product formula, there must be a small neighborhood $\widetilde{\Omega}_{v_1} \subset Y(F_{0,v_1})$ such that

$$Y(F_0) \cap \prod_v \widetilde{\Omega}_v = \{b_0\}.$$

Set $\Omega_{v_1} = \widetilde{\Omega}_{v_1} \cap \mathcal{B}(F_{0,v_1})$ to complete the proof.

□

We are now ready to refine the result of Corollary 13.5.

Proposition 13.8. *Under the assumption of Theorem 13.4, we have, for all* $(g_0, u) \in (\mathrm{U}(V) \times V)(F_0)_{\mathrm{rs}}$ *matching* $(x_0', u') \in (S_n \times V_n')(F_0)_{\mathrm{rs}}$ *such that* $\xi = \mathfrak{q}(u) \neq 0$,

$$\mathrm{Orb}((x_0', u'), \Phi') = \mathrm{Orb}((g_0, u), \Phi).$$

Proof. Let $b_0 \in \mathcal{B}(F_0)$ be the (common) image of (g_0, u) and (x_0', u'). Fix a non-archimedean place v_1 of F_0, split in F. Decompose

$$\mathrm{Orb}((g_0, u), \Phi) = \mathrm{Orb}((g_0, u), \Phi^{v_1})\, \mathrm{Orb}((g_0, u), \Phi_{v_1}),$$

and

$$\mathrm{Orb}((x_0', u'), \Phi') = \mathrm{Orb}((x_0', u'), {\Phi'}^{v_1})\, \mathrm{Orb}((x_0', u'), \Phi'_{v_1}),$$

where the local orbital integral $\mathrm{Orb}((g_0, u), \Phi_{v_1}) = \mathrm{Orb}((x_0', u'), \Phi'_{v_1})$. We may assume that the local orbital integrals at v_1 are nonzero (otherwise both sides vanish). It remains to show

$$\mathrm{Orb}((g_0, u), \Phi^{v_1}) = \mathrm{Orb}((x_0', u'), {\Phi'}^{v_1}). \tag{13.2}$$

For every non-archimedean $v \neq v_1$, we define a compact set $\Omega_v \subset \mathcal{B}(F_{0,v})$ to be the union of the image of the support of Φ_v and Φ'_v.

Now let $v \mid \infty$ (only one such v since we are assuming $F_0 = \mathbb{Q}$). Let $\mathcal{A}_{g_0,\xi;+}$ be the set of $(g_0, \widetilde{u}) \in (\mathrm{U}(V) \times V)(F_{0,v})$ (for the fixed g_0) such that $\mathfrak{q}'(\widetilde{u}) = \xi'$ is *totally positive* and $\mathrm{tr}_{F_0'/F_0}(\xi') = \xi$. Here $\xi = \mathfrak{q}(u)$ is fixed.

We claim that $\mathcal{A}_{g_0,\xi;+}$ is contained in a compact set. To show the claim, since g_0 is fixed, it suffices to show that the refined invariant ξ' lies in a bounded subset of $F_0' \otimes_{F_0,v} \mathbb{R} \simeq \mathbb{R}^n$. By the totally positivity of $\xi' \in F_0'$, we have

$$\sum_{v'|v} |\xi'|_{v'} = \mathrm{tr}_{F_0'/F_0}(\xi') = \xi,$$

and hence ξ' is bounded. This proves the claim.

It follows that we can choose a compact $\Omega_v \subset \mathcal{B}(F_{0,v})$ such that Ω_v contains the image of the set $\mathcal{A}_{g_0,\xi;+}$.

Now apply Lemma 13.7 to choose a small neighborhood $\Omega_{v_1} \subset \mathcal{B}(F_{0,v_1})$ of b_0 such that $\mathcal{B}(F_0) \cap \prod_v \Omega_v = \{b_0\}$. Then we choose a *point-wise non-negative* function Φ_{v_1} with non-empty support whose image in $\mathcal{B}(F_{0,v_1})$ is contained in Ω_{v_1}. Choose Φ'_{v_1} to match Φ_{v_1} in the elementary way (cf. Remark 13.2). Now apply Corollary 13.5, where the sum in each side has only one term left, namely the one with invariant $b_0 \in \mathcal{B}(F_0)$. By the point-wise positivity of Φ_{v_1}, the local orbital integral at the place v_1 does not vanish. We hence deduce the desired equality (13.2). This completes the proof. ☐

13.5. **The proof of FL conjecture.** Now we return to the set up of Conjecture 2.4 in §2.4.

Theorem 13.9. *Conjecture 2.4 holds when* $F_0 = \mathbb{Q}_p$ *and* $p \geq n$.

Proof. By Proposition 2.7, it suffices to prove Conjecture 2.4 part (c). We will do so by induction on $\dim V_0$.

The case $\dim V_0 = 1$ is trivial. Assume now that Conjecture 2.4 part (c) holds when $\dim V_0 = n-1$. Then by Proposition 2.7 part (i), Conjecture 2.4 part (a) holds for S_n, and $\mathbb{Q}_p$ with $p \geq n$.

We now want to apply Proposition 13.8. We start with the following local data

- a place v_0 of $F_0 = \mathbb{Q}$, and an unramified (local) quadratic extension $F_{w_0}/F_{0,v_0}$,
- the split $F_{w_0}/F_{0,v_0}$-hermitian space V_{v_0} of dimension n,
- an element $(g_{v_0}, u_{v_0}) \in (\mathrm{U}(V_{v_0}) \times V_{v_0})(F_{0,v_0})_{\mathrm{srs}}$, we further assume that the characteristic polynomial of g_{v_0} has integral coefficients (in $O_{F_{w_0}}$), $\det(1 - g_{v_0})$ is a unit, and $\langle u_{v_0}, u_{v_0} \rangle \neq 0$,
- an element $(x'_{v_0}, u'_{v_0}) \in (S_n \times V_n')(F_{0,v_0})_{\mathrm{srs}}$ matching (g_{v_0}, u_{v_0}).

To globalize the data, we first use Cayley transform, cf. (4.1). Let $x_{v_0} = \mathfrak{c}^{-1}(g_{v_0}) = \frac{1+g_{v_0}}{1-g_{v_0}}$, an element in the Lie algebra $\mathfrak{u}(V_{v_0}) \subset \mathrm{End}_{F_{w_0}}(V_{v_0})$.

We now choose an imaginary quadratic field $F = F_0[\sqrt{\epsilon}], \epsilon \in F_0^\times$ such that $F \otimes_{F_0} F_{0,v_0} \simeq F_{w_0}$. Consider $x^\natural_{v_0} = \frac{x_{v_0}}{\sqrt{\epsilon}}$. Then the characteristic polynomial of $x^\natural_{v_0}$ has coefficients in the base field F_{v_0}.

Next we choose a totally real field F_0' and an element $x^\natural \in F_0'$ such that, when setting $F' = F_0'[\sqrt{\epsilon}]$ and

$$x = \sqrt{\epsilon}\, x^\natural, \quad g = \mathfrak{c}(x) = -\frac{1-x}{1+x},$$

we have $O_{F_{w_0}}[g] = O_{F_{w_0}}[g_{v_0}]$ as subrings of $F' \otimes_F F_{w_0}$. To achieve this, it suffices to approximate the characteristic polynomial of $x^\natural_{v_0}$ by a polynomial with coefficient in $F_0 = \mathbb{Q}$, and we may prescribe its local behavior at finitely many places by weak approximation. (Here the regular semi-simplicity allows us to determines the isomorphism class of the local field $F[g_{v_0}]$ by the characteristic polynomial of g_{v_0}.)

With such a choice, we have $g \in F'^1$. For the CM extension F'/F_0', there exists a one-dimensional F'/F_0'-hermitian space W such that W is totally positive definite, and $V = \mathrm{Res}_{F'/F} W$, as an n-dimensional F/F_0-hermitian space, is locally at v_0 isometric to V_{v_0}. Such hermitian space W exists because we are only imposing local conditions at finitely many places.

Now we choose $u \in V$ (and possibly replacing g by an element v_0-adically closer to g_{v_0}) such that the pair (g, u) is v_0-adically close to (g_{v_0}, u_{v_0}) and such that

$$\mathrm{Orb}((g,u), \mathbf{1}_{(\mathrm{U}(V_{v_0})\times V_{v_0})(O_{F_0,v_0})}) = \mathrm{Orb}((g_{v_0},u_{v_0}), \mathbf{1}_{(\mathrm{U}(V_{v_0})\times V_{v_0})(O_{F_0,v_0})}). \tag{13.3}$$

This is possible due to the local constancy of orbital integrals near a regular semisimple element. Let $(x_0', u') \in (S_n \times V_n')(F_0)$ be a regular semisimple element matching (g, u). Again by local constancy of orbital integrals we may assume that, possibly replacing (g, u) by an element in $(\mathrm{U}(V) \times V)(F_0)_{\mathrm{srs}}$ that is v_0-adically closer to (g_{v_0}, u_{v_0}),

$$\mathrm{Orb}((x_0', u'), \mathbf{1}_{(S_n\times V_n')(O_{F_0,v_0})}) = \mathrm{Orb}((x_{v_0}', u_{v_0}'), \mathbf{1}_{(S_n\times V_n')(O_{F_0,v_0})}). \tag{13.4}$$

Next, we let S be a finite set of of non-archimedean places of F_0, such that

- $v_0 \notin S$,
- S contains all places with residue characteristic $p < n$,
- for every non-archimedean $v \notin S \cup \{v_0\}$, the order $O_F[g]$ is locally maximal at v, and V_v is a split hermitian space.

Choose functions $\Phi = \otimes_v \Phi_v$ and $\Phi' = \otimes_v \Phi_v'$ satisfying the following conditions:

- for every archimedean v, Φ_v and Φ_v' are the (partial) Gaussian test functions (relative to a small neighborhood of x_0' in $S_n(F_{0,v})$);
- for every non-archimedean $v \in S$, Φ_v' partially (w.r.t. g and x_0') transfers to $\Phi_v \in \mathcal{S}(\mathrm{U}(V_v) \times V_v)(F_{0,v})$ and the local orbital integrals do not vanish at (g, u) and (x_0', u'), cf. Definition 13.1;
- for every non-archimedean $v \notin S$ (in particular at v_0), noting that the hermitian space V_v is split, choose $\Phi_v = \mathbf{1}_{(\mathrm{U}(V)\times V)(O_{F_0,v})}$ (w.r.t. to a self-dual lattice in V_v), and $\Phi_v' = \mathbf{1}_{(S_n\times V')(O_{F_0,v})}$. By enlarging S suitably (while $v_0 \notin S$), we may further assume that, for every $v \notin S$, the image of (g, u) in $\mathcal{B}(F_0)$ lies in the image of the support of $(\mathrm{U}(V) \times V)(O_{F_0,v})$.

By the last condition, for every non-archimedean $v \notin S$ the local orbital integral of Φ_v does not vanish at (g, u) (since the function Φ_v is point-wisely positive on its support). It follows from the special case Proposition 2.6 that the same non-vanishing for Φ_v' for every place $v \notin S \cup \{v_0\}$.

Now we are ready to apply Proposition 13.8 to conclude

$$\mathrm{Orb}((x_0', u'), \Phi') = \mathrm{Orb}((g_0, u), \Phi).$$

By our choices, $\mathrm{Orb}((x_0', u'), \Phi_v') = \mathrm{Orb}((g_0, u), \Phi_v)$ for all $v \neq v_0$, and they do not vanish. It follows that

$$\mathrm{Orb}((x_0', u'), \Phi_{v_0}') = \mathrm{Orb}((g_0, u), \Phi_{v_0}).$$

By (13.3) and (13.4), we have

$$\mathrm{Orb}((x_{v_0}', u_{v_0}'), \mathbf{1}_{(S_n\times V_n')(O_{F_0,v_0})}) = \mathrm{Orb}((g_{v_0},u_{v_0}), \mathbf{1}_{(\mathrm{U}(V_{v_0})\times V_{v_0})(O_{F_0,v_0})}).$$

We have assumed that $g_{v_0} \in \mathrm{U}(V_{v_0})$ is regular semisimple with $\det(1 - g_{v_0}) \in O_{F_{w_0}}^\times$ and $\langle u_{v_0}, u_{v_0}\rangle \neq 0$. We now remove these assumptions. The condition $\det(1 - g_{v_0}) \in O_{F_{w_0}}^\times$ is harmless since we may multiply g_{v_0} by a suitable element in $F_{w_0}^1$ (cf. the proof of Prop. 4.12).

The set of elements $(g_{v_0}, u_{v_0}) \in (\mathrm{U}(V_{v_0}) \times V_{v_0})_{\mathrm{srs}}$ with $\langle u_{v_0}, u_{v_0}\rangle \neq 0$ is dense in $(\mathrm{U}(V_{v_0}) \times V_{v_0})_{\mathrm{rs}}$. By local constancy of orbital integrals at regular semisimple elements, Conjecture 2.4 part (c) holds when $\dim V_0 = n$ (over $\mathbb{Q}_p$ with $p \geq n$). This completes the induction. $\square$

14. The comparison for arithmetic intersections

As a preparation for the proof of AFL conjecture, in this section, we compare $\partial\mathbb{J}(h, \Phi')$ with the arithmetic intersection number $\mathrm{Int}(\tau, \Phi)$ (cf. (9.5) in §9).

Let V be the n-dimensional F/F_0-hermitian space that we use to define the Shimura variety $\mathrm{Sh}_{K_{\widetilde{G}}}\big(\widetilde{G}, \{h_{\widetilde{G}}\}\big)$ in §6.1.1. Let $\Phi = \otimes_{v<\infty}\Phi_v \in \mathcal{S}((\mathrm{U}(V) \times V)(\mathbb{A}_{0,f}))$ be a pure tensor. Let $\Phi' = \otimes_v \Phi'_v \in \mathcal{S}((S_n \times V'_n)(\mathbb{A}_0))$ be a pure tensor such that

- for every $v \mid \infty$, Φ'_v is the partial Gaussian test function, cf. 12.6, and
- for every non-archimedean v, Φ'_v transfers to Φ_v (and the zero function on the other isometric class of hermitian space at v, cf. Definition 2.2).

Remark 14.1. Due to the signature of such V at the archimedean places, there does not exist any *global* F/F_0-hermitian space $\widetilde{V}$ such that Φ' transfer to a function in $\mathcal{S}((\mathrm{U}(\widetilde{V}) \times \widetilde{V})(\mathbb{A}_0))$, cf. [47, §3.2].

Fix a regular elliptic compact element $x'_0 \in S_n(F_0)$ (cf. §12.6), and fix $g_0 \in \mathrm{U}(V)(F_0)$ with the same characteristic polynomial as x'_0.

Lemma 14.2. *Let v be a place of F_0 split in F (necessarily non-archimedean). Then $\partial\mathbb{J}_v(h, \Phi') = 0$ (cf. (12.34)).*

Proof. This follows from the same argument in [47, Prop. 3.6] (also cf. [40, §7.2]). $\square$

If v is non-split (including the archimedean places), let $V(v)$ be the "nearby" F/F_0-hermitian space at v, cf. Theorem 9.4 (resp., Theorem 10.1) for non-archimedean (resp., archimedean) places (the hermitian space V' there). Then the term $\partial\mathbb{J}_v(h, \Phi')$ (cf. (12.34)) is a sum over nonzero $u' \in V'(F_0)/T'_0(F_0)$ such that (x'_0, u') transfers to $(g_0, u) \in \mathrm{U}(V(v)) \times V(v)$. Moreover, we have a Fourier expansion (cf. (11.9))

$$\partial\mathbb{J}_v(h, \Phi') = \sum_{\xi \in F_0} \partial\mathbb{J}_v(h, \xi, \Phi'), \tag{14.1}$$

where $\partial\mathbb{J}_v(h, \xi, \Phi')$ is the sub-sum over terms with $\mathfrak{q}(u') = \xi$,

$$\partial\mathbb{J}_v(h, \xi, \Phi') = \sum_{u' \in V'(F_0)_\xi/T'_0(F_0), u' \neq 0} \partial\mathrm{Orb}((x'_0, u'), \omega(h_v)\Phi'_v) \cdot \mathrm{Orb}((x'_0, u'), \omega(h^v)\Phi'^{v}). \tag{14.2}$$

14.1. **The archimedean places.** Let $v \mid \infty$. Recall that $V' = V_0 \times V_0^*$ for $V_0 = F_0^n$ carries the tautological quadratic form (11.14)

$$\mathfrak{q}\colon\ V_0 \times V_0^* \longrightarrow F_0 ,$$

and an induced quadratic form (11.16)

$$\mathfrak{q}'\colon\ V_0 \times V_0^* \longrightarrow F'_0 ,$$

such that for all $u' \in V'$, $\mathfrak{q}(u') = \mathrm{tr}_{F'_0/F_0}\, \mathfrak{q}'(u')$. Set

$$F'_{0,v} := F'_0 \otimes_{F_0,v} \mathbb{R} \simeq \prod_{v' \in \mathrm{Hom}(F'_0, \mathbb{R}), v' \mid v} \mathbb{R}.$$

Lemma 14.3. *Let $\xi' \in F'_{0,v}$ be an invertible element, and $u' \in V'(F_{0,v})$ with $\mathfrak{q}'(u') = \xi'$.*

(a)

$$\mathrm{Orb}((x'_0, u'), \Phi'_v) = \begin{cases} e^{-\pi\, \mathrm{tr}_{F'_{0,v}/F_{0,v}}(\xi')}, & \text{when } \xi' \in F'_{0,v} \text{ is totally positive,} \\ 0, & \text{otherwise.} \end{cases}$$

(b) Now assume that ξ' is not totally positive. Then $\partial\mathrm{Orb}((x_0', u'), \Phi_v') = 0$, unless ξ' is negative at exactly one place above v, say at v_0', in which case

$$\partial\mathrm{Orb}((x_0', u'), \Phi_v') = \frac{1}{2} e^{-\pi \operatorname{tr}_{F'_{0,v}/F_{0,v}}(\xi')} \operatorname{Ei}(-2\pi|\xi'|_{v_0'}).$$

Proof. This follows from Lemma 12.4, and Lemma 12.7. □

Lemma 14.4. *Let $\xi \in F_0^\times$. Then*

$$\mathrm{Int}_v^{\mathbf{K}}(\xi, \Phi) = -2\partial\mathbb{J}_v(\xi, \Phi').$$

Proof. It follows from the previous Lemma 14.3 that

$$\partial\mathbb{J}_v(\xi, \Phi') = \frac{1}{2} \sum \operatorname{Ei}(-2\pi|\xi'|_{v_0'}) \cdot \mathrm{Orb}\left((x_0', u'), \Phi'\right),$$

where (x_0', u') is the unique orbit with the refined invariant $\mathfrak{q}'(u') = \xi' \in F_0'$, and the sum runs over all $\xi' \in F_0'$ such that

- $\operatorname{tr}_{F_0'/F_0}(\xi') = \xi$, and
- there exists exactly one archimedean place v_0' of F_0' where ξ' is negative, and this place v_0' is above v_0.

Here $\mathrm{Orb}\left((x_0', u'), \Phi'\right)$ depends only on the refined invariant $\langle u', u'\rangle_{F_0'} = \xi'$.

Now the assertion follows from Corollary 10.2, and the fact that Φ^∞ and Φ'^∞ are transfers of each other.

□

For the rest of this section, we assume that $F_0 = \mathbb{Q}$. Recall that the difference $\mathrm{Int}^{\mathbf{K-B}}(h, \Phi)$ between the two Green functions is given by (10.9) and (10.10) (note that this makes sense for any Schwartz function Φ^∞). The following result plays the role of "holomorphic projection" of modular generating function on the analytic side.

Proposition 14.5. *Let $F_0 = \mathbb{Q}$. The sum*

$$\partial\mathbb{J}_{\mathrm{hol}}(h) := 2\partial\mathbb{J}(h, \Phi') + \mathrm{Int}^{\mathbf{K-B}}(h, \Phi), \quad h \in \mathrm{SL}_2(\mathbb{A}_0),$$

lies in $\mathcal{A}_{\mathrm{hol}}(\mathrm{SL}_2(\mathbb{A}_0), K, n)$, where K is the compact open subgroup of $\mathrm{SL}_2(\mathbb{A}_{0,f})$ that acts trivially on both Φ and Φ'.

Proof. First of all the function $h \in \mathrm{SL}_2(\mathbb{A}_0) \mapsto 2\partial\mathbb{J}(h, \Phi')$ belongs to $\mathcal{A}_{\exp}(\mathrm{SL}_2(\mathbb{A}_0), K, n)$. One way to see this is to use the Fourier expansion directly. Another way is to identify it with a linear combination to $\mathrm{SL}_2(\mathbb{A}_{F_0})$ of the restriction of (the first derivative at $s = 0$ of) a degenerate Siegel–Eisenstein series of parallel weight one on $\mathrm{SL}_2(\mathbb{A}_{F_0'})$, cf. Remark 12.11.

By Corollary 10.3, the second summand $\mathrm{Int}^{\mathbf{K-B}}(\cdot, \Phi)$ also belongs to $\mathcal{A}_{\exp}(\mathrm{SL}_2(\mathbb{A}_0), K, n)$. Therefore to complete the proof, it suffices to show the holomorphy of the sum $\partial\mathbb{J}_{\mathrm{hol}}(h)$ on the complex upper half plane $\mathcal{H}$ and at all cusps. Equivalently, for any $h_f \in \mathrm{SL}_2(\mathbb{A}_{0,f})$, the function $\partial\mathbb{J}^\flat_{\mathrm{hol},h_f}$ (associated to $\partial\mathbb{J}_{\mathrm{hol}}$ via (1.7)[8]) is holomorphic, and holomorphic at the cusp $i\infty$. Since we can vary Φ^∞ and Φ'^∞, and by Theorem A.1 the Weil representation commutes with smooth transfer, it suffices to consider the case $h_f = 1$ (but allow all matching Φ^∞ and Φ'^∞).

We claim that the Fourier expansion of $\partial\mathbb{J}^\flat_{\mathrm{hol}}$ takes the form

$$\sum_{\xi \in F_0, \xi \geq 0} A_\xi \, q^\xi, \quad A_\xi \in \mathbb{C}, \tag{14.3}$$

where $A_\xi = 0$ unless ξ lies in a (fractional) ideal of F_0 (depending on Φ', Φ). In other words, the non-holomorphic terms all cancel out. The desired holomorphy follows from the claim.

To show the claim, we use the decomposition (12.35) as a sum over places v of F_0. By Lemma 14.2, $\partial\mathbb{J}_v(h, \Phi') = 0$ if v is a split place.

[8] In (1.7) the requirement for ϕ to be an automorphic form is not necessary to make the definition of $\phi \mapsto \phi^\flat$.

Now let v be a non-archimedean non-split place. By (14.1), (14.2), and the fact that Φ'_∞ is a (partial) Gaussian test function, we obtain

$$\partial\mathbb{J}_v^\flat(\tau,\Phi')=\sum_{\xi\in F_0}\partial\mathbb{J}_v(\xi,\Phi')q^\xi, \tag{14.4}$$

where

$$\partial\mathbb{J}_v(\xi,\Phi')=\sum_{u'\in V'(F_0)_\xi/T'_0(F_0),u'\neq 0}\partial\mathrm{Orb}((x'_0,u'),\omega(h_v)\Phi'_v)\cdot\mathrm{Orb}((x'_0,u'),\omega(h^v)\Phi'^{\infty\cup\{v\}}). \tag{14.5}$$

Since v is non-archimedean, the hermitian space $V(v)$ is totally positive definite. Since in the sum u' is such that (x'_0,u') transfers to $(g_0,u)\in\mathrm{U}(V(v))\times V(v)$, $\partial\mathbb{J}_v(\xi,\Phi')=0$ unless ξ is totally positive. It follows that $2\partial\mathbb{J}_v(h,\Phi')$ has the desired form of Fourier expansion as in (14.3).

Finally let $v\mid\infty$. We observe that by Lemma 14.4, modulo the constant terms, the sum $2\partial\mathbb{J}_v(h,\Phi')+\mathrm{Int}_v^{\mathbf{K-B}}(h,\Phi)$ is equal to $-\mathrm{Int}_v^{\mathbf{B}}(h,\Phi)$, which has the desired form of Fourier expansion. Note that Lemma 14.3 also applies to all $u'\in V'$ with refined invariant $\mathfrak{q}'(u')=\xi'\in F_0'^\times$. Similarly Theorem 10.1 also applies to all $u\in V(v)$ with refined invariant $\mathfrak{q}'(u)=\xi'\in F_0'^\times$ (i.e., $u\neq 0$). Therefore, by the proof of Lemma 14.4, the contribution from null-norm non-zero vectors $u\in V(v)$ cancels that from $u'\in V'$ with $\mathfrak{q}'(u')=\xi'\neq 0\in F'_0$. It follows that the constant term of $2\partial\mathbb{J}_v(h,\Phi')+\mathrm{Int}_v^{\mathbf{K-B}}(h,\Phi)$ is the sum of the nilpotent term $2\,\partial\mathrm{Orb}(0_\pm,\omega(h)\Phi')$ (from $2\partial\mathbb{J}(h,\Phi')$, cf. (12.36)), and the only term that has not been cancelled in (10.9), which is by (8.12)

$$-\mathrm{Orb}((g_0,0),\Phi)\log|a|_v.$$

By Lemma 12.12 and Lemma 12.7, the nilpotent term is

$$2\,\partial\mathrm{Orb}(0_\pm,\Phi')=2\,\mathrm{Orb}((x'_0,0_+),\Phi')\,\log|a|_v+C$$

for some constant C (depending on x'_0,Φ'). Since Φ match Φ', we claim

$$\mathrm{Orb}((x'_0,0_+),\Phi')=-\mathrm{Orb}((x'_0,0_-),\Phi')=\frac{1}{2}\,\mathrm{Orb}((g_0,0),\Phi).$$

In fact, this follows from the argument in [22, (10.4)]. In *loc. cit.*, Jacquet proves the analogous identity in the "coherent" case (here "coherent" is in the sense of Kudla for one-dimensional hermitian spaces). Since the proof verbatim applies to the current setting, we omit the detail.

Therefore the two terms with $\log|a|_v$ cancel, and the sum is a constant independent of a. This shows that $2\partial\mathbb{J}_v(h,\Phi')+\mathrm{Int}_v^{\mathbf{K-B}}(h,\Phi)$ also has the desired form of Fourier expansion when $v\mid\infty$.

This completes the proof.

$\square$

14.2. **The comparison.** Now let $\mathcal{M}=\mathcal{M}_{K_{\widetilde{G}}}(\widetilde{G})$ be the moduli stack introduced in Definition 6.3. Let S be a finite set of non-archimedean places of F_0 such that

- S contains all places $v\mid\mathfrak{d}$, and
- for every non-archimedean $v\notin S$, the hermitian space V_v is split, $\Phi_v=\mathbf{1}_{(\mathrm{U}(V)\times V)(O_{F_0,v})}$ (w.r.t. to a self-dual lattice in V_v), and $\Phi'_v=\mathbf{1}_{(S_n\times V'_n)(O_{F_0,v})}$.

Now we have the FL for all places $v\notin S$ by Theorem 13.9, hence Φ_v and Φ'_v match for every place. Then in Proposition 14.5, we can assume that the compact open subgroup $K\subset\mathrm{SL}_2(\mathbb{A}_{0,f})$ is a principle congruence subgroup $K(N)$ of level N, where the prime factors of N are all contained in S.

We have been assuming that the function $\Phi\in\mathcal{S}((\mathrm{U}(V)\times V)(\mathbb{A}_{0,f}))$ is valued in $\mathbb{Q}$. By Proposition 14.5, $\partial\mathbb{J}^\flat_{\mathrm{hol}}(\cdot,\Phi')$ lies in $\mathcal{A}_{\mathrm{hol}}(\Gamma(N),n)_\mathbb{Q}\otimes_\mathbb{Q}\mathbb{R}$ (the Green's function takes values in $\mathbb{R}$). By passing to the quotient $\mathbb{R}\to\mathbb{R}_S$ (cf. (9.2)), we obtain an element, still denoted by $\partial\mathbb{J}^\flat_{\mathrm{hol}}(\cdot,\Phi')$, in $\mathcal{A}_{\mathrm{hol}}(\Gamma(N),n)_\mathbb{Q}\otimes_\mathbb{Q}\mathbb{R}_S$.

By Theorem 8.5 (cf. (9.7)), $\mathrm{Int}(\cdot,\Phi)$ also belongs to $\mathcal{A}_{\mathrm{hol}}(\Gamma(N),n)_\mathbb{Q}\otimes_\mathbb{Q}\mathbb{R}_S$, hence so does the sum

$$\mathscr{E}^\flat(\tau)=\partial\mathbb{J}^\flat_{\mathrm{hol}}(\tau,\Phi')+\mathrm{Int}(\tau,\Phi),\quad \tau\in\mathcal{H}.$$

Write the Fourier expansion (at the cusp ∞) as

$$\mathscr{E}^\flat(\tau) = \sum_{\xi \geq 0,\, \xi \in N^{-1}\mathbb{Z}} A_\xi\, q^\xi, \quad A_\xi \in \mathbb{R}_S.$$

Theorem 14.6. *Assume that Conjecture 3.8 part* (a) *holds for all p-adic field $\mathbb{Q}_p$ with $p \notin S$ and for S_n. Then*
(a) $\mathscr{E}^\flat = 0$.
(b) For every non-archimedean place $v \notin S$, and $\xi \in F_0^\times$, we have

$$-2\partial\mathbb{J}_v(\xi, \Phi') = \mathrm{Int}_v(\xi, \Phi).$$

Here $\partial\mathbb{J}_v(\xi, \Phi')$ is defined by (14.5).

Proof. The proof of part (a) is analogous to that of Theorem 13.4. Recall from (9.15) and (9.16) that we have a decomposition of of the generating function $\mathrm{Int}(\tau, \Phi)$. We have the following equalities, as formal power series in $\mathbb{R}_S[\![q^{1/N}]\!]$:

$$\begin{aligned}
\mathcal{E}^\flat(\tau) - \mathrm{Int}(0,\Phi) =&(2\partial\mathbb{J}^\flat(\tau,\Phi') + \mathrm{Int}^{\mathbf{K}-\mathbf{B}}(\tau,\Phi)) + (\mathrm{Int}(\tau,\Phi) - \mathrm{Int}(0,\Phi))\\
=&\sum_{v|\infty}\Big(2\partial\mathbb{J}^\flat_v(\tau,\Phi') + \mathrm{Int}^{\mathbf{K}-\mathbf{B}}_v(\tau,\Phi) + \mathrm{Int}_v(\tau,\Phi)\Big)\\
&+\sum_{v<\infty}\Big(2\partial\mathbb{J}^\flat_v(\tau,\Phi') + \mathrm{Int}_v(\tau,\Phi)\Big)\\
=&\sum_{v|\infty}\Big(2\partial\mathbb{J}^\flat_v(\tau,\Phi') + \mathrm{Int}^{\mathbf{K}}_v(\tau,\Phi)\Big) + \sum_{v<\infty}\Big(2\partial\mathbb{J}^\flat_v(\tau,\Phi') + \mathrm{Int}_v(\tau,\Phi)\Big)\\
=&\sum_{v<\infty,\, v\notin S}\Big(2\partial\mathbb{J}^\flat_v(\tau,\Phi') + \mathrm{Int}_v(\tau,\Phi)\Big),
\end{aligned}$$

where the last equality follows from Lemma 14.4. Here the sum $2\partial\mathbb{J}^\flat_v(\tau,\Phi') + \mathrm{Int}^{\mathbf{K}-\mathbf{B}}_v(\tau,\Phi)$ (resp., $2\partial\mathbb{J}^\flat_v(\tau,\Phi') + \mathrm{Int}^{\mathbf{K}}_v(\tau,\Phi)$) both belong to $\mathbb{R}_S[\![q^{1/N}]\!]$, even though each summand does not due to the presence of "non-holomorphic" terms.

Let B be the (finite) set of non-archimedean inert places $v \notin S$ of F_0 where $R_v \colon = O_{F,v}[g_0]$ is not a maximal order in $F_v[g_0] = F'_v$. By the vanishing criterion Lemma 13.6, to show $\mathcal{E}^\flat = 0$, it remains to show the vanishing of the ξ-th Fourier coefficients when $(\xi, B) = 1$.

If $v \notin S$ is split in F, the intersection number $\mathrm{Int}_v(\tau,\Phi) = 0$ vanishes by Corollary 9.3, and $2\partial\mathbb{J}^\flat_v(\tau,\Phi') = 0$ by Lemma 14.2.

If $v \notin S$ is inert, then by Theorem 9.4 and (9.16) we obtain the q-expansion of $\mathrm{Int}_v(\tau,\Phi)$. Similarly, (14.4) and (14.5) give the q-expansion of $2\partial\mathbb{J}^\flat_v(\tau,\Phi')$.

If $v \notin S \cup B$, then R_v is an maximal order and we apply Proposition 3.10 at v to conclude that the v-th summand $2\partial\mathbb{J}^\flat_v(\tau,\Phi') + \mathrm{Int}_v(\tau,\Phi)$ is zero (note that now $\Phi^{(v)}$ and $\Phi'^{(v)}$ match).

Finally, let $v \in B$. Then the v-th term $2\partial\mathbb{J}^\flat_v(\tau,\Phi') + \mathrm{Int}_v(\tau,\Phi)$ is a formal power series in $q^{1/N}$ with coefficients in $\mathbb{Q}\log q_v$ (or its image in $\mathbb{R}_S$). By Proposition 4.12 , our assumption on Conjecture 3.8 part (a) (for all p-adic field $\mathbb{Q}_p$ with $p \notin S$ and for S_n) implies that

$$-\partial\mathrm{Orb}\big((x'_0, u'), \mathbf{1}_{(S_n\times V'_n)(O_{F_{0,v}})}\big) = \mathrm{Int}_v(g_0, u)\cdot \log q_v,$$

whenever $\mathfrak{q}(u,u) = \mathfrak{q}(u') = \xi$ is a unit at v. In other words, the ξ-th Fourier coefficient of $2\partial\mathbb{J}^\flat_v(\tau,\Phi') + \mathrm{Int}_v(\tau,\Phi)$ vanishes if ξ is a unit v, which holds if $(\xi, B) = 1$.

Therefore, whenever $(\xi, B) = 1$, the ξ-th Fourier coefficient of $\mathcal{E}^\flat - \mathrm{Int}(0,\Phi)$ (equivalently $\mathcal{E}^\flat$) vanishes, and this completes the proof of part (a).

Now we turn to part (b). For $\xi \neq 0$, the ξ-th Fourier coefficients of $\mathscr{E}^\flat$ lie in the $\mathbb{Q}$-span of $\log p$ for p in the finite set B. By q-expansion principle, the constant term must also lie in the same span. By the $\mathbb{Q}$-linear independence of $\log p$ for $p \in S \cup B$, we may write the constant term uniquely in the form $\sum_{p_i\in B} a_i \log p_i$, $a_i \in \mathbb{Q}$. We assign the summands as the constant term of $\big(2\partial\mathbb{J}^\flat_v(\tau,\Phi') + \mathrm{Int}_v(\tau,\Phi)\big)$ according to the residue characteristics of $v \in B$. Since the sum belongs to $\mathcal{A}_{\mathrm{hol}}(\Gamma(N), n, \mathbb{R}_S)$, it follows from the $\mathbb{Q}$-linear independence (inside $\mathbb{R}_S$) of

$\{\log p \mid p \in B\}$ that, for every v, the v-th term itself (with the constant term just assigned) also vanishes. Taking the ξ-th Fourier coefficient implies part (b).

$\square$

Remark 14.7. A byproduct of Theorem 14.6 is that the constant term of $\mathcal{E}^\flat$ vanishes. This amounts to an equality relating a certain part of the nilpotent term (12.36) to the arithmetic degree of the restriction of the metrized line bundle $\widehat{\omega}$ to the derived CM cycle. This may be of some independent interest.

Corollary 14.8. *Let $v \notin S$ and assume that Conjecture 3.8 part* (a) *holds for all p-adic field $\mathbb{Q}_p$ with $p \notin S$ and for S_n. Let $(g_0, u) \in (\mathrm{U}(V(v)) \times V(v))(F_0)_{\mathrm{srs}}$ be an element matching $(x_0', u') \in (S_n \times V')(F_0)_{\mathrm{srs}}$. If $\mathfrak{q}(u) \neq 0$, then*

$$-\partial\mathrm{Orb}\big((x_0', u'), \mathbf{1}_{(S_n \times V_n')(O_{F_{0,v}})}\big) = \mathrm{Int}_v(g_0, u) \cdot \log q_v.$$

Proof. We run the same argument as in the proof of Proposition 13.8, where we note that the compactness (modulo $\mathrm{U}(\mathbb{V}_n)(F_{0,v})$) of the support of the function $\mathrm{Int}_v(\cdot,\cdot)$ holds by Theorem 5.5. We then obtain a refinement of the equality in part (ii) of Theorem 14.6

$$-\partial\mathrm{Orb}((x_0', u'), \Phi_v') \cdot \mathrm{Orb}\left((x_0', u'), \Phi'^{(v)}\right) = \mathrm{Int}_v(g_0, u) \cdot \mathrm{Orb}\left((g_0, u), \Phi^{(v)}\right).$$

Here we note that $\mathrm{Int}_v(\xi, \Phi)$ in part (ii) of Theorem 14.6 is given by (9.18). Now the away from v factors on the two sides are equal and can be chosen to be non-zero (e.g., the function $\Phi^{(v)}$ can be chosen point-wise non-negative with non-empty support containing (g_0, u)).

$\square$

15. The proof of AFL

Now we return to the set up of Conjecture 3.8 in §3.

Theorem 15.1. *Conjecture 3.8 holds when $F_0 = \mathbb{Q}_p$ and $p \geq n$.*

Proof. The proof is parallel to that of Theorem 13.9. We prove Conjecture 3.8 part (b) by induction on $\dim \mathbb{V}_n$. The case $n = 1$ is known [47]. Assume now that Conjecture 3.8 part (b) holds for $\mathbb{V}_{n-1}$. Then by Proposition 4.12 part (i), Conjecture 3.8 part (a) holds for S_n. We now want to globalize the situation in order to apply Corollary 14.8.

We start with the following local data

- a place v_0 of $F_0 = \mathbb{Q}$, and an unramified (local) quadratic extension $F_{w_0}/F_{0,v_0}$,
- the non-split $F_{v_0}/F_{0,v_0}$-hermitian space $\mathbb{V}_n$ of dimension n,
- $(g_{v_0}, u_{v_0}) \in (\mathrm{U}(\mathbb{V}_n) \times \mathbb{V}_n)_{\mathrm{srs}}$, we further assume that the characteristic polynomial of g_{v_0} has integral coefficients (in $O_{F_{w_0}}$) and $\det(1 - g_{v_0})$ is a unit, and $\langle u_{v_0}, u_{v_0}\rangle \neq 0$,
- $(x_{v_0}', u_{v_0}') \in (S_n \times V')(F_{0,v_0})_{\mathrm{srs}}$ matching (g_{v_0}, u_{v_0}).

By the proof of Theorem 13.9, there exist the following global data

- an imaginary quadratic field F/F_0 such that $F \otimes_{F_0} F_{0,v_0} \simeq F_{w_0}$.
- a totally real number field F_0', and its quadratic extension $F' = F_0' \otimes_{F_0} F$,
- an element $g \in F'^1$ such that $O_{F_{w_0}}[g] = O_{F_{w_0}}[g_{v_0}]$ as subrings of $F' \otimes_F F_{w_0}$,
- a totally positive definite n-dimensional F/F_0-hermitian space $V(v_0)$ that is locally at v_0 isometric to $\mathbb{V}_n$,
- $u \in V(v_0)$ such that the pair (g, u) is v_0-adically close to (g_{v_0}, u_{v_0}) (in particular $\langle u, u\rangle \neq 0$).

Now we define the Shimura variety and integral models $\mathcal{M}$ for the nearby hermitian space V of $V(v_0)$ at v_0 (that is, non-split at v_0, with signature $(n-1, 1)$ at $v \mid \infty$, and isomorphic to $V(v_0)$ elsewhere). Let $\mathfrak{d}$ be a finite set of places as in §6.2.2 such that $v_0 \nmid \mathfrak{d}$. Let S the set of non-archimedean places such that

- $v_0 \notin S$,
- S contains all places dividing $\mathfrak{d}$,
- for every non-archimedean $v \notin S \cup \{v_0\}$, the ring $O_F[g]$ is locally at v a maximal order.

Then we proceed as the proof of Theorem 13.9 to choose $(x_0', u') \in (S_n \times V_n')(F_0)$ in the unique regular semisimple orbit matching (g, u), and choose (partial) Gaussian test functions Φ and Φ'.

Now we apply Corollary 14.8 to obtain

$$-\partial\mathrm{Orb}((x_0', u'), \Phi'_{v_0}) = \mathrm{Int}_{v_0}(g, u) \log q_{v_0}.$$

Therefore Conjecture 3.8 part (b) holds when $(g, u) \in (\mathrm{U}(\mathbb{V}_n) \times \mathbb{V}_n)_{\mathrm{srs}}$. By the local constancy of the orbital integral, and of the intersection numbers by Theorem 5.5, near a strongly regular semisimple (g, u), we conclude that Conjecture 3.8 part (b) holds when $(g_{v_0}, u_{v_0}) \in (\mathrm{U}(\mathbb{V}_n) \times \mathbb{V}_n)_{\mathrm{srs}}$. This complete the induction.

□

Appendix A. Weil representation commutes with smooth transfer

We retain the notation in §2. Let F/F_0 be a quadratic extension of local fields (the case $E = F \times F$ could also be allowed but in that case the result below is trivial). Recall that $V_n = F_0^n \times (F_0^n)^*$. We have a bijection of regular semisimple orbits, cf. §2.3,

$$\textstyle\coprod_V \big[(\mathrm{U}(V) \times V)(F_0)\big]_{\mathrm{rs}} \xrightarrow{\sim} [S_n(F_0) \times V_n']_{\mathrm{rs}} ,$$

where the disjoint union runs over the set of isometric classes of F/F_0-hermitian spaces V of dimension n. The notion of smooth transfer is as in Definition 2.2 (w.r.t. the transfer factor there). Here let us focus on one hermitian space V at a time.

The Weil representation (for even dimensional quadratic space) is defined in §11. Here we apply the formula (11.1) to the second variable in the functions in $\mathcal{S}(S_n \times V_n')$ and $\mathcal{S}(\mathrm{U}(V) \times V)$ respectively. To fix the set up, we recall that the structure of F_0-bilinear symmetric pairing on V_n' is the tautological pairing

$$\langle u', u' \rangle = 2\, u_2(u_1), \quad u' = (u_1, u_2) \in F_0^n \times (F_0^n)^*.$$

and on V the quadratic form is the induced one, i.e.,

$$\langle u, u \rangle_{F_0} = \mathrm{tr}_{F/F_0} \langle u, u \rangle_F, \quad u \in V$$

where $\langle \cdot, \cdot \rangle_F : V \times V \to F$ is the hermitian pairing (F-linear on the first factor and conjugate F-linear on the second one).

We now deduce the following result from [48] when F is non-archimedean, and [44] when F is archimedean.

Theorem A.1 (Weil representation commutes with smooth transfer)**.** *If $\Phi' \in \mathcal{S}(S_n \times V_n')$ matches a function $\Phi \in \mathcal{S}(\mathrm{U}(V) \times V)$, then $\omega(h)\Phi'$ also matches $\omega(h)\Phi$ for any $h \in \mathrm{SL}_2(F)$.*

Remark A.2. Similar results hold for the partial Fourier transforms on the Lie algebra $\mathfrak{s}_n \times V_n'$ and $\mathfrak{u}(V) \times V$. A similar result for the endoscopic transfer can be deduced from a theorem of Waldspurger.

Proof. We need to check the assertion for h of the form $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, as in (11.1).

The assertion for $h = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, b \in F$ is trivial.

Now let $h_a = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$. Then

$$\mathrm{Orb}((g, u), \omega(h_a)\Phi) = \chi_V(a)|a|^n \,\mathrm{Orb}((g, au), \Phi)$$

for all $(g, u) \in (\mathrm{U}(V) \times V)_{\mathrm{rs}}$. Here

$$\chi_V(a) = (a, (-1)^{\dim_F V} \det(V)),$$

where $\det(V)$ is the discriminant of V as a quadratic space. We claim

$$\chi_V(a) = \eta(a)^{\dim_F V}. \tag{A.1}$$

Since $\det(V_1 \oplus V_2) = \det(V_1)\det(V_2)$ (in $F_0^\times/(F_0^\times)^2$) for orthogonal direct sum $V_1 \oplus V_2$, it suffices to prove the claim when $\dim_F V = 1$. Then there are only two isometric classes and one can check the claim directly.

On the other hand

$$\mathrm{Orb}((\gamma, u'), \omega(h_a)\Phi', s) = \chi_{V_n'}(a)|a|^n \,\mathrm{Orb}((\gamma, au'), \Phi', s).$$

Now $\chi_{V_n'}$ is the trivial character since V_n' is an orthogonal direct sum of n-copies of the hyperbolic 2-space. We now note that the transfer factor (2.15) obeys

$$\omega(\gamma, au') = \eta(a)^n \omega(\gamma, u').$$

This proves the assertion for $\omega(h_a), a \in F^\times$.

Finally, let $h = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. Then

$$\mathrm{Orb}((g, u), \omega(h)\Phi) = \gamma_V \,\mathrm{Orb}((g, u), \widehat{\Phi})$$

where γ_V is the Weil constant. We claim, for our V induced from a hermitian form

$$\gamma_V = \eta(\det(V)_{F/F_0})\, \epsilon(\eta, 1/2, \psi)^{\dim_F V}$$

where $\det(V)_{F/F_0} \in F_0^\times / \mathrm{Nm}\, F^\times$ is the hermitian discriminant of V (as an F/F_0-hermitian space). First note that the right hand side is multiplicative with respect to orthogonal direct sum $V_1 \oplus V_2$

$$\det(V_1 \oplus V_2)_{F/F_0} = \det(V_1)_{F/F_0} \det(V_2)_{F/F_0}.$$

Note that, by definition, the Weil constant γ_V satisfies

$$\widehat{\psi \circ q} = \gamma_V \,\psi \circ (-q),$$

where $\psi \circ q : V \to F_0 \to \mathbb{C}$ (resp., $\psi \circ (-q)$) is the function precomposing ψ with q (resp., $-q$). Here the Fourier transform is understood as applied to distributions. It follows that it is also multiplicative with respect to orthogonal direct sum $V_1 \oplus V_2$:

$$\gamma_{V_1 \oplus V_2} = \gamma_{V_1} \cdot \gamma_{V_2}.$$

Therefore it suffices to show the claim when $\dim_F V = 1$. Then one can check the claim directly. In fact it is easy to see that we have $\gamma_{V_a} = \eta(a)^{\dim_F V} \gamma_V$ where V_a denotes the new hermitian space by multiplying the hermitian form by $a \in F_0^\times$. Hence we may just check the case $\det(V)_{F/F_0} \in \mathrm{Nm}\, F^\times$, which is done in [20, Lemma 1.2] (where the constant $\lambda_{F/F_0}(\psi)$ in *loc. cit.* is the same as $\epsilon(\eta, 1/2, \psi)$).

On the other hand, the Weil constant $\gamma_{V_n'} = 1$ since V_n' is an orthogonal direct sum of n-copies of the hyperbolic 2-space. Hence

$$\mathrm{Orb}((\gamma, u'), \omega(h)\Phi', s) = \mathrm{Orb}((\gamma, u'), \widehat{\Phi}', s).$$

Now the desired assertion follows from [48, Theorem 4.17] [9] when F is non-archimedean, and the proof of [44, Theorem 9.1] when F is archimedean. Note that in [44], $\epsilon(\eta, 1/2, \psi) = \sqrt{-1}$ for his choice of the additive character $\psi(x) = e^{2\pi\sqrt{-1}x}, x \in \mathbb{R}$. □

Appendix B. Grothendieck groups for formal schemes

We collect some facts regarding formal schemes and the Grothendieck group of coherent sheaves, largely following the work by Gillet–Soulé [12]. No result here is new.

[9] Note that in [48], the factor $\eta(\det(V)_{F/F_0})$ is missing.

B.1. **Grothendieck groups.** Let $(X, \mathcal{O}_X)$ be a noetherian formal scheme [16, §10]. Let Y be a closed formal subscheme of X (i.e., closed subscheme of a formal scheme in the terminology in *loc. cit.*). Let $\mathcal{J}$ be the sheaf of ideals defining Y. A coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$–module is said to be *formally supported* on Y if it is annihilated by $\mathcal{J}^n$ for some $n \geq 1$. We make this explicit when $(X, \mathcal{O}_X)$ is an affine formal scheme, say, the formal completion of $\operatorname{Spec} A$ at $\operatorname{Spec} A/I$ for an ideal I of A, where $A = \varprojlim_n A/I^n$ is I-adically complete. Then we may assume that Y is defined by an ideal J of A (i.e., $\mathcal{J} = J^{\Delta}$, cf. [16, §10.10]). Then a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$–module is formally supported on Y if $M = \Gamma(X, \mathcal{F})$ as an A-module (equivalently the sheaf $\widetilde{J}$ of $\mathcal{O}_{\operatorname{Spec} A}$–module) has support contained in the closed subset $\operatorname{Spec}(A/J)$ of $\operatorname{Spec} A$.

Then the definitions in [12, §1] for noetherian schemes carry over to the setting of noetherian formal schemes. Let $K_0'(X)$ denote the Grothendieck group of coherent sheaves of $\mathcal{O}_X$-modules. Let $K_0^Y(X)$ denote the Grothendieck group of finite complex of coherent locally free $\mathcal{O}_X$-modules, acyclic outside Y (i.e., the homology sheaves are supported on Y), cf. [12, §1.2]. Let $K_0(X) = K_0^X(X)$. The tensor product of (complex of) locally free sheaves induces the cup product

$$\cup \colon K_0^Y(X) \times K_0^Z(X) \longrightarrow K_0^{Y \cap Z}(X)$$

by $[\mathcal{F}.] \cup [\mathcal{G}.] = [\mathcal{F}. \otimes \mathcal{G}.]$, cf. [12, §1.4].

There is a descending filtration on $K_0^Y(X)$ by the subgroups

$$F^i K_0^Y(X) = \cup_{Z \subset Y, \operatorname{codim}_X Z \geq i} \operatorname{Im}(K_0^Z(X) \longrightarrow K_0^Y(X)). \tag{B.1}$$

The associated graded groups are

$$\operatorname{Gr}^i K_0^Y(X) = F^i K_0^Y(X) / F^{i+1} K_0^Y(X). \tag{B.2}$$

Similarly, there is an ascending filtration $F_i K_0'(X)$ on $K_0'(X)$

$$F_i K_0'(X) = \cup_{Z \subset X, \dim Z \leq i} \operatorname{Im}(K_0'(Z) \longrightarrow K_0'(X)).$$

From now on we assume that X is regular of pure dimension d. Then we have a natural isomorphism

$$K_0^Y(X) \xrightarrow{\sim} K_0'(Y) \ ,$$

and

$$F^{d-i} K_0^Y(X) \xrightarrow{\sim} F_i K_0'(Y) \ .$$

The construction of the Adam operations $\psi^k, k \in \mathbb{Z}_{\geq 1}$ in [12] still work for $K_0^Y(X)$ and induce a decomposition

$$K_0^Y(X)_{\mathbb{Q}} = \bigoplus_{i \geq 0} K_0^Y(X)^i_{\mathbb{Q}},$$

where ψ^k acts on (the "weight-i" part) $K_0^Y(X)^i_{\mathbb{Q}}$ by the scalar k^i. Moreover, by [12, Prop. 5.3]

$$F^j K_0^Y(X)_{\mathbb{Q}} = \bigoplus_{i \geq j} K_0^Y(X)^i_{\mathbb{Q}},$$

and for $j_1, j_2 \geq 0$, by [12, Prop. 5.5], the cup product has image

$$F^{j_1} K_0^Y(X)_{\mathbb{Q}} \cdot F^{j_2} K_0^Z(X)_{\mathbb{Q}} \subset F^{j_1 + j_2} K_0^{Y \cap Z}(X)_{\mathbb{Q}}.$$

This last inclusion is a result we used in the proof of Proposition 5.3.

Finally, we relax the noetherian hypothesis. For our purpose, we only consider locally noetherian formal schemes $(X, \mathcal{O}_X)$ that can be written as an increasing union

$$(X, \mathcal{O}_X) = \cup_{i \in \mathbb{Z}_{\geq 0}} (X_i, \mathcal{O}_{X_i})$$

of open noetherian formal schemes (the openness means that the transition maps $f_i : X_i \to X_{i+1}$ are open immersions of formal schemes). We then define

$$K_0(X) = \varprojlim_i K_0(X_i), \quad K_0'(X) = \varprojlim_i K_0'(X_i).$$

If Y is a closed formal subscheme of X, setting $Y_i = Y \times_X X_i$ to write Y as the union of Y_i's, we define

$$K_0^Y(X) = \varprojlim_i K_0^{Y_i}(X_i).$$

Similarly, we have the filtration $F^i K_0^Y(X)$, and $F_i K_0'(X)$, and they have the same properties as in the noetherian case.

Now let $\pi : W \to S = \operatorname{Spf} A$ be a morphism of formal schemes, where A is a complete discrete valuation ring. When π is proper [17, III, 3.4.1] and W is a *scheme* (not only a formal scheme), we have a "degree" map

$$\begin{aligned} K_0'(W) &\longrightarrow \mathbb{Z} \\ [\mathcal{E}] &\longmapsto \textstyle\sum_{i\in\mathbb{Z}}(-1)^i \operatorname{length}_{\mathcal{O}_S} \mathrm{R}^i\pi_*\mathcal{E}. \end{aligned}$$

The assumption on π and W implies that all $\mathrm{R}^i\pi_*\mathcal{E}$ are torsion coherent sheaves and hence have finite lengths. It is easy to see that this is independent of the choice of $\mathcal{E}$ in its equivalence class. Now let X be regular with two closed formal subscheme Y and Z. If $\pi : W = Y \cap Z \to S = \operatorname{Spf} A$ is proper and W is a scheme, we obtain a homomorphism

$$\begin{aligned} K_0^Y(X) \times K_0^Z(X) &\longrightarrow \mathbb{Z} \\ ([\mathcal{F}], [\mathcal{G}]) &\longmapsto \chi(X, \mathcal{F} \otimes^{\mathbb{L}} \mathcal{G}) \end{aligned}$$

where the Euler–Poincaré characteristic is defined by

$$\chi(X, \mathcal{F} \otimes^{\mathbb{L}} \mathcal{G}) \colon = \sum_{i,j\in\mathbb{Z}} (-1)^{i+j} \operatorname{length}_{\mathcal{O}_S} \mathrm{R}^i\pi_*(\operatorname{Tor}_j^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})). \tag{B.3}$$

We also denote

$$Y \cap^{\mathbb{L}} Z \colon = \mathcal{O}_Y \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_Z \in K_0'(Y \cap Z) \simeq K_0^{Y\cap Z}(X). \tag{B.4}$$

B.2. **A few lemmas.** For convenience we record the following results.

Lemma B.1. *Let X be a locally noetherian formal schemes of the above type. Let $X = X_1 \cup X_2$ be a union of two closed formal subschemes. Then there is a natural isomorphism* [10]

$$\begin{aligned} \tfrac{K_0'(X)}{K_0'(X_1\cap X_2)} &\xrightarrow{\sim} \tfrac{K_0'(X_1)}{K_0'(X_1\cap X_2)} \bigoplus \tfrac{K_0'(X_2)}{K_0'(X_1\cap X_2)} \\ [\mathcal{E}] &\longmapsto \left([\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_1}], [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_2}]\right). \end{aligned}$$

Proof. We immediately reduce the question to the case when X is noetherian, which we assume now. Let $\mathcal{I}$ and $\mathcal{J}$ be the sheaf of ideals of $\mathcal{O}_X$ defining X_1 and X_2 respectively. Consider two exact sequences of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{O}_X/\mathcal{I}\cap\mathcal{J} \longrightarrow \mathcal{O}_X/\mathcal{I} \oplus \mathcal{O}_X/\mathcal{J} \longrightarrow \mathcal{O}_X/(\mathcal{I}+\mathcal{J}) \longrightarrow 0\ ,$$

and

$$0 \longrightarrow \mathcal{I}\cap\mathcal{J} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{I}\cap\mathcal{J} \longrightarrow 0\ .$$

Tensoring $\mathcal{E}$, we obtain two exact sequences

$$\operatorname{Tor}_1^{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_{X_1\cap X_2}) \longrightarrow \mathcal{E}\otimes\mathcal{O}_X/\mathcal{I}\cap\mathcal{J} \longrightarrow \mathcal{E}\otimes\mathcal{O}_{X_1} \oplus \mathcal{E}\otimes\mathcal{O}_{X_2} \longrightarrow \mathcal{E}\otimes\mathcal{O}_{X_1\cap X_2} \longrightarrow 0\ ,$$

and

$$\mathcal{E}\otimes(\mathcal{I}\cap\mathcal{J}) \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}\otimes\mathcal{O}_X/(\mathcal{I}\cap\mathcal{J}) \longrightarrow 0\ .$$

Since both $\operatorname{Tor}_1^{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_{X_1\cap X_2})$ and $\mathcal{E}\otimes\mathcal{O}_{X_1\cap X_2}$ lie in $K_0'(X_1\cap X_2)$, we have

$$[\mathcal{E}\otimes\mathcal{O}_{X_1}] + [\mathcal{E}\otimes\mathcal{O}_{X_2}] \equiv [\mathcal{E}\otimes\mathcal{O}_X/(\mathcal{I}\cap\mathcal{J})] \in \frac{K_0'(X)}{K_0'(X_1\cap X_2)}.$$

[10]Here $K_0'(X_1\cap X_2) \to K_0'(X_1)$ is not necessarily injective, so the quotient simply denotes the cokernel.

Since $X = X_1 \cup X_2$, we have $\mathcal{I}\mathcal{J} = 0$, and hence $(\mathcal{I} + \mathcal{J}) \cdot (\mathcal{I} \cap \mathcal{J}) = 0$. It follows that $\mathcal{O}_E \otimes \mathcal{I} \cap \mathcal{J} \in K_0'(X_1 \cap X_2)$, and hence

$$[\mathcal{E}] \equiv [\mathcal{E} \otimes \mathcal{O}_X/(\mathcal{I} \cap \mathcal{J})] \in \frac{K_0'(X)}{K_0'(X_1 \cap X_2)}.$$

This completes the proof. □

In the case of "proper intersection", the derived tensor product can be simplified:

Lemma B.2. *Let X be a (locally noetherian) pure dimensional formal scheme of the above type, and Z_1, Z_2 two pure dimensional closed formal subschemes on X. Assume that the closed immersion $Z_1 \to X$ is a regular immersion (e.g., if both X and Z_1 are regular), and Z_2 is Cohen–Macaulay.*

$$\begin{tikzcd} Z_1 \cap Z_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Z_2 \arrow[d] \\ Z_1 \arrow[r] & X. \end{tikzcd}$$

(i) If $Z_1 \cap Z_2$ has the expected dimension (i.e., $\mathrm{codim}_X Y = \mathrm{codim}_X Z_1 + \mathrm{codim}_X Z_2$ at every point of $Z_1 \cap Z_2$), then the higher Tor *sheaves vanish, i.e.,*

$$\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}) = 0, \quad i > 0.$$

In particular, as elements in $K_0'(Z_1 \cap Z_2)$,

$$\mathcal{O}_{Z_1} \otimes^{\mathbb{L}} \mathcal{O}_{Z_2} = \mathcal{O}_{Z_1} \otimes \mathcal{O}_{Z_2}.$$

(ii) Let $Z_1 \cap Z_2 = Y \cup Y'$ such that Y has the expected dimension. Then

$$\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2})|_Y \equiv 0, \quad i > 0,$$

as an element in $K_0'(Y)/K_0'(Y \cap Y')$.

Proof. This follows from the same argument in the proof of [38, Prop. 8.10] regarding the vanishing of higher Tor terms. We prove the first part; the second part is proved similarly by combining Lemma B.1.

Let x be a point on $Z_1 \cap Z_2$. We need to show that $(\mathcal{O}_{Z_1} \otimes^{\mathbb{L}} \mathcal{O}_{Z_2})_x$ is represented by $\mathcal{O}_{Z_1 \cap Z_2, x}$. Let R be the local ring of x on X. Since the closed immersion $Z_1 \to X$ is a regular immersion, by definition Z_1 is defined at x by a regular sequence $f_1, \cdots, f_m$ of R. Then the Koszul complex $K(f_1, \cdots, f_m)$ is a free resolution of the R-module $\mathcal{O}_{Z_1,x}$. It follows that the complex $K(f_1, \cdots, f_m) \otimes_R \mathcal{O}_{Z_2,x}$ represents $(\mathcal{O}_{Z_1} \otimes^{\mathbb{L}} \mathcal{O}_{Z_2})_x$.

Now, since Z_2 is Cohen–Macaulay, the dimension hypothesis implies that the images $\overline{f}_1, \cdots, \overline{f}_m$ of $f_1, \cdots, f_m$ in $\mathcal{O}_{Z_2,x}$ again form a regular sequence which generates the ideal defining $Z_1 \cap Z_2$ at x in Z_2. Hence $K(\overline{f}_1, \cdots, \overline{f}_m)$ is a free resolution of the $\mathcal{O}_{Z_2,x}$-module $\mathcal{O}_{Z_1 \cap Z_2, x}$. On the other hand, we have

$$K(f_1, \cdots, f_m) \otimes_R \mathcal{O}_{Z_2,x} = K(\overline{f}_1, \cdots, \overline{f}_m).$$

It follows that $(\mathcal{O}_{Z_1} \otimes^{\mathbb{L}} \mathcal{O}_{Z_2})_x$ is represented by $K(\overline{f}_1, \cdots, \overline{f}_m)$, or equivalently by $\mathcal{O}_{Z_1 \cap Z_2, x}$. This completes the proof.

□

References

[1] R. Beuzart-Plessis, *Comparison of local spherical characters and the Ichino-Ikeda conjecture for unitary groups*, arXiv:1602.06538. 49, 51

[2] ———, *A new proof of Jacquet-Rallis's fundamental lemma*, arXiv:1901.02653. 3

[3] Bost, J.-B.; Gillet, H.; Soulé, C. *Heights of projective varieties and positive Green forms.* J. Amer. Math. Soc. 7 (1994), no. 4, 903–1027. 42

[4] Burgos, J., Kramer, J., Kuhn, U., *Cohomological arithmetic Chow groups.* J. Inst. Math. Jussieu 6, 1–178 (2007) 41

[5] J. Bruinier, *Regularized theta lifts for orthogonal groups over totally real fields*, J. Reine Angew. Math. 672 (2012), 177–222. 38, 40

[6] J. Bruinier, B. Howard, S. Kudla, M. Rapoport, T. Yang, *Modularity of generating series of divisors on unitary Shimura varieties*, preprint, 2017, `arXiv:1702.07812 [math.NT]`. 3, 26, 27, 31, 40, 41

[7] Jan Hendrik Bruinier, Stephen S. Kudla, and Tonghai Yang, *Special values of Green functions at big CM points,* Int. Math. Res. Not. IMRN (2012), no. 9, 1917–1967. 4, 29, 32

[8] J. H. Bruinier and T. Yang, *Faltings heights of CM cycles and derivatives of L-functions,* Invent. Math. 177 (2009), 631–681. 4, 29, 32

[9] S. Ehlen, S. Sankaran, *On two arithmetic theta lifts.* Compos. Math. 154 (2018), no. 10, 2090–2149. 6, 40

[10] W. T. Gan, B. Gross, and D. Prasad, *Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of classical groups*, Astérisque **346** (2012), 1–109. 1, 7, 27

[11] H. Gillet, *Arithmetic intersection theory on Deligne–Mumford stacks.* In *Motives and algebraic cycles*, Fields Inst. Commun., vol. **56**, Amer. Math. Soc., Providence, RI, 2009, pp. 93–109. 41

[12] H. Gillet and C. Soulé, *Intersection theory using Adams operations.* Invent. Math. 90 (1987), no. 2, 243–277 12, 42, 73, 74

[13] ———, *Arithmetic intersection theory*, Inst. Hautes Etudes Sci. Publ. Math. **72** (1990), 93–174. 41, 43

[14] J. Gordon, *Transfer to characteristic zero*, appendix to [46]. 2, 10

[15] B. Gross and D. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. **84** (1986), no. 2, 225–320. 1

[16] Grothendieck, A. *Éléments de géométrie algébrique.* I. Le langage des schémas. Inst. Hautes Études Sci. Publ. Math., No. 4, 1960, 228 pp. 74

[17] ———, *Éléments de géométrie algébrique.* III. Étude cohomologique des faisceaux cohérents, Premiére partie. Inst. Hautes Études Sci. Publ. Math., No. 11, 1961. 75

[18] Xuhua He, Chao Li, Yihang Zhu, *Fine Deligne-Lusztig varieties and arithmetic fundamental lemmas*, preprint. 4

[19] B. Howard, *Complex multiplication cycles and Kudla–Rapoport divisors*, Ann. of Math. (2) **176** (2012), no. 2, 1097–1171. 4, 14, 27, 29, 32, 33, 34

[20] Jacquet, H.; Langlands, R. P. *Automorphic forms on GL(2).* Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin-New York, 1970. vii+548 pp. 73

[21] H. Jacquet, S. Rallis, *On the Gross-Prasad conjecture for unitary groups*, in On Certain L-Functions, Clay Math. Proc. 13, Amer. Math. Soc., Providence, RI, 2011, pp. 205–264. 1, 10

[22] H. Jacquet, *Sur un résultat de Waldspurger.* Ann. Sci. École Norm. Sup. (4) 19 (1986), no. 2, 185–229 69

[23] S. Kudla, *Algebraic cycles on Shimura varieties of orthogonal type*, Duke Math. J. 86 (1997), 39–78. 38

[24] ———, *Central derivatives of Eisenstein series and height pairings*, Ann. of Math. (2) **146** (1997), no. 3, 545–646. 40

[25] S. Kudla and M. Rapoport, *Special cycles on unitary Shimura varieties I. Unramified local theory*, Invent. Math. **184** (2011), no. 3, 629–682. 3, 12, 26

[26] ———, *Special cycles on unitary Shimura varieties II: Global theory*, J. Reine Angew. Math. **697** (2014), 91–157. 27, 29, 30, 31, 32

[27] Chao Li, Wei Zhang, *Kudla–Rapoport cycles and derivatives of local densities*, arXiv. 1908.01701. 23

[28] Chao Li, Yihang Zhu, *Remarks on the arithmetic fundamental lemma.* Algebra Number Theory, 11 (2017), No. 10, 2425–2445 4

[29] Yifeng Liu, *Arithmetic theta lifting and L-derivatives for unitary groups, I.* Algebra Number Theory 5 (2011), no. 7, 849–921 38, 40

[30] ———, *Relative trace formulae toward Bessel and Fourier-Jacobi periods of unitary groups*, Manuscripta Mathematica, 145 (2014) 1–69 2, 7, 8, 10, 11

[31] ———, *Fourier-Jacobi cycles and arithmetic relative trace formula*, preprint. 2, 3, 12, 13, 30, 31, 32

[32] G. Lusztig, *Coxeter orbits and eigenspaces of Frobenius.* Invent. Math. 38 (1976/77), no. 2, 101–159. 23

[33] A. Mihatsch, *On the arithmetic fundamental lemma conjecture through Lie algebras*, Math. Z. **287** (2017), no. 1–2, 181–197. 4, 13

[34] ———, *Relative unitary RZ-spaces and the arithmetic fundamental lemma*, to appear in the Journal of the Institute of Mathematics of Jussieu. 3, 4, 12, 14

[35] ———, *Local Constancy of Intersection Numbers*, in preparation. 4, 25

[36] B.-C. Ngô, *Le lemme fondamental pour les algebres de Lie*, Publ. Math. Inst. Hautes Études Sci. 111 (2010), 1–169. 3

[37] F. Oberhettinger, *On the derivative of Bessel functions with respect to the order.* J. Math. and Phys. 37 (1958), 75–78. 54

[38] M. Rapoport, B. Smithling, and W. Zhang, *On the arithmetic transfer conjecture for exotic smooth formal moduli spaces*, Duke Math. J. **166** (2017), no. 12, 2183–2336. 2, 22, 76

[39] ———, *Regular formal moduli spaces and arithmetic transfer conjectures*, Math. Ann. **370** (2018), no. 3–4, 1079–1175. 2, 9, 10, 11, 13

[40] ———, *Arithmetic diagonal cycles on unitary Shimura varieties*, arXiv. 1710.06962. 1, 26, 27, 28, 29, 31, 37, 39, 43, 44, 52, 61, 67

[41] ———, *On Shimura varieties for unitary groups*, arXiv:1906.12346. 26, 27, 30

[42] M. Rapoport, U. Terstiege, and W. Zhang, *On the arithmetic fundamental lemma in the minuscule case*, Compos. Math. **149** (2013), no. 10, 1631–1666. 4

[43] I. Vollaard and T. Wedhorn, *The supersingular locus of the Shimura variety for* $\mathrm{GU}(1, n-1)$ *II*, Invent. Math. **184** (2011), no. 3, 591–627. 22

[44] H. Xue, *On the global Gan–Gross–Prasad conjecture for unitary groups: approximating smooth transfer of Jacquet–Rallis*, To appear in J. Reine Angew. Math. 3, 52, 72, 73

[45] X. Yuan, S.-W. Zhang, and W. Zhang, *The Gross–Kohnen–Zagier Theorem over Totally Real Fields*, Compositio Math. 145 (2009), 1147–1162 38, 39

[46] Z. Yun, *The fundamental lemma of Jacquet–Rallis in positive characteristics*, Duke Math. J. **156** (2011), no. 2, 167–228. 2, 3, 10, 11, 13, 77

[47] W. Zhang, *On arithmetic fundamental lemmas*, Invent. Math. **188** (2012), no. 1, 197–252. 1, 2, 4, 8, 10, 24, 44, 61, 67, 71

[48] ———, *Fourier transform and the global Gan–Gross–Prasad conjecture for unitary groups*, Ann. of Math. (2) **180** (2014), no. 3, 971–1049. 2, 3, 52, 61, 62, 64, 72, 73

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