Remark. This is my type-setting of Maria’s handwritten solutions. If this contains any mistakes, they were likely introduced by me.

Assume that there exists a metric $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ that induces $O_\Pi$. We will prove two lemmas

Lemma 0.1. Let $(X, d)$ be a metric space. For any subset $Y \subset X$,

$$\bar{Y} = \{ x \in X : \exists (y_n) \in Y^N \text{ such that } \lim_{n \to \infty} y_n = x \}.$$ 

Proof. We know that $x \in \bar{Y}$ if and only if every open set containing intersects $Y$. Suppose $x \in X$ is such that $x = \lim_{n \to \infty} y_n$ for some $(y_n) \in Y^N$. If $U$ is an open neighborhood of $x$, then there exists an $\varepsilon > 0$ such that $B_{\varepsilon}(x) \subset Y$. The fact that $x = \lim_{n \to \infty} y_n$ means that there is an $N \in \mathbb{N}$ such that $d(x_n, y) < \varepsilon$ whenever $n \geq N$. Thus $x_N \in B_{\varepsilon}(y)$ and thus $U \supset B_{\varepsilon}(y)$ intersects $Y$. This proves that the right-hand side is contained in $\bar{Y}$.

To prove the reverse inclusion, suppose that $x \in \bar{Y}$; that is, every neighborhood of $x$ intersects $Y$. This means, in particular, that the open balls $B_{1/n}(x)$ intersects $Y$. For each $n \in \mathbb{N}$, pick $y_n \in Y \cap B_{1/n}(x)$. This defines a sequence $(y_n) \in Y^N$ with limit $x$.\footnote{To see this, note that if $V$ is an open neighborhood of $x$, then there is a $\delta > 0$ such that $B_{1/\delta}(x) \subset V$; in particular, for sufficiently large $n$ we have $y_n \in Y \cap B_{1/n}(x) \subset B_{1/\delta}(x)$.} Since $x$ was arbitrary, it follows that $\bar{Y}$ is contained in the right-hand side. $\square$

Lemma 0.2. The characteristic function $\chi = \chi_{\mathbb{R}\setminus\mathbb{Q}}$ of $\mathbb{R} \setminus \mathbb{Q}$ is not the pointwise limit of a sequence of continuous functions.

Proof. Suppose $(f_n) \in C(\mathbb{R}, \mathbb{R})^N$ is a sequence of continuous functions such that for all $x \in \mathbb{R}$ we have

$$\lim_{n \to \infty} f_n(x) = \chi_{\mathbb{R}\setminus\mathbb{Q}} = \begin{cases} 0 & x \in \mathbb{Q}, \\ 1 & \text{otherwise}. \end{cases}$$

We can assume that $f_n$ takes values in $[0, 1]$ by replacing $f_n$ with

$$\min(f_n, 0) = \frac{f_n + |f_n|}{2},$$
and then with 
\[ \max\{f_n, 1\}. \]

**Proposition.** There is a nested sequence of intervals \( J_0 \supset J_1 \supset J_2 \supset \cdots \) and an increasing sequence \( n_0 < n_1 < n_2 < \cdots \) such that
\[ f_{n_k}(J_k) = [1/4, 3/4]. \]

We construct these sequences inductively. To begin, pick \( x_0 < y_0 \) and \( x_0 \in \mathbb{Q} \) and \( y_0 \notin \mathbb{Q} \). Since 
\[ \lim_{n \to \infty} f_n(x_0) = 0 \quad \text{and} \quad \lim_{n \to \infty} f_n(y_0) = 1, \]
there exist some \( n_0 \geq 0 \) such that
\[ f_{n_0}(x_0) \leq 1/4 \quad \text{and} \quad f_{n_0}(y_0) \geq 3/4. \]

Since \( f_n \) is continuous, there is an interval \( J_0 \subset [x_0, y_0] \) such that
\[ f_{n_0}(J_0) = [1/4, 3/4]. \]

To see this, note that \( f^{-1}(1/4) \cap [x_0, y_0] \) is compact; hence, there is a largest element \( a_0 \in f^{-1}(1/4) \cap [x_0, y_0] \); similarly, there is a smallest element \( b_0 \in f^{-1}(3/4) \cap [a_0, y_0] \). Set \( J_0 = [a_0, b_0] \).

Since continuous images of connected subsets of \( \mathbb{R} \) are connected and continuous images of compact subsets of \( \mathbb{R} \) are compact, we know that \( f(J_0) \) is of the form \([m, M]\). We know that \( m \leq 1/4 \) and \( M \geq 3/4 \). In fact, we must have given \( m = 1/4 \) and \( M = 3/4 \). To see this note that if \( m < 1/4 \), there is an \( c \in (a_0, b_0) \) with \( f(c) = m < 1/4 \) and thus a \( d \in [c, b_0) \) with \( f(d) = 1/4 \); contradicting the choice of \( a_0 \). (The argument for \( M = 3/4 \) is similar.)

For the induction step, take \( x_{k+1} \in J_k \cap \mathbb{Q} \) and \( y_{k+1} \in J_k \setminus \mathbb{Q} \) with \( x_{k+1} < y_{k+1} \). Arguing as before, there is a \( n_{k+1} > n_k \) such that
\[ f_{n_{k+1}}(x_{k+1}) \leq 1/4 \quad \text{and} \quad f_{n_{k+1}}(y_{k+1}) \geq 3/4, \]
and we can find \( J_{k+1} \subset [x_{k+1}, y_{k+1}] \subset J_k \) such that
\[ f_{n_{k+1}}(J_{k+1}) = [1/4, 3/4]. \]

To complete the proof, observe that because the intervals are compact and nested, their intersection is non-empty. Let \( x \in \bigcap_{k \in \mathbb{N}} I_{N_k} \). Then we have \( f_{n_k}(x) \in [1/4, 3/4] \); but this contradicts the fact that \( \lim_{k \to \infty} f_{n_k}(x) = \lim_{n \to \infty} f_n(x) = \chi_{\mathbb{R} \setminus \mathbb{Q}}(x) \), which is either 0 or 1. \( \square \)

Consider
\[ X = \{ 1 - (\cos(n!2\pi x))^{2m} : n, m \in \{1, 2, \ldots \} \} \subset \mathbb{R}^k. \]

Let \( n \) be fixed. \( F_n(x) = 1 - (\cos(n!2\pi x))^{2m} \) converges to 0 if \( n!\pi x = k\pi \) for some \( k \in \mathbb{Z} \), and to 1 otherwise. Thus \( F_n^m \) converges pointwise to
\[ F_n(x) = \begin{cases} 0 & x = k/n! \text{ for some } k \in \mathbb{Z} \\ 1 & \text{otherwise} \end{cases}. \]
By the first lemma $F_n \in \bar{X}$ for all $n$. If $x \in \mathbb{R} \setminus \mathbb{Q}$, then $\lim_{n \to \infty} F_n(x) = 1$ because $F_n(x) = 1$ for all $n$. If $x \in \mathbb{Q}$, then $x = \frac{p}{q} = \frac{p(q - 1)!}{q!}$ for some $p \in \mathbb{Z}$ and $q \in \{1, 2, \ldots\}$. Therefore, for all $n \geq 1$ $F_n(x) = 0$ and thus $\lim_{n \to \infty} F_n(x) = 0$.

Thus, $(F_n) \in \bar{X}^\mathbb{N}$ converges pointwise to $\chi_{\mathbb{R}\setminus\mathbb{Q}}$. Since $\bar{X} = \tilde{X}$, we have $\chi_{\mathbb{R}\setminus\mathbb{Q}} \in \bar{X}$ by the first lemma. Again by the first lemma it follows that $\chi_{\mathbb{R}\setminus\mathbb{Q}}$ is a pointwise limit of continuous functions, but that contradicts the second lemma.