Problem 1 (20 points). Set
\[ F := \{ f : [0, 1] \to \mathbb{R} : f \text{ is Riemann integrable} \}. \]
This set can be given the structure of a real vector space: if \( f, g \in F \) and \( \lambda \in \mathbb{R} \), then the function \( f + \lambda \cdot g \) (which maps \( x \) to \( f(x) + \lambda \cdot g(x) \)) is also Riemann integrable. Moreover, if \( f \in F \), then the function \( |f| \) (which maps \( x \) to \( |f(x)| \)) is also Riemann integrable. The map \( \|\cdot\|_{L^1} : F \to \mathbb{R} \) defined by
\[ \|f\|_{L^1} := \int_0^1 |f(x)| \, dx \]
defines a semi-norm on \( F \), that is, it satisfies all the properties of a norm except \( \|f\|_{L^1} = 0 \) does not necessarily mean that \( f = 0 \). Nevertheless, the sets
\[ B_r(f) := \{ g \in F : \|g - f\|_{L^1} < r \} \]
still form a base of a topology. We denote this topology by \( O_{\|\cdot\|_{L^1}} \).

Define the map \( \int : F \to \mathbb{R} \) by
\[ \int f := \int_0^1 f(x) \, dx. \]
1. Show that \( \int \) is continuous with respect to the topology \( O_{\|\cdot\|_{L^1}} \) on \( F \) (and the standard topology on \( \mathbb{R} \)).
2. Show that \( \int \) is not continuous with respect to the topology of pointwise convergence on \( F \) (that is: the topology \( F \subset \mathbb{R}^{[0,1]} \) inherits from the product topology on \( F \subset \mathbb{R}^{[0,1]} \)).
Problem 2 (20 points). Let \((X, O)\) be a topological space. Denote by \(N^+ = N \cup \{\infty\}\) the one-point compactification of the locally compact Hausdorff space \((N, O_{\text{discrete}})\). Given a sequence \((x_n) \in X^N\) and \(x_\infty \in X\), define a map \(f : N^+ \to X\) by
\[
f(n) := x_n \quad \text{and} \quad f(\infty) = x_\infty.
\]
Show that \(x_\infty\) is a limit of \((x_n)\) if and only if the map \(f\) is continuous.

Problem 3 (20 points). Let \(\{(X_n, O_n) : n \in \mathbb{N}\}\) be a collection of countably many sequentially compact topological spaces. Set
\[
X := \prod_{n \in \mathbb{N}} X_n
\]
and equip \(X\) with the product topology. Show that \(X\) is sequentially compact. (Hint: Use a diagonal sequence argument.)

Problem 4 (20 points).
1. Give an example of a metric space which is not second countable.
2. Give an example of a second countable Hausdorff space which is not metrizable.

Problem 5 (20 points). This problem introduces an alternative perspective on connectedness (which is closely related to Cantor’s original definition of connectedness).

Let \((X, O)\) be a non-empty topological space. Let \(\mathcal{U} \subset O\) be an open cover of \(X\), that is, \(\bigcup \mathcal{U} = X\). We say that \(x \in X\) and \(y \in X\) are \(\mathcal{U}\)-connected if there exists a \(n \in \mathbb{N}\) and open sets \(U_0, U_1, \ldots, U_n \in \mathcal{U}\) such that
\[
x \in U_0, \quad y \in U_n \quad \text{and} \quad U_k \cap U_{k+1} \neq \emptyset \quad \text{for} \quad k = 0, \ldots, n - 1.
\]
We write \(x \sim_{\mathcal{U}} y\) to denote that \(x\) and \(Y\) are \(\mathcal{U}\)-connected. We write \(x \sim y\) if \(x \sim_{\mathcal{U}} y\) for every cover \(\mathcal{U} \subset O\).

1. Show that \(\sim_{\mathcal{U}}\) is an equivalence relation on \(X\).
2. Show that \(\sim\) is an equivalence relation on \(X\).
3. Let \(x \in X\). Denote by \([x]_{\mathcal{U}} \subset X\) the equivalence class of \(x\) with respect to \(\sim_{\mathcal{U}}\). Show that \([x]_{\mathcal{U}}\) is open and closed.
4. Let \(x \in X\). Denote by \([x] \subset X\) the equivalence class of \(x\) with respect to \(\sim\). Show that \([x]\) is closed.
5. Show that \(X\) is connected if and only if \(X/\sim\) has exactly one element.