18.901: Problem Set #6

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This problem set is due on Friday 2017–04–21 at 12n and to be submitted to the box outside of 4-174.

Problem 1 (20 points).

1. Find an example of a topological space that is Hausdorff but not regular.

2. Find an example of a topological space that is regular but not normal.

Problem 2 (20 points). Show that compact Hausdorff topological spaces are normal.

Problem 3 (40 points). Show that second countable regular topological spaces are normal.
Problem 4 (20 points).

Definition. A group is a set $G$ together with an element $e \in G$, a map $m : G \times G \to G$ and a map $i : G \to G$ such that for any $g, h, k \in G$ we have:

\[
m(g, m(h, k)) = m(m(g, h), k) \\
m(e, g) = g \quad \text{and} \quad m(g, e) = g \\
m(g, i(g)) = e \quad \text{and} \quad m(i(g), g) = e.
\]

We call $e$ the unit, $m$ the multiplication, and $i$ the inverse. One also write $m(g, h)$ as $gh$ and $i(g)$ as $g^{-1}$.

Definition. A topological group is a group $(G, e, m, i)$ together with a topology $O$ on $G$ with respect to which the maps $m$ and $i$ are continuous.

Let $(G, e, m, i; O)$ be a topological group.

1. Given $g \in G$, defined $L_g : G \to G$ by $L_g(h) := m(g, h)$. Show that $L_g$ is a homeomorphism.

2. Let $H \subset G$ be a subgroup, that is a subset such that $e \in H$, $g, h \in H \implies gh \in H$, and $g \in H \implies g^{-1} \in H$.
   Show that $\overline{H}$ (the closure of $H$) is also a subgroup.

3. Show that $G$ is Hausdorff if and only if $\{e\}$ is closed.

4. Assuming $G$ is Hausdorff, show that it is, in fact, regular.
   
   Hint: Given $A$ a closed subset of $G$ with $e \notin A$, try to find an open neighborhood $U$ of $e$ such that $U \cap UA = \emptyset$. Here $UA = \{ua : u \in U, a \in A\}$.

5. Although $(G, O)$ is just a topological space, the group structure of $G$ allows us to still makes sense of Cauchy sequences.

Definition. We call $(x_n)$ a Cauchy sequence if for any neighborhood $U$ of $e$, there exist a $N \in \mathbb{N}$ such that for all $n, m \in \{N, N + 1, \ldots\}$ we have $x_n x_m^{-1} \in U$.

Assuming that $G$ is locally compact, show that any Cauchy sequence $(x_n)$ converges.

Hint: Consider the $N$-th tail $T_N := \{x_n : n \in \{N, N + 1, \ldots\}\}$ and the set $\bigcap_{N \in \mathbb{N}} T_N$.  

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