This problem set is due on Friday 2017–04–14 at 12n and to be submitted to the box outside of 4-174.

Problem 1 (20 points). Let \((X, O)\) and \((Y, U)\) be topological spaces and \(f : X \to Y\) and \(g : X \to Y\) be a continuous maps. Suppose \(A \subset X\) is a dense setsuch that \(f(x) = g(x)\) for all \(x \in A\). Assuming that \(Y\) is Hausdorff, show that \(f = g\).

Problem 2 (20 points).
1. Let \((X, O)\) be a topological space and let \(f : X \to \mathbb{R}\) and \(g : X \to \mathbb{R}\) be continuous functions. Show that the set \(\{x \in X : f(x) < g(x)\}\) is open and the set \(\{x \in X : f(x) \leq g(x)\}\) is closed.
2. Show that any continuous map \(f : [0, 1] \to [0, 1]\) has a fixed point, i.e., an \(x \in [0, 1]\) such that \(f(x) = x\). (Hint: Assuming \(f\) does not have a fixed point, construct a separation of \([0, 1]\).)

Problem 3 (20 points). Show that \([0, 1)\) is not homeomorphic to \(S^1\).

Problem 4 (20 points).
1. Let \((X, d)\) be a metric space. Show that \(\bar{d}(x, y) = \min\{d(x, y), 1\}\) is a metric and that \(O_d = O_{\bar{d}}\).
2. Let \(\{(X_n, d_n) : n \in \mathbb{N}\}\) be a countable collection of metric spaces. Show that
\[
d_{\prod}((x_n), (y_n)) := \sum_{n \in \mathbb{N}} 2^{-n} \bar{d}_n(x_n, y_n)
\]
defines a metric on \(X := \prod_{n \in \mathbb{N}} X_n\).
3. Show that the product topology on \(X\) agrees with \(O_{d_{\prod}}\).
Problem 5 (20 points).

Definition. If $X$ is a set and $f : X \to \mathbb{R}$ is a function, then the zero set of $f$ is the set 
$$Z(f) := \{ x \in X : f(x) = 0 \} = f^{-1}(0).$$
We denote the complement of $Z(f)$ by $D(f)$, that is: 
$$D(f) := X \setminus Z(f).$$

Definition. Let $n \in \{1, 2, \ldots \}$. A polynomial in $n$ variables is a map $p : \mathbb{R}^n \to \mathbb{R}$ which can be written as 
$$p(x_1, \ldots, x_n) = \sum_{(k_1, \ldots, k_n) \in \mathbb{N}^n} a_{k_1, \ldots, k_n} x_1^{k_1} \cdots x_n^{k_n}$$
with $a_{k_1, \ldots, k_n} \in \mathbb{R}$ and $a_{k_1, \ldots, k_n} = 0$ for all but finitely many $(k_1, \ldots, k_n) \in \mathbb{N}^n$. We write $\mathbb{R}[x_1, \ldots, x_n]$ for the set of polynomials in $n$ variables.

1. Show that 
$$D(f) \cap D(g) = D(fg).$$
2. Show that 
$$\mathcal{B} = \{ D(p) : p \in \mathbb{R}[x_1, \ldots, x_n] \}$$
is a base of a topology $\mathcal{O} = O_{\text{Zariski}}$ on $\mathbb{R}^n$. This topology is called the Zariski topology.
3. Show that if $U, V \in O_{\text{Zariski}} \setminus \{ \emptyset \}$, then $U \cap V \neq \emptyset$. Deduce that $(\mathbb{R}^n, O_{\text{Zariski}})$ is connected and not Hausdorff.
4. Given $A \subset \mathbb{R}^n$, set 
$$I(A) := \{ p \in \mathbb{R}[x_1, \ldots, x_n] : p(x) = 0 \text{ for all } x \in A \}.$$Show that 
$$\bar{A} = \bigcap_{p \in I(A)} Z(p).$$
5. What is 
$$\{(n, 1/n) : n \in \mathbb{N} \setminus \{0\}\} \subset \mathbb{R}^2?$$
6. If $\mathcal{P} \subset \mathbb{R}[x_1, \ldots, x_n]$ is any set of polynomials in $n$ variables, set 
$$(\mathcal{P}) := \{ q_1 p_1 + \cdots + q_k p_k : k \in \mathbb{N}, p_1, \ldots, p_k \in \mathcal{P}, q_1, \ldots, q_k \in \mathbb{R}[x_1, \ldots, x_n] \}.$$Hilbert’s Basis Theorem states that for any $\mathcal{P} \subset \mathbb{R}[x_1, \ldots, x_n]$ there is a finite $\mathcal{F} \subset \mathcal{P}$ such that 
$$(\mathcal{P}) = (\mathcal{F}).$$Assuming Hilbert’s Basis Theorem, show that every closed subset $A \subset \mathbb{R}^n$ is compact in the Zariski topology.