This problem set is due on 2017–03–03 at 12n and to be submitted to the box outside of 4-174.

**Problem 1** (20 points). Give an example of a non-metrizable topological space and prove that it is non-metrizable.

**Problem 2** (20 points). Let \((X, \mathcal{O})\) be a topological space. Show that the following hold:

1. \(\emptyset\) and \(X\) are closed.
2. If \(A, B \subset X\) are closed, then \(A \cup B\) is closed.
3. If \((A_i)_{i \in I}\) is a collection of closed subsets of \(X\), then \(\bigcap_{i \in I} A_i\) is closed.

**Problem 3** (20 points). We recall the following definition from the lecture.

**Definition 0.1.** Let \(X\) be a set. A family of neighborhoods is a collection of sets \(\mathcal{V} = \{\mathcal{V}_x : x \in X\}\) such that, for each \(x \in X\), \(\mathcal{V}_x \subset \mathcal{P}(X)\) and

1. For all \(V \in \mathcal{V}_x\) we have \(x \in V\) and \(X \in \mathcal{V}_x\).
2. If \(V \subset W \subset X\) and \(V \in \mathcal{V}_x\), then \(W \in \mathcal{V}_x\).
3. If \(V_1, V_2 \in \mathcal{V}_x\), then \(V_1 \cap V_2 \in \mathcal{V}_x\).
4. For any \(V \in \mathcal{V}_x\), there exists a \(W \in \mathcal{V}_x\) such that \(V \supset W\) and for all \(y \in W\) we have \(W \in \mathcal{V}_y\).

Let \(X\) be a set.

1. Given a topology \(\mathcal{O}\) on \(X\), define \(\mathcal{V}^\mathcal{O} = \{\mathcal{V}_x^\mathcal{O} : x \in X\}\) by decreeing that \(V \in \mathcal{V}_x^\mathcal{O}\) if and only if there exists a \(U \in \mathcal{O}\) such that \(x \in U \subset V\). Show that \(\mathcal{V}^\mathcal{O}\) is a family of neighborhoods.
2. Given family of neighborhoods $\mathcal{V}$ on $X$, define $O_{\mathcal{V}} \subseteq \mathcal{P}(X)$ by decreeing that $U \in O_{\mathcal{V}}$ if and only for each $x \in U$ we have $U \in \mathcal{V}_x$. Show that $O_{\mathcal{V}}$ is a topology on $X$.

3. Show that if $O$ is a topology on $X$, then $O_{\mathcal{V}O} = O$.

4. Show that if $\mathcal{V}$ is a family of neighborhoods on $X$, then $\mathcal{V}^{O_{\mathcal{V}}} = \mathcal{V}$.

**Problem 4** (20 points). We recall the following definition from the lecture.

**Definition 0.2.** Let $X$ be a set. A map $\kappa : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is called a Kuratowski closure operator if the following hold for all $Y, Y_1, Y_2 \subseteq X$:

1. $\kappa(\emptyset) = \emptyset$
2. $Y \subseteq \kappa(Y)$ for all $Y \subseteq X$.
3. $\kappa(Y_1 \cup Y_2) = \kappa(Y_1) \cup \kappa(Y_2)$.
4. $\kappa(\kappa(Y)) = \kappa(Y)$.

Let $X$ be a set and let $\kappa$ be a Kuratowski closure operator on $X$.

1. Show that $O_{\kappa} \subseteq \mathcal{P}(X)$ defined by

   $$ O_{\kappa} = \{ U \subseteq X : \kappa(X \setminus U) = X \setminus U \}. $$

   is a topology on $X$.

2. Show that the closure operator induced by $O_{\kappa}$ agrees with $\kappa$.

---

**Problem 5** (20 points). It is a basic fact in arithmetic that there are infinitely many prime numbers, which is said to have been proved by Euclid in 300 BCE. In 1955 Harry Furstenberg published a proof of the infinitude of primes in American Mathematical Monthly using the language of topology.

1. Denote by $\mathcal{B} \subseteq \mathcal{P}(\mathbb{Z})$ the set of arithmetic progressions, that is,

   $$ \mathcal{B} = \{ a\mathbb{Z} + b : a \in \mathbb{Z} \setminus \{0\}, b \in \mathbb{Z} \}. $$

   Show that $\mathcal{B}$ is a basis of a topology. This topology, denoted by $O_F$, is called Furstenberg’s topology.

2. Show that for any $a \in \mathbb{Z} \setminus \{0\}$ the set $a\mathbb{Z}$ is closed.

3. Set $\mathcal{P} := \{ p \in \mathbb{N} : p \text{ is prime} \}$. Show that

   $$ \mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \in \mathcal{P}} \mathbb{Z}_p. $$

4. Show that $\mathcal{P}$ is infinite.