18.901: Problem Set #1

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This problem set is due on 2017–02–16 at 12n and to be submitted to the box outside of 4-174.

Please, read Munkres’ “Comments on Style” before starting to write up your solutions.

Problem 1 (20 points). Let $X$ be a set and let $(A_i)_{i \in I}$ be a family of subsets indexed by $I$. Prove one of the following distributive laws

$$X \cap \left( \bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} (X \cap A_i) \quad \text{and} \quad X \cup \left( \bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} (X \cup A_i),$$

and prove one of the following De Morgan’s laws

$$X \setminus \left( \bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} X \setminus A_i \quad \text{and} \quad X \setminus \left( \bigcap_{i \in I} A_i \right) = \bigcup_{i \in I} X \setminus A_i.$$

Problem 2 (20 points). Fill in the blanks with the symbols on the right in order to make strongest possible true statement.

1. $A \subset B$ and $A \subset C$ ___ $A \subset (B \cap C)$ ___ $\Rightarrow \Leftarrow \Leftarrow$ none
2. $A \subset B$ or $A \subset C$ ___ $A \subset (B \cup C)$ ___ $\Rightarrow \Leftarrow \Leftarrow$ none
3. $A \subset B$ or $A \subset C$ ___ $A \subset (B \cap C)$ ___ $\Rightarrow \Leftarrow \Leftarrow$ none
4. $A \setminus (A \setminus B)$ ___ $B$ ___ $\subset \supset$ none
5. $A \setminus (B \setminus A)$ ___ $A \setminus B$ ___ $\subset \supset$ none
6. $A \cap (B \setminus C)$ ___ $(A \cap B) \setminus (A \cap C)$ ___ $\subset \supset$ none
7. $A \cup (B \setminus C)$ ___ $(A \cup B) \setminus (A \cup C)$ ___ $\subset \supset$ none
8. $A \subset C$ and $B \subset D$ ___ $(A \times B) \subset (C \times D)$ ___ $\Rightarrow \Leftarrow \Leftarrow$ none
9. $(A \times B) \cup (C \times D)$ ___ $(A \cup C) \times (B \cup D)$ ___ $\subset \supset$ none
10. $(A \times B) \setminus (C \times D)$ ___ $(A \setminus C) \times (B \setminus D)$ ___ $\subset \supset$ none
Problem 3 (20 points). Let $X$ and $Y$ be sets, and let $f : X \to Y$ be a map. Fill in the blanks with $\subset / \supset$ and injective/surjective to make the following statements true and then prove those statements.

1. For each $A \subset X$, we have $f^{-1}(f(A))$ ____ $A$; and equality holds if $f$ is ____.
2. For each $B \subset Y$, we have $f(f^{-1}(B))$ ____ $B$; and equality holds if $f$ is ____.

Problem 4 (20 points). Let $A$ and $B$ be two sets.

1. Denote by $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ the projection maps defined by
   
   $\pi_A(a, b) := a$ and $\pi_B(a, b) := b$.

   Prove that for any set $C$ the map
   
   $\text{Map}(C, A \times B) \to \text{Map}(C, A) \times \text{Map}(C, B) \quad f \mapsto (\pi_A \circ f, \pi_B \circ f)$

   is bijective. **Hint:** Directly define an inverse map.

   **Remark.** This means that any a map into a product is essentially the same as the maps to the factors of the product. This is called the **universal property** of the product.

2. Recall that $A \sqcup B = A \times \{0\} \cup A \times \{1\}$. Define $\iota_A : A \to A \sqcup B$ and $\iota_B : B \to A \sqcup B$ by

   $\iota_A(a) := (a, 0)$ and $\iota_A(b) := (b, 1)$.

   Prove that for any set $C$ the map

   $\text{Map}(A \sqcup B, C) \to \text{Map}(A, C) \times \text{Map}(B, C) \quad f \mapsto (f \circ \iota_A, f \circ \iota_B)$

   is bijective.

   **Remark.** This means that a map from a disjoint union is essentially the same as a map from each summand. This is called the **universal property** of the disjoint union.

Problem 5 (20 points). Prove the following theorem.

**Theorem** (Schröder–Bernstein). Let $A$ and $B$ be two sets. If there are injective maps $f : A \to B$ and $g : B \to A$, then there is a bijective map $h : A \to B$.

**Hint:** There are many ways to prove this and you might want to browse the literature to find one you like. Here is one strategy that will work:

0. Recursively define $C_n \subset A$ for $n = 0, 1, 2, \ldots$ by

   $C_0 := A \setminus g(B)$ and $C_{n+1} := g(f(C_n))$.

   Set

   $C := \bigcup_{n \in \mathbb{N}} C_n$.  

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1. Prove that \( C = C_0 \cup g(f(C)) \).

2. Prove that \( A \setminus C = g(B \setminus f(C)) \).

3. Use the previous step to define the desired map \( h \) (by a certain rule on \( C \) and a different rule on \( A \setminus C \)). Don’t forget to prove that \( h \) is bijective.

After you have finished this problem set, you might also want to watch Serre’s talk on “How to write mathematics badly”.