Proposition 3.43. If $(X,d)$ is a metric space, then $(X,O_d)$ is compact if and only if it is sequentially compact.

Proof. Suppose that $X$ is compact. Let $(x_i) \in X^\mathbb{N}$ be a sequence. Let $A_I = \{x_i : i \in I + \mathbb{N}\}$. Note that if $I \subseteq J$, then $A_I \supseteq A_J$. We show that $\bigcap_{I \in \mathbb{N}} A_I$ is non-empty. If it were empty, then $\{U_I := X \setminus A_I : i \in \mathbb{N}\}$ would an open cover and would thus have a finite subcover. But this means that for some $I \ni 1$, $U_I = X$, which is non-sense. Let $x \in \bigcap_{I \in \mathbb{N}} A_I$. For each $j \in \{1,2,\ldots\}$ and each $I \geq 0$, there exists an $i \geq 1$ such that $x_i \in B_{1/j}(x)$. A diagonal sequence argument constructs a sequence converging to $x$: Pick $i_j$ so that $x_{i_j} \in B_{1/j}(X)$; then $\lim_{j \to \infty} x_{i_j} = x$.

For the converse, suppose $X$ is sequentially compact. Let $\{U_i : i \in I\}$ be an open cover.

Proposition. There is an $\varepsilon > 0$ such that for any $x \in X$ there is an $i \in I$ such that $B_{\varepsilon/2}(x) \subset U_i$.

Suppose not. Then for any $n \in 1 + \mathbb{N}$ we can find an $x_i$ such that $B_{1/n}(x_n)$ is not contained in any of the $U_i$. By sequential compactness a subsequence $(x_{n_m})$ converges, say to $x \in X$. Since $\{U_i : i \in I\}$ is a cover there is an $i \in I$ such that $x \in U_i$. Since $U_i$ is open, there is an $\varepsilon > 0$ such that $B_{\varepsilon}(x) \subset U_i$. For $m$ sufficiently large $1/n_m < \varepsilon/2$ and since $x = \lim x_{n_m}$ we can also assume that $x_{n_m} \in B_{\varepsilon/2}(x)$. But then by the triangle inequality

$$B_{1/n_m}(x_{n_m}) \subset B_{\varepsilon/2}(x_{n_m}) \subset B_{\varepsilon}(x) \subset U_i.$$  

This contradicts the choice of $x_{n_m}$ however.

Proposition. Let $\varepsilon > 0$ be as in the proposition. There is a finite set $\{x_n : n = 1, \ldots, N\}$ such that $X = \bigcup_n B_{\varepsilon}(x_n)$. In particular, if $i_n \in I$ is such that $B_{\varepsilon}(x_n) \subset U_{i_n}$, then $\{U_{i_n} : n = 1, \ldots, N\}$ is a finite subcover.

Pick $x_1 \in X$. If $B_{\varepsilon}(x_1) = X$, we are done. If not, pick $x_2 \in X \setminus B_{\varepsilon}(x_1)$. Again, if $B_{\varepsilon}(x_1) \cup B_{\varepsilon}(x_2) = X$, we are done. Otherwise, keep going and pick

$$x_{n+1} \in X \setminus \bigcup_{k=1,\ldots,n} B_{\varepsilon}(x_k).$$

If this process terminates, we are done. If not, we get a sequence $(x_n)$. By sequential compactness, a subsequence $(x_{n_m})$ has a limit, say, $x$. Since $(x_{n_m})$ converges it is Cauchy, thus for $m < k$ sufficiently large $d(x_{n_m}, x_{n_k}) \leq \varepsilon/2$; but that contradicts the choice of $x_{n_k}$. \qed

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