Semi-classical analysis

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Chapter 1

Introduction.

Let $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ with coordinates $(x^1, \ldots, x^n, t)$. Let

$$P = P\left(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)$$

be a k-th order partial differential operator. Suppose that we want to solve the partial differential equation

$$Pu = 0$$

with initial conditions

$$u(x,0) = \delta_0, \quad \frac{\partial^i}{\partial t^i} u(x,0) = 0, \quad i = 1, \ldots, k-1.$$

Let ρ be a C^∞ function of x of compact support which is identically one near the origin. We can write

$$\delta_0(x) = \frac{1}{(2\pi)^n} \rho(x) \int_{\mathbb{R}^n} e^{ix\cdot\xi} d\xi.$$

Let us introduce polar coordinates in ξ space:

$$\xi = \omega \cdot r, \quad \|\omega\| = 1, \quad r = \|\xi\|$$

so we can rewrite the above expression as

$$\delta_0(x) = \frac{1}{(2\pi)^n} \rho(x) \int_{\mathbb{R}_+} \int_{S^{n-1}} e^{i(x\cdot\omega)r} r^{n-1} dr d\omega$$

where $d\omega$ is the measure on the unit sphere S^{n-1}. This shows that we are interested in solving the partial differential equation $Pu = 0$ with the initial conditions

$$u(x,0) = \rho(x) e^{i(x\cdot\omega)r} r^{n-1}, \quad \frac{\partial^i}{\partial t^i} u(x,0) = 0, \quad i = 1, \ldots, k-1.$$

1.1 The problem.

More generally, set

$$r = \hbar^{-1}$$

and let

$$\psi \in C^\infty(\mathbb{R}^n).$$

We look for solutions of the partial differential equation with initial conditions

$$Pu(x,t) = 0, \quad u(x,0) = \rho(x)e^{i\frac{\psi(x)}{\hbar}}\hbar^{-\ell} \quad \frac{\partial^i}{\partial t^i}u(x,0) = 0, \quad i = 1, \dots, k-1. \tag{1.1}$$

1.2 The eikonal equation.

Look for solutions of (1.1) of the form

$$u(x,t) = a(x,t,\hbar)e^{i\phi(x,t)/\hbar} \tag{1.2}$$

where

$$a(x,t,\hbar) = \hbar^{-\ell}\sum_{i=0}^{\infty} a_i(x,t)\hbar^i. \tag{1.3}$$

1.2.1 The principal symbol.

Define the **principal symbol** $H(x,t,\xi,\tau)$ of the differential operator P by

$$\hbar^k e^{-i\frac{x.\xi+t\tau}{\hbar}} P e^{i\frac{x.\xi+t\tau}{\hbar}} = H(x,t,\xi,\tau) + O(\hbar). \tag{1.4}$$

We think of H as a function on $T^*\mathbb{R}^{n+1}$.

If we apply P to $u(x,t) = a(x,t,\hbar)e^{i\phi(x,t)/\hbar}$, then the term of degree $\hbar^{-k}$ is obtained by applying all the differentiations to $e^{i\phi(x,t)/\hbar}$. In other words,

$$\hbar^k e^{-i\phi/\hbar} Pa(x,t)e^{i\phi/\hbar} = H\left(x,t,\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial t}\right)a(x,t) + O(\hbar). \tag{1.5}$$

So as a first step we must solve the first order non-linear partial differential equation

$$H\left(x,t,\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial t}\right) = 0 \tag{1.6}$$

for ϕ. Equation (1.6) is known as the **eikonal equation** and a solution ϕ to (1.6) is called an **eikonal**. The Greek word eikona $\epsilon\iota\kappa\omega\nu\alpha$ means image.

1.2.2 Hyperbolicity.

For all (x,t,ξ) the function

$$\tau \mapsto H(x,t,\xi,\tau)$$

is a polynomial of degree (at most) k in τ. We say that P is **hyperbolic** if this polynomial has k distinct real roots

$$\tau_i = \tau_i(x,t,\xi).$$

These are then smooth functions of (x,t,ξ).

We assume from now on that P is hyperbolic. For each $i=1,\dots,k$ let

$$\Sigma_i \subset T^*\mathbb{R}^{n+1}$$

be defined by

$$\Sigma_i = \{(x,0,\xi,\tau) | \xi = d_x\psi,\ \tau = \tau_i(x,0,\xi)\} \tag{1.7}$$

where ψ is the function occurring in the initial conditions in (1.1). The classical method for solving (1.6) is to reduce it to solving a system of ordinary differential equations with initial conditions given by (1.7). We recall the method:

1.2.3 The canonical one form on the cotangent bundle.

If X is a differentiable manifold, then its cotangent bundle T^*X carries a **canonical one form** $\alpha = \alpha_X$ defined as follows: Let

$$\pi : T^*X \to X$$

be the projection sending any covector $p \in T^*_xX$ to its base point x. If $v \in T_p(T^*X)$ is a tangent vector to T^*X at p, then

$$d\pi_p v$$

is a tangent vector to X at x. In other words, $d\pi_p v \in T_xX$. But $p \in T^*_xX$ is a linear function on T_xX, and so we can evaluate p on $d\pi_p v$. The canonical linear differential form α is defined by

$$\langle \alpha, v\rangle := \langle p, d\pi_p v\rangle \quad \text{if} \quad v \in T_p(T^*X). \tag{1.8}$$

For example, if our manifold is $\mathbb{R}^{n+1}$ as above, so that we have coordinates (x,t,ξ,τ) on $T^*\mathbb{R}^{n+1}$ the canonical one form is given in these coordinates by

$$\alpha = \xi \cdot dx + \tau dt = \xi_1 dx^1 + \cdots \xi_n dx^n + \tau dt. \tag{1.9}$$

1.2.4 The canonical two form on the cotangent bundle.

This is defined as

$$\omega_X = -d\alpha_X. \tag{1.10}$$

Let $q^1,\ldots,q^n$ be local coordinates on X. Then $dq^1,\ldots,dq^n$ are differential forms which give a basis of T^*_xX at each x in the coordinate neighborhood U. In other words, the most general element of T^*_xX can be written as $p_1(dq^1)_x+\cdots+p_n(dq^n)_x$. Thus $q^1,\ldots,q^n,p_1,\ldots,p_n$ are local coordinates on

$$\pi^{-1}U\subset T^*X.$$

In terms of these coordinates the canonical one form is given by

$$\alpha = p\cdot dq = p_1dq^1+\cdots p_ndq^n$$

Hence the canonical two form has the local expression

$$\omega = dq\wedge\cdot dp = dq^1\wedge dp_1+\cdots+dq^n\wedge dp_n. \tag{1.11}$$

The form ω is closed and is of maximal rank, i.e. ω defines an isomorphism between the tangent space and the cotangent space at every point of T^*X.

1.2.5 Symplectic manifolds.

A two form which is closed and is of maximal rank is called **symplectic**. A manifold M equipped with a symplectic form is called a **symplectic manifold**. We shall study some of the basic geometry of symplectic manifolds in Chapter 2. But here are some elementary notions which follow directly from the definitions: A diffeomorphism $f: M\to M$ is called a **symplectomorphism** if $f^*\omega=\omega$. More generally if (M,ω) and (M',ω') are symplectic manifolds then a diffeomorphism

$$f: M\to M'$$

is called a symplectomorphism if

$$f^*\omega'=\omega.$$

If v is a vector field on M, then the general formula for the Lie derivative of a differential form Ω with respect to v is given by

$$D_v\Omega = i(v)d\Omega + di(v)\Omega.$$

This is known as Weil's identity. See (**??**) in Chapter **??** below. If we take Ω to be a symplectic form ω , so that $d\omega=0$, this becomes

$$D_\xi\omega = di(v)\omega.$$

So the flow $t\mapsto \exp tv$ generated by v consists of symplectomorphisms if and only if

$$di(v)\omega = 0.$$

1.2.6 Hamiltonian vector fields.

In particular, if H is a function on a symplectic manifold M, then the **Hamiltonian vector field** v_H associated to H and defined by

$$i(v_H)\omega = -dH \tag{1.12}$$

satisfies

$$(\exp tv_H)^*\omega = \omega.$$

Also

$$D_{v_H}H = i(v_H)dH = -i(v_H)i(v_H)\omega = \omega(v_H.v_H) = 0.$$

Thus

$$(\exp tv_H)^*H = H. \tag{1.13}$$

So the flow $\exp tv_H$ preserves the level sets of H. In particular, it carries the zero level set - the set $H = 0$ - into itself.

1.2.7 Isotropic submanifolds.

A submanifold Y of a symplectic manifold is called **isotropic** if the restriction of the symplectic form ω to Y is zero. So if

$$\iota_Y : \ Y \to M$$

denotes the injection of Y as a submanifold of M, then the condition for Y to be isotropic is

$$\iota_Y^*\omega = 0$$

where ω is the symplectic form of M.

For example, consider the submanifold Σ_i of $T^*(\mathbb{R}^{n+1})$ defined by (1.7). According to (1.9), the restriction of $\alpha_{\mathbb{R}^{n+1}}$ to Σ_i is given by

$$\frac{\partial\psi}{\partial x_1}dx_1 + \cdots \frac{\partial\psi}{\partial x_n}dx_n = d_x\psi$$

since $t \equiv 0$ on Σ_i. So

$$\iota_{\Sigma_i}^*\omega_{\mathbb{R}^{n+1}} = -d_xd_x\psi = 0$$

and hence Σ_i is isotropic.

Let H be a smooth function on a symplectic manifold M and let Y be an isotropic submanifold of M contained in a level set of H. For example, suppose that

$$H_{|Y} \equiv 0. \tag{1.14}$$

Consider the submanifold of M swept out by Y under the flow $\exp tv_\xi$. More precisely suppose that

- v_H is transverse to Y in the sense that for every $y \in Y$, the tangent vector $v_H(y)$ does *not* belong to T_yY and

- there exists an open interval I about 0 in $\mathbb{R}$ such that $\exp tv_H(y)$ is defined for all $t \in I$ and $y \in Y$.

We then get a map

$$j : Y \times I \to M, \quad j(y,t) := \exp tv_H(y)$$

which allows us to realize $Y \times I$ as a submanifold Z of M. The tangent space to Z at a point (y,t) is spanned by

$$(\exp tv_H)_* TY_y \quad \text{and} \quad v_H(\exp tv_H y)$$

and so the dimension of Z is $\dim Y + 1$.

Proposition 1 *With the above notation and hypotheses, Z is an isotropic submanifold of M.*

Proof. We need to check that the form ω vanishes when evaluated on

1. two vectors belonging to $(\exp tv_H)_* TY_y$ and
2. $v_H(\exp tv_H y)$ and a vector belonging to $(\exp tv_H)_* TY_y$.

For the first case observe that if $w_1, w_2 \in T_y Y$ then

$$\omega((\exp tv_H)_* w_1, (\exp tv_H)_* w_2) = (\exp tv_H)^* \omega(w_1, w_2) = 0$$

since

$$(\exp tv_H)^* \omega = \omega$$

and Y is isotropic.

For the second case observe that $i(v_H)\omega = -dH$ and so for $w \in T_y Y$ we have

$$\omega((\exp tv_H)_* w, i(v_H(\exp tv_H y)) = dH(w) = 0$$

since H is constant on Y. □

If we consider the function H arising as the symbol of a hyperbolic equation, i.e. the function H given by (1.4) then H is a homogeneous polynomial in ξ and τ of the form $b(x,t,\xi)\prod_i(\tau - \tau_i)$, with $b \neq 0$ so

$$\frac{\partial H}{\partial \tau} \neq 0 \quad \text{along} \quad \Sigma_i.$$

But the coefficient of $\partial/\partial t$ in v_H is $-\partial H/\partial \tau$. Now $t \equiv 0$ along Σ_i so v_H is transverse to Σ_i. Our transversality condition is satisfied. We can arrange that the second of our conditions, the existence of solutions for an interval I can be satisfied locally. (In fact, suitable compactness conditions that are frequently satisfied will guarantee the existence of global solutions.)

Thus, at least locally, the submanifold of $T^*(\mathbb{R}^{n+1})$ swept out from Σ_i by $\exp tv_H$ is an $n+1$ dimensional isotropic submanifold.

1.2.8 Lagrangian submanifolds.

A submanifold of a symplectic manifold which is isotropic and whose dimension is one half the dimension of M is called **Lagrangian**. We shall study Lagrangian submanifolds in detail in Chapter 2. Here we shall show how they are related to our problem of solving the eikonal equation (1.6).

The submanifold Σ_i of $T^*\mathbb{R}^{n+1}$ is isotropic and of dimension n. It is transversal to v_H. Therefore the submanifold Λ_i swept out by Σ_i under $\exp tv_H$ is Lagrangian. Also, near $t = 0$ the projection

$$\pi : T^*\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

when restricted to Λ_i is (locally) a diffeomorphism. It is (locally) **horizontal** in the sense of the next section.

1.2.9 Lagrangian submanifolds of the cotangent bundle.

To say that a submanifold $\Lambda \subset T^*X$ is Lagrangian means that Λ has the same dimension as X and that the restriction to Λ of the canonical one form α_X is closed.

Suppose that Z is a submanifold of T^*X and that the restriction of $\pi : T^*X \to X$ to Z is a diffeomorphism. This means that Z is the image of a section

$$s : \ X \to T^*X.$$

Such a section is the same as assigning a covector at each point of X, in other words it is a linear differential form. For the purposes of the discussion we temporarily introduce a redundant notation and call the section s by the name β_s when we want to think of it as a linear differential form. We claim that

$$s^*\alpha_X = \beta_s.$$

Indeed, if $w \in T_xX$ then $d\pi_{s(x)} \circ ds_x(w) = w$ and hence

$$s^*\alpha_X(w) = \langle (\alpha_X)_{s(x)}, ds_x(w)\rangle = \langle s(x), d\pi_{s(x)} ds_x(w)\rangle = \langle s(x), w\rangle = \beta_s(x)(w).$$

Thus the submanifold Z is Lagrangian if and only if $d\beta_s = 0$. Let us suppose that X is connected and simply connected. Then $d\beta = 0$ implies that $\beta = d\phi$ where ϕ is determined up to an additive constant.

With some slight abuse of language, let us call a Lagrangian submanifold of T^*X **horizontal** if the restriction of $\pi : T^*X \to X$ to Λ is a diffeomorphism. We have proved

Proposition 2 *Suppose that X is connected and simply connected. Then every horizontal Lagrangian submanifold of T^*X is given by a section γ_ϕ : $X \to T^*X$ where γ_ϕ is of the form*

$$\gamma_\phi(x) = d\phi(x)$$

where ϕ is a smooth function determined up to an additive constant.

1.2.10 Local solution of the eikonal equation.

We have now found a local solution of the eikonal equation! Starting with the initial conditions Σ_i given by (1.7) at $t = 0$, we obtain the Lagrangian submanifold Λ_i. Locally (in x and in t near zero) the manifold Λ_i is given as the image of γ_{ϕ_i} for some function ϕ_i. The fact that Λ_i is contained it the set $H = 0$ then implies that ϕ_i is a solution of (1.6).

1.2.11 Caustics.

What can go wrong globally? One problem that might arise is with integrating the vector field v_H. As is well known, the existence theorem for non-linear ordinary differential equations is only local - solutions might "blow up" in a finite interval of time. In many applications this is not a problem because of compactness or boundedness conditions. A more serious problem - one which will be a major concern of this book - is the possibility that after some time the Lagrangian manifold is no longer horizontal.

If $\Lambda \subset T^*X$ is a Lagrangian submanifold, we say that a point $m \in \Lambda$ is a **caustic** if

$$d\pi_m T_m\Lambda \to T_xX. \quad x = \pi(m)$$

is *not* surjective. A key ingredient in what we will need to do is to describe how to choose convenient parametrizations of a Lagrangian manifolds near caustics. The first person to deal with this problem (through the introduction of so-called "angle characteristics") was Hamilton (1805-1865) in a paper he communicated to Dr. Brinkley in 1823, by whom, under the title "Caustics" it was presented in 1824 to the Royal Irish Academy.

We shall deal with caustics in a more general manner, after we have introduced some categorical language.

1.3 The transport equations.

Let us return to our project of looking for solutions of the form (1.2) to the partial differential equation and initial conditions (1.1). Our first step was to find the Lagrangian manifold $\Lambda = \Lambda_\phi$ which gave us, locally, a solution of the eikonal equation (1.6). This determines the "phase function" ϕ up to an overall additive constant, and also guarantees that no matter what $a_i's$ enter into the expression for u given by (1.2) and (1.3), we have

$$Pu = O(\hbar^{-k-\ell+1}).$$

The next step is obviously to try to choose a_0 in (1.3) such that

$$P\left(a_0 e^{i\phi(x,t)/\hbar}\right) = O(\hbar^{-k+2}).$$

In other words, we want to choose a_0 so that there are no terms of order $\hbar^{-k+1}$ in $P\left(a_0 e^{i\phi(x,t)/\hbar}\right)$. Such a term can arise from three sources:

1. We can take the terms of degree $k-1$ and apply all the differentiations to $e^{i\phi/\hbar}$ with none to a or to ϕ. We will obtain an expression C similar to the principal symbol but using the operator Q obtained from P by eliminating all terms of degree k. This expression C will then multiply a_0.

2. We can take the terms of degree k in P, apply all but one differentiation to $e^{i\phi/\hbar}$ and the remaining differentiation to a partial derivative of ϕ. The resulting expression B will involve the second partial derivatives of ϕ. This expression will also multiply a_0.

3. We can take the terms of degree k in P, apply all but one differentiation to $e^{i\phi/\hbar}$ and the remaining differentiation to a_0. So we get a first order differential operator
$$\sum_{i=1}^{n+1} A_i \frac{\partial}{\partial x_i}$$
applied to a_0. In the above formula we have set $t = x_{n+1}$ so as to write the differential operator in more symmetric form.

So the coefficient of $\hbar^{-k+1}$ in $P\left(a_0 e^{i\phi(x,t)/\hbar}\right)$ is
$$(Ra_0)\, e^{i\phi(x,t)/\hbar}$$
where R is the first order differential operator
$$R = \sum A_i \frac{\partial}{\partial x_i} + B + C.$$
We will derive the explicit expressions for the A_i, B and C below.

The strategy is the to look for solutions of the first order homogenous linear partial differential equation
$$Ra_0 = 0.$$
This is known as the **first order transport equation**.

Having found a_0, we next look for a_1 so that
$$P\left((a_0 + a_1\hbar)e^{i\phi/\hbar}\right) = O(h^{-k+3}).$$
From the above discussion it is clear that this amounts to solving and inhomogeneous linear partial differential equation of the form
$$Ra_1 = b_0$$
where b_0 is the coefficient of $\hbar^{-k+2}e^{i\phi/\hbar}$ in $P(a_0 e^{i\phi/\hbar})$ and where R is the same operator as above. Assuming that we can solve all the equations, we see that we have a recursive procedure involving the operator R for solving (1.1) to all orders, at least locally - up until we hit a caustic!

We will find that when we regard P as acting on $\frac{1}{2}$-densities (rather than on functions) then the operator R has an invariant (and beautiful) expression as a differential operator acting on $\frac{1}{2}$-densities on Λ. In fact, the differentiation part of the differential operator will be given by the vector field v_H which we know to be tangent to Λ. The differential operator on Λ will be defined even at caustics. This fact will be central in our study of global asymptotic solutions of hyperbolic equations.

In the next section we shall assume only the most elementary facts about $\frac{1}{2}$-densities - the fact that the product of two $\frac{1}{2}$-densities is a density and hence can be integrated if this product has compact support. Also that the concept of the Lie derivative of a $\frac{1}{2}$-density with respect to a vector field makes sense. If the reader is unfamiliar with these facts they can be found with many more details in Chapter 6.

1.3.1 A formula for the Lie derivative of a $\frac{1}{2}$-density.

We want to consider the following situation: H is a function on T^*X and Λ is a Lagrangian submanifold of T^*X on which $H = 0$. This implies that the corresponding Hamiltonian vector field is tangent to Λ. Indeed, for any $w \in T_m\Lambda$ we have

$$\omega_X(v_H, w) = -dH(w) = 0$$

since H is constant on Λ. Since Λ is Lagrangian, this implies that $v_H(m) \in T_m(\Lambda)$.

If τ is a smooth $\frac{1}{2}$-density on Λ, we can consider its Lie derivative with respect to the vector field v_H restricted to Λ. We want an explicit formula for this Lie derivative in terms of local coordinates on X on a neighborhood over which Λ is horizontal.

Let

$$\iota : \Lambda \to T^*X$$

denote the embedding of Λ as submanifold of X so we are assuming that

$$\pi \circ \iota : \Lambda \to X$$

is a diffeomorphism. (We have replaced X by the appropriate neighborhood over which Λ is horizontal and on which we have coordinates $x^1, \ldots, x^m$.) We let $dx^{\frac{1}{2}}$ denote the standard $\frac{1}{2}$-density relative to these coordinates. Let a be a function on X, so that

$$\tau := (\pi \circ \iota)^* \left(a dx^{\frac{1}{2}} \right)$$

is a $\frac{1}{2}$-density on Λ, and the most general $\frac{1}{2}$-density on Λ can be written in this form. Our goal in this section is to compute the Lie derivative $D_{v_H}\tau$ and express it in a similar form. We will prove:

Proposition 3 *If* $\Lambda = \Lambda_\phi = \gamma_\phi(X)$ *then*

$$D_{v_H|\Lambda}(\pi \circ \iota)^* \left(a dx^{\frac{1}{2}}\right) = b(\pi \circ \iota)^* \left(dx^{\frac{1}{2}}\right)$$

where

$$b = D_{v_H} a + \left[\frac{1}{2}\sum_{i,j} \frac{\partial^2 H}{\partial \xi_i \partial \xi_j} \frac{\partial^2 \phi}{\partial x^i \partial x^j} + \frac{1}{2}\sum_i \frac{\partial^2 H}{\partial \xi_i \partial x^i}\right] a. \tag{1.15}$$

Proof. Since $D_v(f\tau) = (D_v f)\tau + f D_v \tau$ for any vector field v, function f and any $\frac{1}{2}$-density τ, it suffices to prove (1.15) for the case the $a \equiv 1$ in which case the first term disappears. By Leibnitz's rule,

$$D_{v_H}(\pi \circ \iota)^* \left(dx^{\frac{1}{2}}\right) = \frac{1}{2} c (\pi \circ \iota)^* \left(dx^{\frac{1}{2}}\right)$$

where

$$D_{v_H}(\pi \circ \iota)^* |dx| = c(\pi \circ \iota)^* |dx|.$$

Here we are computing the Lie derivative of the density $(\pi \circ \iota)^*|dx|$, but we get the same function c if we compute the Lie derivative of the m-form

$$D_{v_H}(\pi \circ \iota)^*(dx^1 \wedge \cdots \wedge dx^m) = c(\pi \circ \iota)^*(dx^1 \wedge \cdots \wedge dx^m).$$

Now $\pi^*(dx^1 \wedge \cdots \wedge dx^m)$ is a well defined m-form on T^*X and

$$D_{v_H|\Lambda}(\pi \circ \iota)^*(dx^1 \wedge \cdots \wedge dx^m) = \iota^* D_{v_H} \pi^*(dx^1 \wedge \cdots \wedge dx^m).$$

We may write dx^j instead of $\pi^* dx^j$ with no risk of confusion and we get

$$\begin{aligned}
D_{v_H}(dx^1 \wedge \cdots \wedge dx^m) &= \sum_j dx^1 \wedge \cdots \wedge d(i(v_H) dx^j \wedge \cdots \wedge dx^m \\
&= \sum_j dx^1 \wedge \cdots \wedge d\frac{\partial H}{\partial \xi_j} \wedge \cdots \wedge dx^m \\
&= \sum_j \frac{\partial^2 H}{\partial \xi_j \partial x^j} dx^1 \wedge \cdots \wedge dx^m + \\
&\qquad \sum_{jk} dx^1 \wedge \cdots \wedge \frac{\partial^2 H}{\partial \xi_j \partial \xi_k} d\xi_k \wedge \cdots \wedge dx^m.
\end{aligned}$$

We must apply ι^* which means that we must substitute $d\xi_k = d\left(\frac{\partial \phi}{\partial x^k}\right)$ into the last expression. We get

$$c = \sum_{i,j} \frac{\partial^2 H}{\partial \xi_i \partial \xi_j} \frac{\partial^2 \phi}{\partial x^i \partial x^j} + \sum_i \frac{\partial^2 H}{\partial \xi_i \partial x^i}$$

proving (1.15). $\square$

1.3.2 The total symbol, locally.

Let U be an open subset of $\mathbb{R}^m$ and $x_1, \ldots x_m$ the standard coordinates. We will let D_j denote the differential operator

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j} = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_j}.$$

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$ where the α_j are non-negative integers, we let

$$D^\alpha := D_1^{\alpha_1} \cdots D_m^{\alpha_m}$$

and

$$|\alpha| := \alpha_1 + \cdots + \alpha_m.$$

So the most general k-th order linear differential operator P can be written as

$$P = P(x, D) = \sum_{|\alpha| \le k} a_\alpha(x) D^\alpha.$$

The **total symbol** of P is defined as

$$e^{-i\frac{x\cdot\xi}{\hbar}} P e^{i\frac{x\cdot\xi}{\hbar}} = \sum_{r=0}^{k} \hbar^{-k} p_j(x, \xi)$$

so that

$$p_j(x, \xi) = \sum_{|\alpha| = j} a_\alpha(x) \xi^\alpha. \tag{1.16}$$

So p_k is exactly the principal symbol as defined in (1.4).

Since we will be dealing with operators of varying orders, we will denote the principal symbol of P by

$$\sigma(P).$$

We should emphasize that the definition of the total symbol is heavily coordinate dependent: If we make a non-linear change of coordinates, the expression for the total symbol in the new coordinates will not look like the expression in the old coordinates. However the principal symbol does have an invariant expression as a function on the cotangent bundle which a polynomial in the fiber variables.

1.3.3 The transpose of P.

We continue our study of linear differential operators on an open subset $U \subset \mathbb{R}^n$. If f and g are two smooth functions of compact support on U then

$$\int_U (Pf) g dx = \int_U f P^t g dx$$

where, by integration by parts,

$$P^t g = \sum (-1)^{|\alpha|} D^\alpha (a_\alpha g).$$

(Notice that in this definition, following convention, we are using g and not $\overline{g}$ in the definition of P^t.) Now

$$D^\alpha (a_\alpha g) = a_\alpha D^\alpha g + \cdots$$

where the $\cdots$ denote terms with fewer differentiations in g. In particular, the principal symbol of P^t is

$$p_k^t(x,\xi) = (-1)^k p_k(x,\xi). \tag{1.17}$$

Hence the operator

$$Q := \frac{1}{2}(P - (-1)^k P^t) \tag{1.18}$$

is of order $= k-1$ The **sub-principal symbol** is defined as the principal symbol of Q (considered as an operator of degree $(k-1)$). So

$$\sigma_{sub}(P) := \sigma(Q)$$

where Q is given by (1.18).

1.3.4 The formula for the sub-principal symbol.

We claim that

$$\sigma_{sub}(P)(x,\xi) = p_{k-1}(x,\xi) + \frac{i}{2} \sum_i \frac{\partial^2}{\partial x_i \partial \xi_i} p_k(x,\xi). \tag{1.19}$$

Proof. If $p_k(x,\xi) \equiv 0$, i.e. if P is actually an operator of degree $k-1$, then it follows from (1.18) that the principal symbol of Q is p_{k-1} which is the first term on the right in (1.19). So it suffices to prove (1.19) for operators which are strictly of order k. By linearity, it suffices to prove (1.19) for operators of the form

$$a_\alpha(x) D^\alpha.$$

By polarization it suffices to prove (1.19) for operators of the form

$$a(x)D^k, \quad D = \sum_{j=1}^k c_j D_j, \qquad c_i \in \mathbb{R}$$

and then, by making a linear change of coordinates, for an operator of the form

$$a(x) D_1^k.$$

For this operator

$$p_k(x,\xi) = a(x)\xi_1^k.$$

By Leibnitz's rule,

$$\begin{aligned}
P^t f &= (-1)^k D_1^k(af) \\
&= (-1)^k \sum_j \binom{k}{j} D_1^j a D_1^{k-j} f \\
&= (-1)^k \left(a D_1^k f + \frac{k}{i} \left(\frac{\partial a}{\partial x_1} \right) D^{k-1} + \cdots \right) \quad \text{so} \\
Q &= \frac{1}{2}(P - (-1)^k P^t) \\
&= -\frac{k}{2i} \left(\frac{\partial a}{\partial x_1} D^{k-1} + \cdots \right)
\end{aligned}$$

and therefore

$$\begin{aligned}
\sigma(Q) &= \frac{ik}{2} \frac{\partial a}{\partial x_1} \xi_1^{k-1} \\
&= \frac{i}{2} \frac{\partial}{\partial x_1} \frac{\partial}{\partial \xi_1} (a \xi_1^k) \\
&= \frac{i}{2} \sum_i \frac{\partial^2 p_k}{\partial x_i \partial \xi_i}(x, \xi). \qquad \square
\end{aligned}$$

1.3.5 The local expression for the transport operator R.

We claim that

$$\hbar^k e^{-i\phi/\hbar} P(u e^{i\phi/\hbar}) = p_k(x, d\phi) u + \hbar R u + \cdots$$

where R is the first order differential operator

$$Ru =$$

$$\sum_j \frac{\partial p_k}{\partial \xi_j}(x, d\phi) D_j u + \left[\frac{1}{2\sqrt{-1}} \sum_{ij} \frac{\partial^2 p_k}{\partial \xi_i \partial \xi_j}(x, d\phi) \frac{\partial^2 \phi}{\partial x_i \partial x_j} u + p_{k-1}(x, d\phi) \right] u. \tag{1.20}$$

Proof. The term coming from p_{k-1} is clearly the result of applying

$$\sum_{|\alpha| = k-1} a_\alpha D^\alpha.$$

So we only need to deal with a homogeneous operator of order k. Since the coefficients a_α are not going to make any difference in this formula, we need only prove it for the differential operator

$$P(x, D) = D^\alpha$$

which we will do by induction on $|\alpha|$.

For $|\alpha| = 1$ have an operator of the form D_j and Leibnitz's rule gives

$$\hbar e^{-i\phi/\hbar} D_j(ue^{i\phi/\hbar}) = \frac{\partial\phi}{\partial x_j} u + \hbar D_j u$$

which is exactly (1.20) as $p_1(\xi) = \xi_j$, and so the second and third terms in (1.20) do not occur.

Suppose we have verified (1.20) for D^α and we want to check it for

$$D_r D^\alpha = D^{\alpha+\delta_r}.$$

So

$$\hbar^{|\alpha|+1} e^{-i\phi/\hbar}\left(D_r D^\alpha(ue^{i\phi/\hbar})\right) = \hbar e^{-i\phi/\hbar} D_r[(d\phi)^\alpha ue^{i\phi/\hbar}) + \hbar(R_\alpha u)e^{i\phi/\hbar}] + \cdots$$

where R_α denotes the operator in (1.20) corresponding to D^α. A term involving the zero'th power of $\hbar$ can only come from applying the D_r to the exponential in the first expression and this will yield

$$(d\phi)^{\alpha+\delta_r} u$$

which $p_{|\alpha|+1}(d\phi)u$ as desired. In applying D_r to the second term in the square brackets we get

$$\hbar^2 D_r(R_\alpha u) + \hbar\frac{\partial\phi}{\partial x_r} R_\alpha u$$

and we ignore the first term as we are ignoring all powers of $\hbar$ higher than the first. So all have to do is collect coefficients:

We have

$$D_r((d\phi^\alpha)u) = (d\phi)^\alpha D_r u + \frac{1}{i}\left[\alpha_1(d\phi)^{\alpha-\delta_1}\frac{\partial^2\phi}{\partial x_1\partial x_r} + \cdots + \alpha_m(d\phi)^{\alpha-\delta_m}\frac{\partial^2\phi}{\partial x_m\partial x_r}\right]u.$$

Also

$$\frac{\partial\phi}{\partial x_r} R_\alpha u =$$

$$\sum \alpha_i(d\phi)^{\alpha-\delta_i+\delta_r} D_i u + \frac{1}{2\sqrt{-1}}\sum_{ij}\alpha_i(\alpha_j - \delta_{ij})(d\phi)^{\alpha-\delta_i-\delta_j+\delta_r}\frac{\partial^2\phi}{\partial x_i\partial x_j}u.$$

The coefficient of $D_j u, \quad j \neq r$ is

$$\alpha_j(d\phi)^{(\alpha+\delta_r-\delta_j)}$$

as desired. The coefficient of $D_r u$ is

$$(d\phi)^\alpha + \alpha_r(d\phi)^\alpha = (\alpha+\delta_r)(d\phi)^{(\alpha+\delta_r)-\delta_r}$$

as desired.

Let us now check the coefficient of $\frac{\partial^2\phi}{\partial x_i\partial x_j}$. If $i\neq r$ and $j\neq r$ then the desired result is immediate.

If $j=r$, there are two sub-cases to consider: 1) $j=r, j\neq i$ and 2) $i=j=r$.

If $j=r,\ j\neq i$ remember that the sum in R_α is over *all* i and j, so the coefficient of $\frac{\partial^2\phi}{\partial x_i\partial x_j}$ in

$$\sqrt{-1}\frac{\partial\phi}{\partial x_r}R_\alpha u$$

is

$$\frac{1}{2}\left(\alpha_i\alpha_j+\alpha_j\alpha_i\right)(d\phi)^{\alpha-\delta_i}=\alpha_i\alpha_j(d\phi)^{\alpha-\delta_i}$$

to which we add

$$\alpha_i(d\phi)^{\alpha-\delta_i}$$

to get

$$\alpha_i(\alpha_j+1)(d\phi)^{\alpha-\delta_i}=(\alpha+\delta_r)_i(\alpha+\delta_r)_j(d\phi)^{\alpha-\delta_i}$$

as desired.

If $i=j=r$ then the coefficient of $\frac{\partial^2\phi}{(\partial x_i)^2}$ in

$$\sqrt{-1}\frac{\partial\phi}{\partial x_r}R_\alpha u$$

is

$$\frac{1}{2}\alpha_i(\alpha_i-1)(d\phi)^{\alpha-\delta_i}$$

to which we add

$$\alpha_i(d\phi)^{\alpha-\delta_i}$$

giving

$$\frac{1}{2}\alpha_i(\alpha_i+1)(d\phi)^{\alpha-\delta_i}$$

as desired.

This completes the proof of (1.20).

1.3.6 Putting it together locally.

We have the following three formulas, some of them rewritten with H instead of p_k so as to conform with our earlier notation: The formula for the transport operator R given by (1.20):

$$\sum_j\frac{\partial H}{\partial\xi_j}(x,d\phi)D_ja+\left[\frac{1}{2i}\sum_{ij}\frac{\partial^2H}{\partial\xi_i\partial\xi_j}(x,d\phi)\frac{\partial^2\phi}{\partial x_i\partial x_j}+p_{k-1}(x,d\phi)\right]a,$$

and the formula for the Lie derivative with respect to v_H of the pull back $(\pi \circ \iota)^*(adx^{\frac{1}{2}})$ given by $(\pi \circ \iota) * bdx^{\frac{1}{2}}$ where b is

$$\sum_j \frac{\partial H}{\partial \xi_j}(x, d\phi)\frac{\partial a}{\partial x_j} + \left[\frac{1}{2}\sum_{i,j} \frac{\partial^2 H}{\partial \xi_i \partial \xi_j}\frac{\partial^2 \phi}{\partial x^i \partial x^j} + \frac{1}{2}\sum_i \frac{\partial^2 H}{\partial \xi_i \partial x^i}\right] a.$$

This is equation (1.15). Our third formula is the formula for the sub-principal symbol, equation (1.19) which says that

$$\sigma_{sub}(P)(x,\xi)a = \left[p_{k-1}(x,\xi) + \frac{i}{2}\sum_i \frac{\partial^2 H}{\partial x_i \partial \xi_i}(x,\xi)\right] a.$$

As first order partial differential operators on a, if we multiply the first expression above by i we get the second plus i times the third! So we can write the transport operator as

$$(\pi \circ \iota)^*[(Ra)dx^{\frac{1}{2}}] = \frac{1}{i}\left[D_{v_H} + i\sigma_{sub}(P)(x, d\phi)\right](\pi \circ \iota)^*(adx^{\frac{1}{2}}). \tag{1.21}$$

The operator inside the brackets on the right hand side of this equation is a perfectly good differential operator on $\frac{1}{2}$-densities on Λ. We thus have two questions to answer: Does this differential operator have invariant significance when Λ is horizontal - but in terms of a general coordinate transformation? Since the first term in the brackets comes from H and the symplectic form on the cotangent bundle, our question is one of attaching some invariant significance to the sub-principal symbol. We will deal briefly with this question in the next section and at more length in Chapter 6.

The second question is how to deal with the whole method - the eikonal equation, the transport equations, the meaning of the series in $\hbar$ etc. when we pass through a caustic. The answer to this question will occupy us for the whole book.

1.3.7 Differential operators on manifolds.

Differential operators on functions.

Let X be an m-dimensional manifold. An operator

$$P: \ C^\infty(X) \to C^\infty(X)$$

is called a differential operator of order k if, for every coordinate patch $(U, x_1, \ldots, x_m)$ the restriction of P to $C_0^\infty(U)$ is of the form

$$P = \sum_{|\alpha| \le k} a_\alpha D^\alpha, \qquad a_\alpha \in C^\infty(U).$$

As mentioned above, the total symbol of P is no longer well defined, but the principal symbol *is* well defined as a function on T^*X. Indeed, it is defined as in Section 1.2.1: The value of the principal symbol H at a point $(x, d\phi(x))$ is determined by

$$H(x, d\phi(x))u(x) = \hbar^k e^{-i\frac{\phi}{\hbar}}(P(ue^{i\frac{\phi}{\hbar}})(x) + O(\hbar).$$

What about the transpose and the sub-principal symbol?

Differential operators on vector bundles.

Let $E \to X$ and $F \to X$ be vector bundles. Let E be of dimension p and F be of dimension q. We can find open covers of X by coordinate patches $(U, x_1, \ldots, x_m)$ over which E and F are trivial. So we can find sections $r_1, \ldots, r_p$ of E so that every smooth section of E over U can be written as

$$f_1 r_1 + \cdots f_p r_p$$

where the f_i are smooth functions on U and every smooth section of F over U can be written as

$$g_1 s_1 + \cdots + g_q s_q$$

over U. An operator

$$P : \; C^\infty(X, E) \to C^\infty(X, F)$$

is called a differential operator of order k if, for every such U the restriction of P to smooth sections of compact support supported in U is given by

$$P\left(f_1 r_1 + \cdots f_p r_p\right) = \sum_{j=1}^{q} \sum_{i=1}^{p} P_{ij} f_i s_j$$

where the P_{ij} are differential operators of order k.

In particular if E and F are line bundles so that $p = q = 1$ it makes sense to talk of differential operators of order k from smooth sections of E to smooth sections of F. In a local coordinate system with trivializations r of E and s of F a differential operator locally is given by

$$fr \mapsto (Pf)s.$$

If $E = F$ and $r = s$ it is easy to check that the principal symbol of P is independent of the trivialization. (More generally the matrix of principal symbols in the vector bundle case is well defined up to appropriate pre and post multiplication by change of bases matrices, i.e. is well defined as a section of $\mathrm{Hom}(E, F)$ pulled up to the cotangent bundle. See Chaper II of [**?**] for the general discussion.)

In particular it makes sense to talk about a differential operator of degree k on the space of smooth $\frac{1}{2}$-densities and the principal symbol of such an operator.

The transpose and sub-principal symbol of a differential operator on $\frac{1}{2}$-densities.

If μ and ν are $\frac{1}{2}$-densities on a manifold X, their product $\mu \cdot \nu$ is a density (of order one). If this product has compact support, for example if μ or ν has compact support then the integral

$$\int_X \mu \cdot \nu$$

is well defined. See Chapter 6 for details. So if P is a differential operator of degree k on $\frac{1}{2}$-densities, its transpose P^t is defined via

$$\int_X (P\mu) \cdot \nu = \int_X \mu \cdot (P^t \nu)$$

for all μ and ν one of which has compact support. Locally, in terms of a coordinate neighborhood $(U, x_1, \ldots, x_m)$, every $\frac{1}{2}$-density can be written as $f dx^{\frac{1}{2}}$ and then the local expression for P^t is given as in Section 1.3.3. We then define the operator Q as in equation (1.18) and the sub-principal symbol as the principal symbol of Q as an operator of degree $k-1$ just as in Section 1.3.3.

1.4 The plan.

Chapter 2

Symplectic geometry.

2.1 Symplectic vector spaces.

Let V be a (usually finite dimensional) vector space over the real numbers. A symplectic structure on V consists of an antisymmetric bilinear form

$$\omega : V \times V \to \mathbf{R}$$

which is non-degenerate. So we can think of ω as an element of $\wedge^2 V^*$ when V is finite dimensional, as we shall assume until further notice. A vector space equipped with a symplectic structure is called a symplectic vector space.

A basic example is $\mathbf{R}^2$ with

$$\omega_{\mathbf{R}^2}\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) := \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

We will call this the standard symplectic structure on $\mathbf{R}^2$.

2.1.1 Special kinds of subspaces.

If W is a subspace of symplectic vector space V then $W^\perp$ denotes the symplectic orthocomplement of W:

$$W^\perp := \{v \in V | \ \omega(v, w) = 0, \ \forall w \in W\}.$$

A subspace is called

1. **symplectic** if $W \cap W^\perp = \{0\}$,
2. **isotropic** if $W \subset W^\perp$,
3. **coisotropic** if $W^\perp \subset W$, and
4. **Lagrangian** if $W = W^\perp$.

Since $(W^\perp)^\perp = W$ by the non-degeneracy of ω, it follows that W is symplectic if and only if $W^\perp$ is. Also, the restriction of ω to any symplectic subspace W is non-degenerate, making W into a symplectic vector space. Conversely, to say that the restriction of ω to W is non-degenerate means precisely that $W \cap W^\perp = \{0\}$.

2.1.2 Normal forms.

For any non-zero $e \in V$ we can find an $f \in V$ such that $\omega(e,f) = 1$ and so the subspace W spanned by e and f is a two dimensional symplectic subspace. Furthermore the map

$$e \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

gives a symplectic isomorphism of W with $\mathbf{R}^2$ with its standard symplectic structure. We can apply this same construction to $W^\perp$ if $W^\perp \neq 0$. Hence by induction, we can decompose any symplectic vector space into a direct sum of two dimensional symplectic subspaces:

$$V = W_1 \oplus \cdots W_d$$

where dim $V = 2d$ (proving that every symplectic vector space is even dimensional) and where the W_i are pairwise (symplectically) orthogonal and where each W_i is spanned by e_i, f_i with $\omega(e_i, f_i) = 1$. In particular this shows that all $2d$ dimensional symplectic vector spaces are isomorphic, and isomorphic to a direct sum of d copies of $\mathbf{R}^2$ with its standard symplectic structure.

2.1.3 Existence of Lagrangian subspaces.

Let us collect the $e_1, \ldots, e_d$ in the above construction and let L be the subspace they span. It is clearly isotropic. Also, $e_1, \ldots, e_n, f_1, \ldots, f_d$ form a basis of V. If $v \in V$ has the expansion

$$v = a_1 e_1 + \cdots a_d e_d + b_1 f_1 + \cdots + b_d f_d$$

in terms of this basis, then $\omega(e_i, v) = b_i$. So $v \in L^\perp \Rightarrow v \in L$. Thus L is Lagrangian. So is the subspace M spanned by the f's.

Conversely, if L is a Lagrangian subspace of V and if M is a complementary Lagrangian subspace, then ω induces a non-degenerate linear pairing of L with M and hence any basis $e_1, \cdots e_d$ picks out a dual basis $f_1, \cdots .f_d$ of M giving a basis of V of the above form.

2.1.4 Consistent Hermitian structures.

In terms of the basis $e_1, \ldots, e_n, f_1, \ldots, f_d$ introduced above, consider the linear map

$$J: \quad e_i \mapsto -f_i, \quad f_i \mapsto e_i.$$

It satisfies

$$\begin{aligned} J^2 &= -I, & (2.1)\\ \omega(Ju,Jv) &= \omega(u,v), \quad \text{and} & (2.2)\\ \omega(Ju,v) &= \omega(Jv,u). & (2.3) \end{aligned}$$

Notice that any J which satisfies two of the three conditions above automatically satisfies the third. Condition (2.1) says that J makes V into a d-dimensional complex vector space. Condition (2.2) says that J is a symplectic transformation, i.e acts so as to preserve the symplectic form ω. Condition (2.3) says that $\omega(Ju,v)$ is a real symmetric bilinear form.

All three conditions (really any two out of the three) say that $(\ ,\ ) = (\ ,\ )_{\omega,J}$ defined by

$$(u,v) = \omega(Ju,v) + i\omega(u,v)$$

is a semi-Hermitian form whose imaginary part is ω. For the J chosen above this form is actually Hermitian, that is the real part of $(\ ,\ )$ is positive definite.

2.1.5 Choosing Lagrangian complements.

The results of this section are purely within the framework of symplectic linear algebra. Hence their logical place is here. However their main interest is that they serve as lemmas for more geometrical theorems, for example the Weinstein isotropic embedding theorem. The results here all have to do with making choices in a "consistent" way, so as to guarantee, for example, that the choices can be made to be invariant under the action of a group.

For any a Lagrangian subspace $L \subset V$ we will need to be able to choose a complementary Lagrangian subspace L', and do so in a consistent manner, depending, perhaps, on some auxiliary data. Here is one such way, depending on the datum of a symmetric positive definite bilinear form B on V. (Here B has nothing to do with with the symplectic form.)

Let L^B be the orthogonal complement of L relative to the form B. So

$$\dim L^B = \dim L = \frac{1}{2}\dim V$$

and any subspace $W \subset V$ with

$$\dim W = \frac{1}{2}\dim V \quad \text{and } W \cap L = \{0\}$$

can be written as

$$\text{graph}(A)$$

where $A : L^B \to L$ is a linear map. That is, under the vector space identification

$$V = L^B \oplus L$$

the elements of W are all of the form

$$w + Aw, \quad w \in L^B.$$

We have

$$\omega(u + Au, w + Aw) = \omega(u, w) + \omega(Au, w) + \omega(u, Aw)$$

since $\omega(Au, Aw) = 0$ as L is Lagrangian. Let C be the bilinear form on L^B given by

$$C(u, w) := \omega(Au, w).$$

Thus W is Lagrangian if and only if

$$C(u, w) - C(w, u) = -\omega(u, w).$$

Now

$$\mathrm{Hom}(L^B, L) \sim L \otimes L^{B*} \sim L^{B*} \otimes L^{B*}$$

under the identification of L with L^{B*} given by ω. Thus the assignment $A \leftrightarrow C$ is a bijection, and hence the space of all Lagrangian subspaces complementary to L is in one to one correspondence with the space of all bilinear forms C on L^B which satisfy $C(u, w) - C(w, u) = -\omega(u, w)$ for all $u, w \in L^B$. An obvious choice is to take C to be $-\frac{1}{2}\omega$ restricted to L^B. In short,

Proposition 4 *Given a positive definite symmetric form on a symplectic vector space V, there is a consistent way of assigning a Lagrangian complement L' to every Lagrangian subspace L.*

Here the word "consistent" means that the choice depends only on B. This has the following implication: Suppose that T is a linear automorphism of V which preserves both the symplectic form ω and the positive definite symmetric form B. In other words, suppose that

$$\omega(Tu, Tv) = \omega(u, v) \quad \text{and} \quad B(Tu, Tv) = B(u, v) \quad \forall\, u, v \in V.$$

Then if $L \mapsto L'$ is the correspondence given by the proposition, then

$$TL \mapsto TL'.$$

More generally, if $T : V \to W$ is a symplectic isomorphism which is an isometry for a choice of positive definite symmetric bilinear forms on each, the above equation holds.

Given L and B (and hence L') we determined the complex structure J by

$$J : L \to L', \quad \omega(u, Jv) = B(u, v) \quad u, v \in L$$

and then

$$J := -J^{-1} : L' \to L$$

and extending by linearity to all of V so that

$$J^2 = -I.$$

Then for $u, v \in L$ we have

$$\omega(u, Jv) = B(u, v) = B(v, u) = \omega(v, Ju)$$

while

$$\omega(u, JJv) = -\omega(u, v) = 0 = \omega(Jv, Ju)$$

and

$$\omega(Ju, JJv) = -\omega(Ju, v) = -\omega(Jv, u) = \omega(Jv, JJu)$$

so (2.3) holds for all $u, v \in V$. We should write $J_{B,L}$ for this complex structure, or J_L when B is understood

Suppose that T preserves ω and B as above. We claim that

$$J_{TL} \circ T = T \circ J_L \tag{2.4}$$

so that T is complex linear for the complex structures J_L and J_{TL}. Indeed, for $u, v \in L$ we have

$$\omega(Tu, J_{TL}Tv) = B(Tu, Tv)$$

by the definition of J_{TL}. Since B is invariant under T the right hand side equals $B(u, v) = \omega(u, J_L v) = \omega(Tu, TJ_L v)$ since ω is invariant under T. Thus

$$\omega(Tu, J_{TL}Tv) = \omega(Tu, TJ_L v)$$

showing that

$$TJ_L = J_{TL}T$$

when applied to elements of L. This also holds for elements of L'. Indeed every element of L' is of the form $J_L u$ where $u \in L$ and $TJ_L u \in TL'$ so

$$J_{TL}TJ_L u = -J_{TL}^{-1}TJ_L u = -Tu = TJ_L(J_L u). \quad \square$$

Let I be an isotropic subspace of V and let $I^\perp$ be its symplectic orthogonal subspace so that $I \subset I^\perp$. Let

$$I_B = (I^\perp)^B$$

be the B-orthogonal complement to $I^\perp$. Thus

$$\dim I_B = \dim I$$

and since $I_B \cap I^\perp = \{0\}$, the spaces I_B and I are non-singularly paired under ω. In other words, the restriction of ω to $I_B \oplus I$ is symplectic. The proof of the preceding proposition gives a Lagrangian complement (inside $I_B \oplus I$) to I which, as a subspace of V has zero intersection with $I^\perp$. We have thus proved:

Proposition 5 *Given a positive definite symmetric form on a symplectic vector space V, there is a consistent way of assigning an isotropic complement I' to every co-isotropic subspace $I^{\perp}$.*

We can use the preceding proposition to prove the following:

Proposition 6 *Let V_1 and V_2 be symplectic vector spaces of the same dimension, with $I_1 \subset V_1$ and $I_2 \subset V_2$ isotropic subspaces, also of the same dimension. Suppose we are given*

- *a linear isomorphism $\lambda : I_1 \to I_2$ and*
- *a symplectic isomorphism $\ell : I_1^{\perp}/I_1 \to I_2^{\perp}/I_2$.*

Then there is a symplectic isomorphism

$$\gamma : V_1 \to V_2$$

such that

1. *$\gamma : I_1^{\perp} \to I_2^{\perp}$ and (hence) $\gamma : I_1 \to I_2$,*
2. *The map induced by γ on $I_1^{\perp}/I_1$ is ℓ and*
3. *The restriction of γ to I_1 is λ.*

Furthermore, in the presence of positive definite symmetric bilinear forms B_1 on V_1 and B_2 on V_2 the choice of γ can be made in a "canonical" fashion.

Indeed, choose isotropic complements I_{1B} to $I_1^{\perp}$ and I_{2B} to $I_2^{\perp}$ as given by the preceding proposition, and also choose B orthogonal complements Y_1 to I_1 inside $I_1^{\perp}$ and Y_2 to I_2 inside $I_2^{\perp}$. Then Y_i $(i = 1, 2)$ is a symplectic subspace of V_i which can be identified as a symplectic vector space with $I_i^{\perp}/I_i$. We thus have

$$V_1 = (I_1 \oplus I_{1B}) \oplus Y_1$$

as a direct sum decomposition into the sum of the two symplectic subspaces $(I_1 \oplus I_{1B})$ and Y_1 with a similar decomposition for V_2. Thus ℓ gives a symplectic isomorphism of $Y_1 \to Y_2$. Also

$$\lambda \oplus (\lambda^*)^{-1} : I_1 \oplus I_{1B} \to I_2 \oplus I_{2B}$$

is a symplectic isomorphism which restricts to λ on I_1. QED

2.2 Equivariant symplectic vector spaces.

Let V be a symplectic vector space. We let $Sp(V)$ denote the group of all all symplectic automorphisms of V, i.e all maps T which satisfy $\omega(Tu, Tv) = \omega(u, v)\ \forall\ u, v \in V$.

A representation $\tau : G \to \mathrm{Aut}(V)$ of a group G is called symplectic if in fact $\tau : G \to Sp(V)$. Our first task will be to show that if G is compact, and τ is symplectic, then we can find a J satisfying (2.1) and (2.2), which commutes with all the $\tau(a)$, $a \in G$ and such that the associated Hermitian form is positive definite.

2.2.1 Invariant Hermitian structures.

Once again, let us start with a positive definite symmetric bilinear form B. By averaging over the group we may assume that B is G invariant. (Here is where we use the compactness of G.) Then there is a unique linear operator K such that

$$B(Ku,v) = \omega(u,v) \quad \forall\, u,v \in V.$$

Since both B and ω are G-invariant, we conclude that K commutes with all the $\tau(a),\ a \in G$. Since $\omega(v,u) = -\omega(u,v)$ we conclude that K is skew adjoint relative to B, i.e. that

$$B(Ku,v) = -B(u,Kv).$$

Also K is non-singular. Then K^2 is symmetric and non-singular, and so V can be decomposed into a direct sum of eigenspaces of K^2 corresponding to distinct eigenvalues, all non-zero. These subspaces are mutually orthogonal under B and invariant under G. If $K^2u = \mu u$ then

$$\mu B(u,u) = B(K^2u,u) = -B(Ku,Ku) < 0$$

so all these eigenvalues are negative; we can write each μ as $\mu = -\lambda^2,\ \lambda > 0$. Furthermore, if $K^2u = -\lambda^2 u$ then

$$K^2(Ku) = KK^2u = -\lambda^2 Ku$$

so each of these eigenspaces is invariant under K. Also, any two subspaces corresponding to different values of λ^2 are orthogonal under ω. So we need only define J on each such subspace so as to commute with all the $\tau(a)$ and so as to satisfy (2.1) and (2.2), and then extend linearly. On each such subspace set

$$J := \lambda K^{-1}.$$

Then (on this subspace)

$$J^2 = \lambda^2 K^{-2} = -I$$

and

$$\omega(Ju,v) = \lambda\omega(K^{-1}u,v) = \lambda B(u,v)$$

is symmetric in u and v. Furthermore $\omega(Ju,u) = \lambda B(u,u) > 0$. QED

Notice that if τ is irreducible, then the Hermitian form $(\ ,\ ) = \omega(J\cdot,\cdot) + i\omega(\cdot,\cdot)$ is uniquely determined by the property that its imaginary part is ω.

2.2.2 The space of fixed vectors for a compact group is symplectic.

If we choose J as above, if $\tau(a)u = u$ then $\tau(a)Ju = Ju$. So the space of fixed vectors is a complex subspace for the complex structure determined by J.

But the restriction of a positive definite Hermitian form to any (complex) subspace is again positive definite, in particular non-singular. Hence its imaginary part, the symplectic form ω, is also non-singular. QED

This result need not be true if the group is not compact. For example, the one parameter group of shear transformations

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

in the plane is symplectic as all of these matrices have determinant one. But the space of fixed vectors is the x-axis.

2.2.3 Toral symplectic actions.

Suppose that $G = \mathbf{T}^n$ is an n-dimensional torus, and that $\mathbf{g}$ denotes its Lie algebra. Then exp:$\mathbf{g} \to G$ is a surjective homomorphisms, whose kernel $\mathbf{Z}_G$ is a lattice.

If $\tau : G \to U(V)$ as above, we can decompose V into a direct sum of one dimensional complex subspaces

$$V = V_1 \oplus \cdots \oplus V_d$$

where the restriction of τ to each subspace is given by

$$\tau_{|V_k}(\exp\, \xi)v = e^{2\pi i \alpha_k(\xi)}v$$

where

$$\alpha_k \in \mathbf{Z}^*_G,$$

the dual lattice.

2.3 Symplectic manifolds.

A manifold M is called **symplectic** if it comes equipped with a closed non-degenerate two form ω. A diffeomorphism is called symplectic if it preserves ω and a vector field v is called symplectic if

$$D_v\omega = 0.$$

Since $D_v\omega = d\iota(v)\omega + \iota(v)d\omega = d\iota(v)\omega$ as $d\omega = 0$, a vector field v is symplectic if and only if $\iota(v)\omega$ is closed.

A vector field v is called **Hamiltonian** if $\iota(v)\omega$ is exact. If θ is a closed one form, and v a vector field, then $D_v\theta = d\iota(v)\theta$ is exact. Hence if v_1 and v_2 are symplectic vector fields

$$D_{v_1}\iota(v_2)\omega = \iota([v_1, v_2])\omega$$

so $[v_1, v_2]$ is Hamiltonian with

$$\iota([v_1, v_2])\omega = d\omega(v_2, v_1).$$

2.4 Darboux style theorems.

These are theorems which state that two symplectic structures on a manifold are the same or give a normal form near a submanifold etc. via the Moser-Weinstein method. This method hinges on the basic formula of differential calculus: If $f_t : X \to Y$ is a smooth family of maps and ω_t is a one parameter family of differential forms on Y then

$$\frac{d}{dt} f_t^* \omega_t = f_t^* \frac{d}{dt} \omega_t + Q_t d\omega_t + dQ_t\omega_t \tag{2.5}$$

where

$$Q_t : \Omega^k(Y) \to \Omega^{k-1}(X)$$

is given by

$$Q_t\tau(w_1, \ldots, w_{k-1}) := \tau(v_t, df_t(w_1), \ldots, df_t(w_{k-1}))$$

where

$$v_t : X \to T(Y), \quad v_t(x) := \frac{d}{dt} f_t(x).$$

If ω_t does not depend explicitly on t then the first term on the right of (2.5) vanishes, and integrating (2.5) with respect to t from 0 to 1 gives

$$f_1^* - f_0^* = dQ + Qd, \quad Q := \int_0^1 Q_t dt. \tag{2.6}$$

Here is the first Darboux type theorem:

2.4.1 Compact manifolds.

Theorem 1 *Let M be a compact manifold, ω_0 and ω_1 two symplectic forms on M in the same cohomology class so that*

$$\omega_1 - \omega_0 = d\alpha$$

for some one form α. Suppose in addition that

$$\omega_t := (1-t)\omega_0 + t\omega_1$$

is symplectic for all $0 \leq t \leq 1$. Then there exists a diffeomorphism $f : M \to M$ such that

$$f^*\omega_1 = \omega_0.$$

Proof. Solve the equation

$$\iota(v_t)\omega_t = -\alpha$$

which has a unique solution v_t since ω_t is symplectic. Then solve the time dependent differential equation

$$\frac{df_t}{dt} = v_t(f_t), \quad f_0 = \mathrm{id}$$

which is possible since M is compact. Since

$$\frac{d\omega_t}{dt} = d\alpha,$$

the fundamental formula (2.5) gives

$$\frac{df_t^*\omega_t}{dt} = f_t^*\left[d\alpha + 0 - d\alpha\right] = 0$$

so

$$f_t^*\omega_t \equiv \omega_0.$$

In particular, set $t = 1$. QED

This style of argument was introduced by Moser and applied to Darboux type theorems by Weinstein.

Here is a modification of the above:

Theorem 2 *Let M be a compact manifold, and ω_t, $0 \le t \le 1$ a family of symplectic forms on M in the same cohomology class.*

Then there exists a diffeomorphism $f : M \to M$ such that

$$f^*\omega_1 = \omega_0.$$

Proof. Break the interval $[0, 1]$ into subintervals by choosing $t_0 = 0 < t_1 < t_2 < \cdots < t_N = 1$ and such that on each subinterval the "chord" $(1-s)\omega_{t_i} + s\omega_{t_{i+1}}$ is close enough to the curve $\omega_{(1-s)t_i+st_{i+1}}$ so that the forms $(1-s)\omega_{t_i} + s\omega_{t_{i+1}}$ are symplectic. Then successively apply the preceding theorem. QED

2.4.2 Compact submanifolds.

The next version allows M to be non-compact but has to do with with behavior near a compact submanifold. We will want to use the following proposition:

Proposition 7 *Let X be a compact submanifold of a manifold M and let*

$$i : X \to M$$

denote the inclusion map. Let $\gamma \in \Omega^k(M)$ be a k-form on M which satisfies

$$\begin{aligned} d\gamma &= 0 \\ i^*\gamma &= 0. \end{aligned}$$

Then there exists a neighborhood U of X and a $k-1$ form β defined on U such that

$$\begin{aligned} d\beta &= \gamma \\ \beta_{|X} &= 0. \end{aligned}$$

(This last equation means that at every point $p \in X$ we have

$$\beta_p(w_1, \ldots, w_{k-1}) = 0$$

for all tangent vectors, not necessarily those tangent to X. So it is a much stronger condition than $i^*\beta = 0$.)

Proof. By choice of a Riemann metric and its exponential map, we may find a neighborhood of W of X in M and a smooth retract of W onto X, that is a one parameter family of smooth maps

$$r_t : W \to W$$

and a smooth map $\pi : W \to X$ with

$$r_1 = \mathrm{id}, \quad r_0 = i \circ \pi, \;\; \pi : W \to X, \;\; r_t \circ i \equiv i.$$

Write

$$\frac{dr_t}{dt} = w_t \circ r_t$$

and notice that $w_t \equiv 0$ at all points of X. Hence the form

$$\beta := Q\gamma$$

has all the desired properties where Q is as in (2.6). QED

Theorem 3 *Let X, M and i be as above, and let ω_0 and ω_1 be symplectic forms on M such that*

$$i^*\omega_1 = i^*\omega_0$$

and such that

$$(1-t)\omega_0 + t\omega_1$$

is symplectic for $0 \leq t \leq 1$. Then there exists a neighborhood U of M and a smooth map

$$f : U \to M$$

such that

$$f_{|X} = id \quad \text{ and } \;\; f^*\omega_0 = \omega_1.$$

Proof. Use the proposition to find a neighborhood W of X and a one form α defined on W and vanishing on X such that

$$\omega_1 - \omega_0 = d\alpha$$

on W. Let v_t be the solution of

$$\iota(v_t)\omega_t = -\alpha$$

where $\omega_t = (1-t)\omega_0 + t\omega_1$. Since v_t vanishes identically on X, we can find a smaller neighborhood of X if necessary on which we can integrate v_t for $0 \leq t \leq 1$ and then apply the Moser argument as above. QED

A variant of the above is to assume that we have a curve of symplectic forms ω_t with $i^*\omega_t$ independent of t.

Finally, a very useful variant is Weinstein's

Theorem 4 *X, M, i as above, and ω_0 and ω_1 two symplectic forms on M such that $\omega_{1|X} = \omega_{0|X}$. Then there exists a neighborhood U of M and a smooth map*

$$f : U \to M$$

such that

$$f_{|X} = id \quad \text{and} \quad f^*\omega_0 = \omega_1.$$

Here we can find a neighborhood of X such that

$$(1-t)\omega_0 + t\omega_1$$

is symplectic for $0 \leq t \leq 1$ since X is compact. QED

One application of the above is to take X to be a point. The theorem then asserts that all symplectic structures of the same dimension are locally symplectomorphic. This is the original theorem of Darboux.

2.4.3 The isotropic embedding theorem.

Another important application of the preceding theorem is Weinstein's isotropic embedding theorem: Let (M, ω) be a symplectic manifold, X a compact manifold, and $i : X \to M$ an isotropic embedding, which means that $di_x(TX)_x$ is an isotropic subspace of $TM_{i(x)}$ for all $x \in X$. Thus

$$di_x(TX)_x \subset (di_x(TX)_x)^\perp$$

where $(di_x(TX)_x)^\perp$ denotes the orthogonal complement of $di_x(TX)_x$ in $TM_{i(x)}$ relative to $\omega_{i(x)}$. Hence

$$(di_x(TX)_x)^\perp / di_x(TX)_x$$

is a symplectic vector space, and these fit together into a symplectic vector bundle (i.e. a vector bundle with a symplectic structure on each fiber). We will call this the symplectic normal bundle of the embedding, and denote it by

$$SN_i(X)$$

or simply by $SN(X)$ when i is taken for granted.

Suppose that U is a neighborhood of $i(X)$ and $g : U \to N$ is a symplectomorphism of U into a second symplectic manifold N. Then $j = g \circ i$ is an isotropic embedding of X into N and f induces an isomorphism

$$g_* : NS_i(X) \to NS_j(X)$$

of symplectic vector bundles. Weinstein's isotropic embedding theorem asserts conversely, any isomorphism between symplectic normal bundles is in fact induced by a symplectomorphism of a neighborhood of the image:

Theorem 5 *Let* (M, ω_M, X, i) *and* (N, ω_N, X, j) *be the data for isotropic embeddings of a compact manifold* X*. Suppose that*

$$\ell : SN_i(X) \to SN_j(X)$$

is an isomorphism of symplectic vector bundles. Then there is a neighborhood U *of* $i(X)$ *in* M *and a symplectomorphism* g *of* U *onto a neighborhood of* $j(X)$ *in* N *such that*

$$g_* = \ell.$$

For the proof, we will need the following extension lemma:

Proposition 8 *Let*

$$i : X \to M, \quad j : Y \to N$$

be embeddings of compact manifolds X *and* Y *into manifolds* M *and* N*. suppose we are given the following data:*

- *A smooth map* $f : X \to Y$ *and, for each* $x \in X$*,*
- *A linear map* $A_x TM_{i(x)} \to TN_{j(f(x))}$ *such that the restriction of* A_x *to* $TX_x \subset TM_{i(x)}$ *coincides with* df_x*.*

Then there exists a neighborhood W *of* X *and a smooth map* $g : W \to N$ *such that*

$$g \circ i = f \circ i$$

and

$$dg_x = A_x \quad \forall\, x \in X.$$

Proof. If we choose a Riemann metric on M, we may identify (via the exponential map) a neighborhood of $i(X)$ in M with a section of the zero section of X in its (ordinary) normal bundle. So we may assume that $M = \mathcal{N}_i X$ is this normal bundle. Also choose a Riemann metric on N, and let

$$\exp : \mathcal{N}_j(Y) \to N$$

be the exponential map of this normal bundle relative to this Riemann metric. For $x \in X$ and $v \in N_i(i(x))$ set

$$g(x, v) := \exp_{j(x)}(A_x v).$$

Then the restriction of g to X coincides with f, so that, in particular, the restriction of dg_x to the tangent space to T_x agrees with the restriction of A_x to this subspace, and also the restriction of dg_x to the normal space to the zero section at x agrees A_x so g fits the bill. QED

Proof of the theorem. We are given linear maps $\ell_x : (I_x^\perp / I_x) \to J_x^\perp / J_x$ where $I_x = di_x(TX)_x$ is an isotropic subspace of $V_x := TM_{i(x)}$ with a similar notation involving j. We also have the identity map of

$$I_x = TX_x = J_x.$$

So we may apply Proposition 6 to conclude the existence, for each x of a unique symplectic linear map

$$A_x : \; TM_{i(x)} \to TN_{j(x)}$$

for each $x \in X$. We may then extend this to an actual diffeomorphism, call it h on a neighborhood of $i(X)$, and since the linear maps A_x are symplectic, the forms

$$h^*\omega_N \quad \text{and} \quad \omega_M$$

agree at all points of X. We then apply Theorem 4 to get a map k such that $k^*(h^*\omega_N) = \omega_M$ and then $g = h \circ k$ does the job. QED

Notice that the constructions were all determined by the choice of a Riemann metric on M and of a Riemann metric on N. So if these metrics are invariant under a group G, the corresponding g will be a G-morphism. If G is compact, such invariant metrics can be constructed by averaging over the group, as will be recalled in the next section.

An important special case of the isotropic embedding theorem is where the embedding is not merely isotropic, but is Lagrangian. Then the symplectic normal bundle is trivial, and the theorem asserts that all Lagrangian embeddings of a compact manifold are locally equivalent, for example equivalent to the embedding of the manifold as the zero section of its cotangent bundle.

Chapter 3

The language of category theory.

3.1 Categories.

We briefly recall the basic definitions:

A **category C** consists of the following data:

(i) A set, $Ob(\mathbf{C})$, whose elements are called the **objects** of **C** ,

(ii) For every pair (X,Y) of $Ob(\mathbf{C})$ a set, $\mathrm{Morph}(X,Y)$, whose elements are called the **morphisms** or **arrows** from X to Y,

(iii) For every triple (X,Y,Z) of $Ob(\mathbf{C})$ a map from $\mathrm{Morph}(X,Y)\times\mathrm{Morph}(Y,Z)$ to $\mathrm{Morph}(X,Z)$ called the **composition map** and denoted $(f,g) \rightsquigarrow g \circ f$.

These data are subject to the following conditions:

(iv) The composition of morphisms is *associative*

(v) For each $X \in Ob(\mathbf{C})$ there is an $id_X \in \mathrm{Morph}(X,X)$ such that

$$f \circ id_X = f, \ \forall f \in \mathrm{Morph}(X,Y)$$

(for any Y) and

$$id_X \circ f = f, \ \forall f \in \mathrm{Morph}(Y,X)$$

(for any Y).

It follows from the axioms that id_X is unique.

3.2 Functors and morphisms.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, a *functor* F from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

(vi) a map $F : Ob(\mathcal{C}) \to Ob(\mathcal{D})$

and

(vii) for each pair (X, Y) of $Ob(\mathcal{C})$ a map

$$F : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$$

subject to the rules

(viii)

$$F(id_X) = id_{F(X)}$$

and

(ix)

$$F(g \circ f) = F(g) \circ F(f).$$

This is what is usually called a **covariant functor**.

A **contravariant functor** would have $F : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(Y), F(X))$ in (vii) and $F(f) \circ F(g)$ on the right hand side of (ix).)

Here is an important example, valid for any category $\mathcal{C}$. Let us fix an $X \in Ob(\mathcal{C})$. We get a functor

$$F_X : \mathcal{C} \to \mathbf{Set}$$

by the rule which assigns to each $Y \in Ob(\mathcal{C})$ the set $F_X(Y) = \mathrm{Hom}(X, Y)$ and to each $f \in \mathrm{Hom}(Y, Z)$ the map $F_X(f)$ consisting of composition (on the left) by f. In other words, $F_X(f) : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$ is given by

$$g \in \mathrm{Hom}(X, Y) \mapsto f \circ g \in \mathrm{Hom}(X, Z).$$

Let F and G be two functors from $\mathcal{C}$ to $\mathcal{D}$. A **morphism**, $\mathfrak{m}$, from F to G (older name: "natural transformation") consists of the following data:

(x) for each $X \in Ob(\mathcal{C})$ an element $\mathfrak{m}(X) \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$ subject to the "naturality condition"

(xi) for any $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the diagram in Figure 3.1 commutes.

$$\begin{tikzcd}
F(X) \arrow[r, "\mathfrak{m}(X)"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\mathfrak{m}(Y)"'] & G(Y)
\end{tikzcd}$$

Figure 3.1:

3.2.1 Involutory functors and involutive functors.

Consider the category $\mathcal{V}$ whose objects are finite dimensional vector spaces (over some given field $\mathbb{K}$) and whose morphisms are linear transformations. We can consider the "transpose functor" $F : \mathcal{V} \to \mathcal{V}$ which assigns to every vector space V its dual space

$$V^* = \mathrm{Hom}(V, \mathbb{K})$$

and which assigns to every linear transformation $\ell : V \to W$ its transpose

$$\ell^* : V^* \to W^*.$$

In other words,

$$F(V) = V^*, \quad F(\ell) = \ell^*.$$

This is a contravariant functor which has the property that F^2 is naturally equivalent to the identity functor. There does not seem to be a standard name for this type of functor. We will call it an **involutory** functor.

A special type of involutory functor is one in which $F(X) = X$ for all objects X and $F^2 = \mathrm{id}$ (not merely naturally equivalent to the identity). For example, let $\mathcal{H}$ denote the category whose objects are Hilbert spaces and whose morphisms are bounded linear transformations. We take $F(X) = X$ on objects and $F(L) = L^\dagger$ on maps where $L^\dagger$ denotes the adjoint of L in the Hilbert space sense. We shall call such a functor a **involutive** functor.

3.3 Example: Sets, maps and relations.

The category **Set** is the category whose objects are ("all") sets and and whose morphisms are ("all") maps between sets. For reasons of logic, the word "all" must be suitably restricted to avoid contradiction.

We will take the extreme step in this section of restricting our attention to the class of finite sets. Our main point is to examine a category whose objects are finite sets, but whose morphisms are much more general than maps. Some of the arguments and constructions that we use in the study of this example will be models for arguments we will use later on, in the context of the symplectic category

3.3.1 The category of finite relations.

We will consider the category whose objects are finite sets. But we enlarge the set of morphisms by defining

$$\mathrm{Morph}(X,Y) = \text{the collection of all subsets of } X \times Y.$$

A subset of $X \times Y$ is called a **relation**. We must describe the map

$$\mathrm{Morph}(X,Y) \times \mathrm{Morph}(Y,Z) \to \mathrm{Morph}(X,Z)$$

and show that this composition law satisfies the axioms of a category. So let

$$\Gamma_1 \in \mathrm{Morph}(X,Y) \quad \text{and } \Gamma_2 \in \mathrm{Morph}(Y,Z).$$

Define

$$\Gamma_2 \circ \Gamma_1 \subset X \times Z$$

by

$$(x,z) \in \Gamma_2 \circ \Gamma_1 \;\Leftrightarrow\; \exists\, y \in Y \text{ such that } (z,y) \in \Gamma_1 \text{ and } (y,z) \in \Gamma_2. \tag{3.1}$$

Notice that if $f : X \to Y$ and $g : Y \to Z$ are maps, then

$$\mathrm{graph}(f) = \{(x,f(x)\} \in \mathrm{Morph}(X,Y) \quad \text{and} \quad \mathrm{graph}(g) \in \mathrm{Morph}(Y,Z)$$

with

$$\mathrm{graph}(g) \circ \mathrm{graph}(f) = \mathrm{graph}(g \circ f).$$

So we have indeed enlarged the category of finite sets and maps.

We still must check the axioms. Let $\Delta_X \subset X \times X$ denote the diagonal:

$$\Delta_X = \{(x,x),\ x \in X\}.$$

If $\Gamma \in \mathrm{Morph}(X,Y)$ then

$$\Gamma \circ \Delta_X = \Gamma \quad \text{and } \Delta_Y \circ \Gamma = \Gamma.$$

So Δ_X satisfies the conditions for id_X. Let us now check the associative law. Suppose that $\Gamma_1 \in \mathrm{Morph}(X,Y), \Gamma_2 \in \mathrm{Morph}(Y,Z)$ and $\Gamma_3 \in \mathrm{Morph}(Z,W)$. Then both $\Gamma_3 \circ (\Gamma_2 \circ \Gamma_1)$ and $(\Gamma_3 \circ \Gamma_2) \circ \Gamma_1$ consist of all $(x,w) \in X \times W$ such that there exist $y \in Y$ and $z \in Z$ with

$$(x,y) \in \Gamma_1,\ (y,z) \in \Gamma_2, \quad \text{and } (z,w) \in \Gamma_3.$$

This proves the associative law.

3.3.2 Categorical "points".

Let us pick a distinguished one element set and call it "pt.". Giving a map from pt. to any set X is the same as picking a point of X. So in the category of sets and maps, the points of X are the same as the morphisms from our distinguished object pt. to X.

In a more general category, where the objects are not necessarily sets, we can not talk about the points of an object X. However if we have a distinguished object pt., then we can *define* a **"point"** of any object X to be an element of $\mathrm{Morph}(\mathrm{pt.}, X)$. Thus in the category we are currently studying, the category of finite sets and relations, an element of $\mathrm{Morph}(\mathrm{pt.}, X)$, i.e a subset of $\mathrm{pt.} \times X$ is the same as a subset of X (by projection onto the second factor). So in this category, the "points" of X are the subsets of X.

A morphism $\Gamma \in \mathrm{Morph}(X, Y)$ yields a map from "points" of X to "points" of Y.

Consider the following example: For three objects X, Y, Z in

$$X \times X \times Y \times Y \times Z \times Z$$

we have the subset

$$\Delta_X \times \Delta_Y \times \Delta_Z.$$

Let us move the first X factor past the others until it lies to immediate left of the right Z factor, so consider the subset

$$\tilde{\Delta}_{X,Y,Z} \subset X \times Y \times Y \times Z \times X \times Z, \quad \tilde{\Delta}_{X,Y,Z} = \{(x, y, y, z, x, z)\}.$$

By introducing parentheses around the first four and last two factors we can write

$$\tilde{\Delta}_{X,Y,Z} \subset (X \times Y \times Y \times Z) \times (X \times Z).$$

In other words,

$$\tilde{\Delta}_{X,Y,Z} \in \mathrm{Morph}(X \times Y \times Y \times Z, X \times Z).$$

Let $\Gamma_1 \in \mathrm{Morph}(X, Y)$ and $\Gamma_2 \in \mathrm{Morph}(Y, Z)$. Then

$$\Gamma_1 \times \Gamma_2 \subset X \times Y \times Y \times Z$$

is a "point" of $X \times Y \times Y \times Z$. We can think of it as an element of

$$\mathrm{Morph}(\mathrm{pt.}, X \times Y \times Y \times Z).$$

So we can form

$$\tilde{\Delta}_{X,Y,Z} \circ (\Gamma_1 \times \Gamma_2)$$

which consists of all (x, z) such that

$$\exists (x_1, y_1, y_2, z_1, x, z) \text{ with}$$

$$\begin{array}{rcl} (x_1, y_1) & \in & \Gamma_1, \\ (y_2, z_1) & \in & \Gamma_2 \\ x_1 & = & x \\ y_1 & = & y_2 \\ z_1 & = & z. \end{array}$$

Thus

$$\tilde{\Delta}_{X,Y,Z} \circ (\Gamma_1 \times \Gamma_2) = \Gamma_2 \circ \Gamma_1. \tag{3.2}$$

Similarly, given four sets X, Y, Z, W we can form

$$\tilde{\Delta}_{X,Y,Z,W} \subset (X \times Y \times Y \times Z \times Z \times W) \times (X \times W)$$

$$\tilde{\Delta}_{X,Y,Z,W} = \{(x, y, y, z, z, w, x, w)\}$$

so

$$\tilde{\Delta}_{X,Y,Z,W} \in \mathrm{Morph}(X \times Y \times Y \times Z \times Z \times W, X \times W).$$

If $\Gamma_1 \in \mathrm{Morph}(X, Y)$, $\Gamma_2 \in \mathrm{Morph}(Y, Z)$, and $\Gamma_3 \in \mathrm{Morph}(Z, W)$ then

$$\Gamma_3 \circ (\Gamma_2 \circ \Gamma_1) = (\Gamma_3 \circ \Gamma_2) \circ \Gamma_1 = \tilde{\Delta}_{X,Y,Z,W}(\Gamma_1 \times \Gamma_2 \times \Gamma_3).$$

From this point of view the associative law is a reflection of the fact that

$$(\Gamma_1 \times \Gamma_2) \times \Gamma_3 = \Gamma_1 \times (\Gamma_2 \times \Gamma_3) = \Gamma_1 \times \Gamma_2 \times \Gamma_3.$$

3.3.3 The transpose.

In our category of sets and relations, if $\Gamma \in \mathrm{Morph}(X, Y)$ define $\Gamma^\dagger \in \mathrm{Morph}(Y, X)$ by

$$\Gamma^\dagger := \{(y, x) | (x, y) \in \Gamma\}.$$

We have defined a map

$$\dagger : \mathrm{Morph}(X, Y) \to \mathrm{Morph}(Y, X) \tag{3.3}$$

for all objects X and Y which clearly satisfies

$$\dagger^2 = \mathrm{id} \tag{3.4}$$

and

$$(\Gamma_2 \circ \Gamma_1)^\dagger = \Gamma_1^\dagger \circ \Gamma_2^\dagger. \tag{3.5}$$

This makes our category of finite sets and relations into an involutive category.

3.3.4 The finite Radon transform.

This is a contravariant functor $\mathcal{F}$ from the category of finite sets and relations to the category of finite dimensional vector spaces over a field $\mathbb{K}$. It is defined as follows: On objects we let

$$\mathcal{F}(X) := \mathcal{F}(X, \mathbb{K}) = \text{the space of all } \mathbb{K}\text{-valued functions on } X.$$

If $\Gamma \subset X \times Y$ is a relation and $g \in \mathcal{F}(Y)$ we set

$$(\mathcal{F}(\Gamma)(g))(x) := \sum_{y|(x,y)\in\Gamma} g(y).$$

(It is understood that the empty sum gives zero.) It is immediate to check that this is indeed a contravariant functor.

In case $\mathbb{K} = \mathbb{C}$ we can be more precise: Let us make $\mathcal{F}(X)$ into a (finite dimensional) Hilbert space by setting

$$(f_1, f_2) := \sum_{x\in X} f_1(x)\overline{f_2(x)}.$$

Then for $\Gamma \in \mathrm{Morph}(X, Y), f \in \mathcal{F}(X), g \in \mathcal{F}(Y)$ we have

$$(f, \mathcal{F}(\Gamma)g) = \sum_{(x,y)\in\Gamma} f(x)\overline{g(y)} = (\mathcal{F}(\Gamma^\dagger)f, g).$$

So

$$\mathcal{F}(\Gamma^\dagger) = \mathcal{F}(\Gamma)^\dagger.$$

The functor $\mathcal{F}$ carries the involutive structure of the category of finite sets and relations into the involutive structure of the category of finite dimensional Hilbert spaces.

3.3.5 Enhancing the category of finite sets and relations.

By a vector bundle over a finite set we simply mean a rule which assigns a vector space E_x (which we will assume to be finite dimensional) to each point x of X. We are going to consider a category whose objects are vector bundles over finite sets. We will denote such an object by $E \to X$.

Following Atiyah and Bott, we will define the morphisms in this category as follows: If $E \to X$ and $F \to Y$ are objects in our category, and $\Gamma \subset X \times Y$ we consider the vector bundle over Γ which assigns to each point $(x, y) \in \Gamma$ the vector space $\mathrm{Hom}(F_y, E_x)$. A morphism in our category will be a section of this vector bundle. So a morphism in our category will be a subset Γ of $X \times Y$ together with a map

$$r_{x,y} : F_y \to F_x$$

given for each $(x,y) \in \Gamma$. Suppose that $(\Gamma_1, r) \in \mathrm{Morph}(E \to X, F \to Y)$ and $(\Gamma_2, s) \in \mathrm{Morph}(F \to Y, G \to Z)$. Their composition is defined to be $(\Gamma_2 \circ \Gamma_1, t)$ where t is the section of the vector bundle over $\Gamma_2 \circ \Gamma_1$ given by

$$t(x,z) = \sum_{y|(x,y)\in\Gamma_1,(y,z)\in\Gamma_2} r(x,y) \circ s(y,z).$$

The verification of the category axioms is immediate.

We have **enhanced** the category of finite sets and relations to the category of vector bundles over finite sets.

We also have a generalization of the functor $\mathcal{F}$: we now define $\mathcal{F}(E \to X)$ to be the space of sections of the vector bundle $E \to X$ and if $M \in \mathrm{Morph}(E \to X, F \to Y)$ then

$$\mathcal{F}(g)(x) = \sum_{y|(x,y)\in\Gamma} r(x,y)g(y).$$

This generalizes the Radon functor of the preceding section.

3.4 The linear symplectic category.

3.4.1 Linear Lagrangian squares.

Let V and W be symplectic vector spaces with symplectic forms ω_V and ω_W. We put the direct sum symplectic form on $V \oplus W$ and denote it by $\omega_{V\oplus W}$. Let L be a Lagrangian subspace of V and set

$$H := L \oplus W.$$

Let Λ be a Lagrangian subspace of $V \oplus W$. Consider the exact square

$$\begin{array}{ccc} F & \longrightarrow & \Lambda \\ \downarrow & & \downarrow \iota_\Lambda \\ H & \xrightarrow[\iota_H]{} & V \oplus W \end{array} \tag{3.6}$$

This means that we have the exact sequence

$$0 \to F \to H \oplus \Lambda \xrightarrow{\tau} V \oplus W \to \mathrm{Coker}(\tau) \to 0 \tag{3.7}$$

where the middle map

$$\tau : H \oplus \Lambda \to V \oplus W$$

is given by

$$\tau(h,\lambda) = \iota_H(h) - \iota_\Lambda(\lambda).$$

Let

$$\mathrm{pr} : F \to \Lambda$$

denote projection of $F \subset H \oplus \Lambda$ onto the second component. Let

$$\rho : \Lambda \to W$$

denote the projection of $\Lambda \subset V \oplus W$ onto the second component. So

$$\alpha := \rho \circ \mathrm{pr} : F \to W. \tag{3.8}$$

Theorem 6 *The image of α is a Lagrangian subspace of W.*

Proof. If $w_1 = \alpha((v_1, w_1))$ and $w_2 = \alpha((v_2, w_2))$ then

$$\omega_W(w_1, w_2) = \omega_{V \oplus W}\left((v_1, w_1), (v_2, w_2)\right) - \omega_V(v_1, v_2) = 0.$$

The first term vanishes because Λ is Lagrangian, and the second term vanishes because L is Lagrangian. We have proved that α maps F onto an isotropic subspace of W. We want to prove that this subspace is Lagrangian. We do this by a dimension count:

We have the exact sequence

$$0 \to \ker(\alpha) \to F \to \mathrm{im}(\alpha) \to 0. \tag{3.9}$$

Write

$$\lambda = (v_1, w_1), \qquad h = (v_2, w_2)$$

so that $(h, \lambda) \in F$ when these two expressions are equal. To say that $\rho(\lambda) = 0$ means that $\lambda = (v, 0)$ so we may identify $\ker(\alpha)$ with the set of all $v \in L$ such that

$$(v, 0) \in \Lambda.$$

In this way we identify $\ker \alpha$ as a subspace of $V \oplus W$ consisting of all $(v, 0) \in \Lambda$ with $v \in L$.

On the other hand, $u \in \mathrm{im}(\tau)$ when

$$u = \iota_H(v_2, w_2) - \iota_\Lambda(v_1, w_1)$$

for

$$(v_1, w_1) \in \Lambda, \quad (v_2, w_2) \in H.$$

So

$$\mathrm{im}(\tau) = H + \Lambda$$

and hence

$$\mathrm{im}(\tau)^\perp = H^\perp \cap \Lambda^\perp.$$

But $\Lambda^\perp = \Lambda$ since Λ is Lagrangian, and $H^\perp = L \oplus \{0\}$ since L is Lagrangian and $H = L \oplus W$. So when we think of $\ker \alpha$ as a subspace of $V \oplus W$ we have $\ker \alpha = (\mathrm{im}\,\tau)^\perp$. Hence

$$\ker(\alpha)^\perp = \mathrm{im}(\tau) \quad \text{in } V \oplus W. \tag{3.10}$$

In other words, the symplectic form on $V \oplus W$ induces a non-degenerate pairing between $\ker(\alpha)$ and $\mathrm{Coker}(\tau)$. Thus we can write

$$\dim\ \mathrm{im}(\alpha) = \dim F - \dim\ker(\alpha) = \dim F - \dim\ \mathrm{Coker}\,\tau.$$

From (3.7) we have

$$\dim F - \dim\ \mathrm{Coker}(\tau) = \dim H + \dim\Lambda - \dim V - \dim W.$$

Since $\dim H = \frac{1}{2}\dim V + \dim W$ and $\dim\Lambda = \frac{1}{2}\dim V + \frac{1}{2}\dim W$ we obtain

$$\dim\ \alpha(F) = \frac{1}{2}\dim W$$

as desired. □

From (3.7) it follows that $\mathrm{Coker}(\tau) = \{0\}$ if and only if $H + \Lambda = V \oplus W$, in other words, if and only if the spaces H and Λ are transverse. We have thus proved

Proposition 9 *α is injective if and only if Λ and H are transverse.*

Whether or not α is injective, we may identify $\mathrm{im}(\alpha)$ with the set of all $w \in W$ such that there exists a $v \in V$ such that $(v, w) \in \Lambda$ with $v \in L$. In terms of the notation we shall introduce in the next section, it will be convenient to denote this Lagrangian subspace of W as $\Lambda \circ L$, and think of it as "the image of L under Λ".

3.4.2 The category of symplectic vector spaces and linear Lagrangians.

Let X,Y, and Z be symplectic vector spaces with symplectic forms ω_X, ω_Y, and ω_Z. We will let X^- denote the vector space X equipped with the symplectic form $-\omega_X$. So $X^- \oplus Y$ denotes the vector space $X \oplus Y$ equipped with the symplectic form $-\omega_X \oplus \omega_Y$ and similarly $Y^- \oplus Z$ denotes the vector space $Y \oplus Z$ equipped with the symplectic form $-\omega_Y \oplus \omega_Z$. Let

$$\Lambda_1 \text{ be a Lagrangian subspace of } \quad X^- \oplus Y$$

and let

$$\Lambda_2 \text{ be a Lagrangian subspace of } \quad Y^- \oplus Z.$$

Consider the exact square

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \Lambda_2 \arrow[d] \\ \Lambda_1 \arrow[r] & Y \end{tikzcd}. \tag{3.11}$$

This means that we have the exact sequence

$$0 \to F \to \Lambda_1 \oplus \Lambda_2 \xrightarrow{\tau} Y \to \mathrm{Coker}(\tau) \to 0 \tag{3.12}$$

where

$$\tau : \Lambda_1 \oplus \Lambda_2 \to Y$$

is given by

$$\tau\left((u, y_1), (y_2, z)\right) = y_1 - y_2.$$

Thus the points of F consist of elements of the form

$$(x, y, y, z)$$

where $(x, y) \in \Lambda_1$ and $(y, z) \in \Lambda_2$. Let

$$\alpha : F \to X \oplus Z$$

be defined by

$$\alpha((x, v, v, z)) = (x, z).$$

The image of α is the set of all $(x, z) \in X \oplus Z$ such that there exists a $y \in Y$ with $(x, y) \in \Lambda_1$ and $(y, z) \in \Lambda_2$. If we think of Λ_1 and Λ_2 as *linear relations*, the image of α is just the composite of the two relations in the sense of Section 3.3.1. We may denote it by $\Lambda_2 \circ \Lambda_1$ Then we have

Theorem 7 *The composite $\Lambda_2 \circ \Lambda_1$ is a Lagrangian subspace of $X^- \oplus Z$.*

Proof. Take $V := Y \oplus Y$ with the symplectic form $\omega_{Y1} - \omega_{Y2}$ where ω_{Y2} denotes the pullback of ω_Y to $Y \oplus Y$ via projections onto the second factor, with ω_{Y1} the pullback via projection onto the first factor. Take $L = \Delta$ to be the diagonal in $Y \oplus Y$ so $\Delta := \{(y, y)\}$. Take $W = X \oplus Z$. Identify $V \oplus W$ with $X \oplus Y \oplus Y \oplus Z$ (by putting the two Y components in the middle), and let $\Lambda := \Lambda_1 \oplus \Lambda_2$. So in the terminology Theorem 6, $H = \{(x, y, y, z)\}$ and $\alpha(F) = \Lambda_2 \circ \Lambda_1$. Thus Theorem 7 is a consequence of Theorem 6. $\square$

Theorem 7 means that we get a category if we take as our objects the symplectic vector spaces, and, if X and Y are symplectic vectors spaces define the morphisms from X to Y to consist of the Lagrangian subspaces of $X^- \oplus Y$.

This category is a vast generalization of the symplectic group because of the following observation: Suppose that the Lagrangian subspace $\Lambda \subset X^- \oplus Y$ projects bijectively onto X under the projection of $X \oplus Y$ onto the first factor. This means that Λ is the graph of a linear transformation T from X to Y:

$$\Lambda = \{(x, Tx)\}.$$

T must be injective. Indeed, if $Tx = 0$ the fact that Λ is isotropic implies that $x \perp X$ so $x = 0$. Also T is surjective since if $y \perp \mathrm{im}(T)$, then $(0, y) \perp \Lambda$. This implies that $(0, y) \in \Lambda$ since Λ is maximal isotropic. By the bijectivity of the projection this implies that $y = 0$. In other words T is a bijection. The fact that Λ is isotropic then says that

$$\omega_Y(Tx_1, Tx_2) = \omega(x_1, x_2),$$

i.e. T is a symplectic isomorphism. If $\Lambda_1 = \operatorname{graph} T$ and $\Lambda_2 = \operatorname{graph} S$ then

$$\Lambda_2 \circ \Lambda_1 = \operatorname{graph} S \circ T$$

so composition of Lagrangian relations reduces to composition of symplectic isomorphisms in the case of graphs. In particular, If we take $Y = X$ we see that $\operatorname{Symp}(X)$ is a subgroup of Morph (X, X) in our category.

Chapter 4

The Symplectic "Category".

Let M be a symplectic manifold with symplectic form ω. Then $-\omega$ is also a symplectic form on M. We will frequently write M instead of (M, ω) and by abuse of notation we well let M^- denote the manifold M with the symplectic form $-\omega$.

Let (M_i, ω_i) $i = 1, 2$ be symplectic manifolds. A Lagrangian submanifold of

$$\Gamma \subset M_1^- \times M_2$$

is called a **canonical relation**. So Γ is a subset of $M_1 \times M_2$ which is a Lagrangian submanifold relative to the symplectic form $\omega_2 - \omega_1$ in the obvious notation. So a canonical relation is a relation which is a Lagrangian submanifold.

For example, if $f : M_1 \to M_2$ is a symplectomorphism, then $\Gamma_f =$ graph f is a canonical relation.

If $\Gamma_1 \subset M_1 \times M_2$ and $\Gamma_2 \subset M_2 \times M_3$ we can form their composite

$$\Gamma_2 \circ \Gamma_1 \subset M_1 \times M_3$$

in the sense of the composite of relations. So $\Gamma_2 \circ \Gamma_1$ consists of all points (x, z) such that there exists a $y \in M_2$ with $(x, y) \in \Gamma_1$ and $(y, z) \in \Gamma_2$. Let us put this in the language of fiber products: Let

$$\pi : \Gamma_1 \to M_2$$

denote the restriction to Γ_1 of the projection of $M_1 \times M_2$ onto the second factor. Let

$$\rho : \Gamma_2 \to M_2$$

denote the restriction to Γ_2 of the projection of $M_2 \times M_3$ onto the first factor. Let

$$F \subset M_1 \times M_2 \times M_2 \times M_3$$

be defined by

$$F = (\pi \times \rho)^{-1}\Delta_{M_2}.$$

In other words, F is defined as the fiber product

$$\begin{array}{ccc} F & \xrightarrow{\iota_1} & \Gamma_1 \\ {\scriptstyle \iota_2}\downarrow & & \downarrow{\scriptstyle \pi}. \\ \Gamma_2 & \xrightarrow[\rho]{} & M_2 \end{array} \tag{4.1}$$

so

$$F \subset \Gamma_1 \times \Gamma_2 \subset M_1 \times M_2 \times M_2 \times M_3.$$

Let pr_{13} denote the projection of $M_1 \times M_2 \times M_2 \times M_3$ onto $M_1 \times M_3$ (projection onto the first and last components). Let π_{13} denote the restriction of pr_{13} to F. Then, as a set,

$$\Gamma_2 \circ \Gamma_1 = \pi_{13}(F). \tag{4.2}$$

The map pr_{13} is smooth, and hence its restriction to any submanifold is smooth. The problems are that

1. F defined as
$$F = (\pi \times \rho)^{-1}\Delta_{M_2},$$
i.e. by (4.1), need not be a submanifold, and

2. that the restriction π_{13} of pr_{13} to F need not be an embedding.

So we need some additional hypotheses to ensure that $\Gamma_2 \circ \Gamma_1$ is a submanifold of $M_1 \times M_3$. Once we impose these hypotheses we will find it easy to check that $\Gamma_2 \circ \Gamma_1$ is a Lagrangian submanifold of $M_1^- \times M_3$ and hence a canonical relation.

4.1 Clean intersection.

Assume that the maps

$$\pi : \Gamma_1 \to M_2 \quad \text{and} \quad \rho : \Gamma_2 \to M_2$$

defined above intersect cleanly.

Notice that $(m_1, m_2, m_2', m_3') \in F$ if and only if

- $m_2 = m_2'$,
- $(m_1, m_2) \in \Gamma_1$, and
- $(m_2', m_3) \in \Gamma_2$.

So we can think of F as the subset of $M_1 \times M_2 \times M_3$ consisting of all points (m_1, m_2, m_3) with $(m_1, m_2) \in \Gamma_1$ and $(m_2, m_3) \in \Gamma_2$. The clean intersection hypothesis involves two conditions. The first is that F be a manifold. The second is that the derived square be exact at all points. Let us state this second condition more explicitly: Let $m = (m_1, m_2, m_3) \in F$. We have the following vector spaces:

$$\begin{aligned} V_1 &:= T_{m_1} M_1, \\ V_2 &:= T_{m_2} M_2, \\ V_3 &:= T_{m_3} M_3, \\ \Gamma_1^m &:= T_{(m_1,m_2)} \Gamma_1, \quad \text{and} \\ \Gamma_2^m &:= T_{(m_2,m_3)} \Gamma_2. \end{aligned}$$

So

$$\Gamma_1^m \subset T_{(m_1,m_2)}(M_1 \times M_2) = V_1 \oplus V_2$$

is a linear Lagrangian subspace of $V_1^- \oplus V_2$. Similarly, Γ_2^m is a linear Lagrangian subspace of $V_2^- \oplus V_3$. The clean intersection hypothesis asserts that $T_m F$ is given by the exact square

$$\begin{array}{ccc} T_m F & \xrightarrow{d(\iota_1)_m} & \Gamma_1^m \\ {\scriptstyle d(\iota_2)_m}\big\downarrow & & \big\downarrow{\scriptstyle d\pi_{m_1,m_2)}} \\ \Gamma_2^m & \xrightarrow[d\rho_{(m_2,m_3)}]{} & T_{m_2} M_2 \end{array} \tag{4.3}$$

In other words, $T_m F$ consists of all $(v_1, v_2, v_3) \in V_1 \oplus V_2 \oplus V_3$ such that

$$(v_1, v_2) \in \Gamma_1^m \quad \text{and} \quad (v_2, v_3) \in \Gamma_2^m.$$

The exact square (4.3) is of the form (3.11) that we considered in Section 3.4.2. We know from Section 3.4.2 that $\Gamma_2^m \circ \Gamma_1^m$ is a linear Lagrangian subspace of $V_1^- \oplus V_3$. In particular its dimension is $\frac{1}{2}(\dim M_1 + \dim M_3)$ which does not depend on the choice of $m \in F$. This implies the following: Let

$$\iota : F \to M_1 \times M_2 \times M_3$$

denote the inclusion map, and let

$$\kappa_{13} : M_1 \times M_2 \times M_3 \to M_1 \times M_3$$

denote the projection onto the first and third components. So

$$\kappa \circ \iota : F \to M_1 \times M_3$$

is a smooth map whose differential at any point $m \in F$ maps $T_m F$ onto $\Gamma_2^m \circ \Gamma_1^m$ and so has locally constant rank. Furthermore, the image os $T_m F$ is a Lagrangian subspace of $T_{(m_1,m3)}(M_1^- \times M_3)$. We have proved:

Theorem 8 *If the canonical relations* $\Gamma_1 \subset M_1^- \times M_2$ *and* $\Gamma_2 \subset M_2^- \times M_3$ *intersect cleanly, then their composition* $\Gamma_2 \circ \Gamma_1$ *is an immersed Lagrangian submanifold of* $M_1^- \times M_3$.

We must still impose conditions that will ensure that $\Gamma_2 \circ \Gamma_1$ is an honest submanifold of $M_1 \times M_3$. We will do this in the next section.

We will need a name for the manifold F we created out of Γ_1 and Γ_2 above. We will call it $\Gamma_2 \star \Gamma_1$.

4.2 Composable canonical relations.

Victor: What is a reference for this?

We recall a theorem from differential topology:

Theorem 9 *Let* X *and* Y *be smooth manifolds and* $f : X \to Y$ *is a smooth map of constant rank. Let* $W = f(X)$. *Suppose that* f *is proper and that for every* $w \in W$, $f^{-1}(w)$ *is connected. Then* W *is a smooth submanifold of* Y.

We apply this theorem to the map $\kappa_{13} \circ \iota : F \to M_1 \times M_3$. To shorten the notation, let us define

$$\kappa := \kappa_{13} \circ \iota. \tag{4.4}$$

Theorem 10 *Suppose that the canonical relations* Γ_1 *and* Γ_2 *intersect cleanly. Suppose in addition that the map* κ *is proper and that the inverse image of every* $\gamma \in \Gamma_2 \circ \Gamma_1 = \kappa(\Gamma_2 \star \Gamma_1)$ *is connected. Then* $\Gamma_2 \circ \Gamma_1$ *is a canonical relation. Furthermore*

$$\kappa : \Gamma_2 \star \Gamma_1 \to \Gamma_2 \circ \Gamma_1 \tag{4.5}$$

is a smooth fibration with compact connected fibers.

So we are in the following situation: We can not always compose the canonical relations $\Gamma_2 \subset M_2^- \times M_3$ with $\Gamma_1 \subset M_1^- \times M_2$ to obtain a canonical relation $\Gamma_2 \circ \Gamma_1 \subset M_1^- \times M_3$. We must impose some additional conditions, for example those of the theorem. So following Weinstein we put quotation maps around the word category to indicate this fact.

We will let $\mathcal{S}$ denote the "category" whose objects are symplectic manifolds and whose morphisms are canonical ralations. We will call $\Gamma_1 \subset M_1^- \times M_2$ and $\Gamma_2 \subset M_2^- \times M_3$ **composable** if they satisfy the hypotheses of Theorem 10.

If $\Gamma \subset M_1^- \times M_2$ is a canonical relation, we will sometimes use the notation

$$\Gamma \in \mathrm{Morph}(M_1, M_2)$$

and sometimes use the notation

$$\Gamma : \ M_1 \twoheadrightarrow M_2$$

to denote this fact.

4.3 Transverse composition.

A special case of clean intersection is transverse intersection. In fact, in applications, this is a convenient hypothesis, and it has some special properties:

Suppose that the maps π and ρ are transverse. This means that

$$\pi \times \rho : \Gamma_1 \times \Gamma_2 \to M_2 \times M_2$$

intersects Δ_{M_2} transversally, which implies that the codimension of

$$\Gamma_2 \star \Gamma_1 = (\pi \times \rho)^{-1}(\Delta_{M_2})$$

in $\Gamma_1 \times \Gamma_2$ is $\dim M_2$. So

$$\begin{aligned}
\dim F &= \dim \Gamma_1 + \dim \Gamma_2 - \dim M_2 \\
&= \frac{1}{2}\dim M_1 + \frac{1}{2}\dim M_2 + \frac{1}{2}\dim M_2 + \frac{1}{2}\dim M_3 - \dim M_2 \\
&= \frac{1}{2}\dim M_1 + \frac{1}{2}\dim M_3 \\
&= \dim \Gamma_2 \circ \Gamma_1.
\end{aligned}$$

So under the hypothesis of transversality, the map κ is an immersion. If we add the hypotheses of Theorem 10, we see that κ is a diffeomorphism.

For example, if Γ_2 is the graph of a symplectomorphism of M_2 with M_3 then $d\rho_{(m_2,m_3)} : T_{(m_2,m_3)}(M_2 \times M_3) \to T_{m_2}M_2$ is surjective at all points $(m_2, m_3) \in \Gamma_2$. So if $m = (m_1, m_2, m_2, m_3) \in \Gamma_1 \times \Gamma_2$ the image of $d(\pi \times \rho)_m$ contains all vectors of the form $(0, w)$ in $T_{m_2}M_2 \oplus T_{m_2}M_2$ and so is transverse to the diagonal. The manifold $\Gamma_2 \star \Gamma_1$ consists of all points of the form $(m_1, m_2, g(m_2))$ with $(m_1, m_2) \in \Gamma_1$, and

$$\kappa : (m_1, m_2, g(m_2)) \mapsto (m_1, g(m_2)).$$

Since g is one to one, so is κ. So the graph of a symplectomorphism is transversally composible with any canonical relation.

We will need the more general concept of "clean composability" described in the preceding section for certain applications.

4.4 Lagrangian submanifolds as canonical relations.

We can consider the "zero dimensional symplectic manifold" consisting of the distinguished point that we call "pt.". Then a canonical relation between pt. and a symplectic manifold M is a Lagrangian submanifold of pt. $\times M$ which may be identified with a symplectic submanifold of M. These are the "points" in our "category" $\mathcal{S}$.

Suppose that Λ is a Lagrangian submanifold of M_1 and $\Gamma \in \mathrm{Morph}(M_1, M_2)$ is a canonical relation. If we think of Λ as an element of $\mathrm{Morph}(\mathrm{pt.}, M_1)$, then if Γ and Λ are composible, we can form $\Gamma \circ \Lambda \in \mathrm{Morph}(\mathrm{pt.}, M_2)$ which may be identified with a Lagrangian submanifold of M_2. If we want to think of it this way, we may sometimes write $\Gamma(\Lambda)$ instead of $\Gamma \circ \Lambda$.

We can mimic the construction of composition given in Section 3.3.2 for the category of finite sets and relations. Let M_1, M_2 and M_3 be symplectic manifolds and let $\Gamma_1 \in \mathrm{Morph}(M_1, M_2)$ and $\Gamma_2 \in \mathrm{Morph}(M_2, M_3)$ be canonical relations. So

$$\Gamma_1 \times \Gamma_2 \subset M_1^- \times M_2 \times M_2^- \times M_3$$

is a Lagrangian submanifold. Let

$$\tilde{\Delta}_{M_1,M_2,M_3} = \{(x,y,y,z,x,z)\} \subset M_1 \times M_2 \times M_2 \times M_3 \times M_1 \times M_3. \quad (4.6)$$

We endow the right hand side with the symplectic structure

$$M_1 \times M_2^- \times M_2 \times M_3^- \times M_1^- \times M_3 = (M_1^- \times M_2 \times M_2^- \times M_3)^- \times (M_1^- \times M_3).$$

Then $\tilde{\Delta}_{M_1,M_2,M_3}$ is a Lagrangian submanifold, i.e. an element of

$$\mathrm{Morph}(M_1^- \times M_2 \times M_2^- \times M_3, M_1^- \times M_3).$$

Just as in Section 3.3.2,

$$\tilde{\Delta}_{M_1,M_2,M_3}(\Gamma_1 \times \Gamma_2) = \Gamma_2 \circ \Gamma_1.$$

Victor: I don't know if you want more details here.

It is easy to check that Γ_2 and Γ_1 are composible if and only if $\tilde{\Delta}_{M_1,M_2,M_3}$ and $\Gamma_1 \times \Gamma_2$ are composible.

4.5 The involutive structure on $\mathcal{S}$.

Let $\Gamma \in \mathrm{Morph}(M_1, M_2)$ be a canonical relation. Just as in the category of finite sets and relations, define

$$\Gamma^\dagger = \{(m_2, m_1) | (m_1, m_2) \in \Gamma\}.$$

As a set it is a subset of $M_2 \times M_1$ and it is a Lagrangian sumbanifold of $M_2 \times M_1^-$. But then it is also a Lagrangian submanifold of

$$(M_2 \times M_1^-)^- = M_2^- \times M_1.$$

So

$$\Gamma^\dagger \in \mathrm{Morph}(M_2, M_1).$$

Therefore $M \mapsto M, \Gamma \mapsto \Gamma^\dagger$ is a involutive functor on $\mathcal{S}$.

4.6 Canonical relations between cotangent bundles.

In this section we want to discuss some special properties of our "category" $\mathcal{S}$ when we restrict the objects to be cotangent bundles (which are, after all, special kinds of symplectic manifolds). One consequence of our discussion will be that $\mathcal{S}$ contains the category $\mathcal{C}^\infty$ whose objects are smooth manifolds and whose morphisms are smooth maps as a (tiny) subcategory. Another consequence will be a local description of Lagrangian submanifolds of the cotangent bundle which generalizes the description of horizontal Lagrangian submanifolds of the cotangent bundle that we gave in Chapter 1. We will use this local description to deal with the problem of passage through caustics that we encountered in Chapter 1.

We recall the following definitions from Chapter 1: Let X be a smooth manifold and T^*X its cotangent bundle, so that we have the projection $\pi : T^*X \to X$. The canonical one form α_X is defined by (1.8). We repeat the definition: If $\xi \in T^*X, x = \pi(x)$, and $v \in T_\xi(T^*X)$ then the value of α_X at v is given by

$$\langle \alpha_X, v \rangle := \langle \xi, d\pi_\xi v \rangle. \tag{1.8}$$

The symplectic form ω_X is given by

$$\omega_X = -d\alpha_X. \tag{1.10}$$

So if Λ is a submanifold of T^*X on which α_X vanishes and whose dimension is $\dim X$ then Λ is (a special kind of) Lagrangian submanifold of T^*X. An instance of this is the conormal bundle of a submanifold: Let $Y \subset X$ be a submanifold. Its conormal bundle

$$N^*Y \subset T^*X$$

consists of all $(x, \xi) \in T^*X$ such that $x \in Y$ and ξ vanishes on T_xY.If $v \in T_\xi(N^*Y)$ then $d\pi_\xi(v) \in Y$ so by (1.8) $\langle \alpha_X, v \rangle = 0$.

4.7 The canonical relation associated to a map.

Let X_1 and X_2 be manfolds and $f : X_1 \to X_2$ be a smooth map. We set

$$M_1 := T^*X_1 \quad \text{and} \quad M_2 := T^*X_2$$

with their canonical symplectic structures. We have the identification

$$M_1 \times M_2 = T^*(X_1 \times X_2).$$

The graph of f is a submanifold of $X_1 \times X_2$:

$$X_1 \times X_2 \supset \mathrm{graph}(f) = \{(x_1, f(x_1))\}.$$

So the conormal bundle of the graph of f is a Lagrangian submanifold of $M_1 \times M_2$. Explicitly,

$$N^*(\mathrm{graph}(f)) = \{(x_1, \xi_1, x_2, \xi_2)|x_2 = f(x_1), \quad \xi_1 = -df^*_{x_1}\xi_2\}. \tag{4.7}$$

Let

$$\varsigma_1 : T^*X_1 \to T^*X_1$$

be defined by

$$\varsigma_1(x, \xi) = (x, -\xi).$$

Then $\varsigma_1^*(\alpha_{X_1}) = -\alpha_{X_1}$ and hence

$$\varsigma_1^*(\omega_{X_1}) = -\omega_{X_1}.$$

We can think of this as saying that ς_1 is a symplectomorphism of M_1 with M_1^- and hence

$$\varsigma_1 \times \mathrm{id}$$

is a symplectomorphism of $M_1 \times M_2$ with $M_1^- \times M_2$. Let

$$\Gamma_f := (\varsigma_1 \times \mathrm{id})(N^*(\mathrm{graph}(f)). \tag{4.8}$$

Then Γ_f is a Lagrangian submanifold of $M_1^- \times M_2$. In other words,

$$\Gamma_f \in \mathrm{Morph}(M_1, M_2).$$

Explicitly,

$$\Gamma_f = \{(x_1, \xi_1, x_2, \xi_2)|x_2 = f(x_1), \quad \xi_1 = df^*_{x_1}\xi_2\}. \tag{4.9}$$

Suppose that $g : X_2 \to X_3$ is a smooth map so that $\Gamma_g \in \mathrm{Morph}(M_2, M_3)$. So

$$\Gamma_g = \{(x_2, \xi_2, x_3, \xi_3)|x_3 = g(x_2), \xi_2 = dg^*_{x_2}\xi_3.\}.$$

The the maps

$$\pi : \Gamma_f \to M_2, \qquad (x_1, \xi_1, x_2, \xi_2) \mapsto (x_2, \xi_2)$$

and

$$\rho : \Gamma_g \to M_2, \qquad (x_2, \xi_2, x_3, \xi_3) \mapsto (x_2, \xi_2)$$

are transverse. Indeed at any point $(x_1, \xi_1, x_2, \xi_2, x_2, \xi_2, x_3, \xi_3)$ the image of $d\pi$ contains all vectors of the form $(0, w)$ and the image of $d\rho$ contains all vectors of the form $(v, 0)$. So Γ_g and Γ_f are transversely composible. Their composite $\Gamma_g \circ \Gamma_f$ consists of all (x_1, ξ_1, x_3, ξ_3) such that there exists an x_2 such that $x_2 = f(x_1)$ and $x_3 = g(x_2)$ and a ξ_2 such that $\xi_1 = df^*_{x_1}\xi_2$ and $\xi_2 = dg^*_{x_2}\xi_3$. But this is precisely the condition that $(x_1, \xi_1, x_3, \xi_3) \in \Gamma_{g\circ f}$! We have proved:

Theorem 11 *The assignments*

$$X \mapsto T^*X$$

and

$$f \mapsto \Gamma_f$$

define a covariant functor from the category $\mathcal{C}^\infty$ *of manifolds and smooth maps to the symplectic "category"* $\mathcal{S}$. *As a consequence the assignments* $X \mapsto T^*X$ *and*

$$f \mapsto (\Gamma_f)^\dagger$$

defines a contravariant functor from the category $\mathcal{C}^\infty$ *of manifolds and smooth maps to the symplectic "category"* $\mathcal{S}$.

We now study these functors in a little more detail:

4.8 Pushforward of Lagrangian submanifolds of the cotangent bundle.

Let $f : X_2 \to X_2$ be a smooth map, and $M_1 := T^*X_1$, $M_2 := T^*X_2$ as before. The Lagrangian submanifold $\Gamma_f \subset M_1^- \times M_2$ is defined by (4.9). In particular, it is a subset of $T^*X_1 \times T^*X_2$ and hence a particular kind of relation (in the sense of Chapter 3). So if A is any subset of T^*X_1 then $\Gamma_f(A)$ is a subset of T^*X_2 which we shall also denote by $df_*(A)$. So

$$df_*(A) := \Gamma_f(A), \qquad A \subset T^*X_1.$$

Explicitly,

$$df_*A = \{(y,\eta) \in T^*X_2 | \exists (x,\xi) \in A \text{ with } y = f(x) \text{ and } (x, df_x^*\eta) \in A\}.$$

Now suppose that $A = \Lambda$ is a Lagrangian submanifold of T^*X_1. Considering Λ as an element of $\mathrm{Morph}(\mathrm{pt.}, T^*X_1)$ we may apply Theorem 8. Let

$$\pi_1 : \ N^*(\mathrm{graph}(f)) \to T^*X_1$$

denote the restriction to $N^*(\mathrm{graph}(f))$ of the projection of $T^*X_1 \times T^*X_2$ onto the first component. Notice that $N^*(\mathrm{graph}(f))$ is stable under the map $(x,\xi,y,\eta) \mapsto (x,-\xi,y,-\eta$ and hence π_1 intersects Λ cleanly if and only if $\pi_1 \circ (\varsigma \times \mathrm{id}) : \Gamma_f \to T^*X_1$ intersects Λ cleanly where, by abuse of notation, we have also denoted by π_1 restriction of the projection to Γ_f. So

Theorem 12 *If* Λ *is a Lagrangian submanifold and* $\pi_1 : N^*(\mathrm{graph}(f)) \to T^*X_1$ *intersects* Λ *transversally then* $df_*(\Lambda)$ *is an immersed Lagrangian submanifold of* T^*X_2.

If f has constant rank, then the dimension of $df_x^*T^*(X_2)_{f(x)}$ does not vary, so that $df^*(T^*X_2)$ is a sub-bundle of T^*X_1. If Λ intersects this subbundle transversally, then our conditions are certainly satisified. So

Theorem 13 *Suppose that $f : X_1 \to X_2$ has constant rank. If Λ is a Lagrangian submanifold of T^*X_1 which intersects $df^*T^*X_2$ transversaly then $df_*(\Lambda)$ is a Lagrangian submanifold of T^*X_2.*

For example, if f is an immersion, then $df^*T^*X_2 = T^*X_1$ so all Lagrangian submanifolds are transverse to $df^*T^*X_2$.

Corollary 14 *If f is an immersion, then $df_*(\Lambda)$ is a Lagrangian submanifold of T^*X_2.*

At the other extreme, suppose that $f : X_1 \to X_2$ is a fibration. Then $H^*(X_1) := df^*T^*N$ consists of the "horizontal sub-bundle", i.e those covectors which vanish when restricted to the tangent space to the fiber. So

Corollary 15 *Let $f : X_1 \to X_2$ be a fibration, and let $H^*(X_1)$ be the bundle of the horizontal covectors in T^*X_1. If Λ is a Lagrangian submanifold of T^*X_1 which intersects $H^*(X_1)$ transversaly, then $df_*(\Lambda)$ is a Lagrangian submanifold of T^*X_2.*

An important special case of this corollary for us will be when $\Lambda = \text{graph } d\phi$. Then $\Lambda \cap H^*(X_1)$ consists of those points where the "vertical derivative", i.e. the derivative in the fiber direction vanishes. At such points $d\phi$ descends to give a covector at $x_2 = f(x_1)$. If the intersection is transverse, the set of such covectors is then a Lagrangian submanifold of T^*N. All of the next chapter will be devoted to the study of this special case of Corollary 15.

4.8.1 Envelopes.

Another important special case of Corollary 15 is the theory of envelopes, a classical subject which has more or less disappeared from the standard curriculum:

Let

$$X_1 = X \times S, \qquad X_2 = X$$

where X and S are manifolds and let $f = \pi : X \times S \to X$ be projection onto the first component.

Let

$$\phi : X \times S \to \mathbb{R}$$

be a smooth function having 0 as a regular value so that

$$Z := \phi^{-1}(0)$$

is a submanifold of $X \times S$. In fact, we will make a stronger assumption: Let $\phi_s : N \to \mathbf{R}$ be the map obtained by holding s fixed:

$$\phi_s(x) := \phi(x, s).$$

We make the stronger assumption that each ϕ_s has 0 as a regular value, so that

$$Z_s := \phi_s^{-1}(0) = Z \cap (N \times \{s\})$$

is a submanifold and

$$Z = \bigcup_s Z_s$$

as a set. The Lagrangian submanifold $N^*(Z) \subset T^*(X\times)$ consists of all points of the form

$$(x, s, td\phi_X(x,s), td_S\phi(x,s)) \text{ such that } \phi(x,s) = 0.$$

Here t is an arbitrary real number. The sub-bundle $H^*(X \times S)$ consists of all points of the form

$$(x, s, \xi, 0).$$

So the transversality condition of Corollary 15 asserts that the map

$$z \mapsto d\left(\frac{\partial \phi}{\partial s}\right)$$

have rank equal to dim S on Z. The image Lagrangian submanifold $df_*N^*(Z))$ then consists of all covectors $td_X\phi$ where

$$\phi(x,s) = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial s}(x,s) = 0,$$

a system of $p{+}1$ equations in $n{+}p$ variables, where $p = \dim S$ and $n = \dim X$

Our transversality assumptions say that these equations define a submanifold of $N \times S$. If we make the stronger hypothesis that the last p equations can be solved for s as a function of x, then the first equation becomes

$$\phi(x, s(x)) = 0$$

which defines a hypersurface $\mathcal{E}$ called the envelope of the surfaces Z_s. Furthermore, by the chain rule,

$$d\phi(\cdot, s(\cdot)) = d_X\phi(\cdot, s(\cdot)) + d_S\phi(\cdot, s(\cdot))d_Xs(\cdot) = d_X\phi(\cdot, s(\cdot))$$

since $d_S\phi = 0$ at the points being considered. So if we set

$$\psi := \phi(\cdot, s(\cdot))$$

we see that under these restrictive hypotheses $df_*N^*(Z))$ consists of all multiples of $d\psi$, i.e.

$$df_*(N^*(Z)) = N^*(\mathcal{E})$$

is the normal bundle to the envelope.

In the classical theory, the envelope "develops singularities". But form our point of view it is natural to consider the Lagrangian submanifold $df_*(Z)$. This will not be globally a normal bundle to a hypersurface because its projection on N (from T^*N) may have singularities. But as a submanifold of T^*N it is fine:

Examples:

- Suppose that S is an oriented curve in the plane, and at each point $s \in S$ we draw the normal ray to S at s. We might think of this line as a light ray propagating down the normal. The initial curve is called an “initial wave front” and the curve along which the the light tends to focus is called the “caustic”. Focusing takes place where “nearby normals intersect” i.e. at the envelope of the family of rays. These are the points which are the loci of the centers of curvature of the curve, and the corresponding curve is called the evolute.

- We can let S be a hypersurface in n- dimensions, say a surface in three dimensions. We can consider a family of lines emanating from a point source (possible at infinity), and reflected by by S. The corresponding envelope is called the “caustic by reflection”. In Descartes' famous theory of the rainbow he considered a family of parallel lines (light rays from the sun) which were refracted on entering a spherical raindrop, internally reflected by the opposite side and refracted again when exiting the raindrop. The corresponding “caustic” is the Descartes cone of 42 degrees.

- If S is a submanifold of $\mathbf{R}^n$ we can consider the set of spheres of radius r centered at points of S. The corresponding envelope consist of “all points at distance r from S”. But this develops singularities past the radii of curvature. Again, from the Lagrangian or “upstaris” point of view there is no problem.

4.9 Pullback of Lagrangian submanifolds of the cotangent bundle.

We now investigate the contravariant functor which assigns to the smooth map $f : X_1 \to X_2$ the canonical relation

$$\Gamma_f^\dagger : \quad T^*X_2 \twoheadrightarrow T^*X_1.$$

As a subset of $T^*(X_2) \times T^*(X_1)m$, $\Gamma_f^\dagger$ consists of all

$$(y, \eta, x, \xi)|\ y = f(x), \ \text{ and } \ \xi = df_x^*(\eta). \tag{4.10}$$

If B is a subset of T^*X_2 we can form $\Gamma_f^\dagger(B) \subset T^*X_1$ which we shall denote by $df^*(B)$. So

$$df^*(B) := \Gamma_f^\dagger(B) = \{(x, \xi)|\exists\ b = (y, \eta) \in B \text{ with } f(x) = y,\ df_x^*\eta = \xi\}. \tag{4.11}$$

If $B = \Lambda$ is a Lagrangian submanifold, Once again we may apply Theorem 8 to obtain a sufficient condition for $df^*(\Lambda)$ to be a Lagrangian submanifold of T^*X_1. Notice that in the description of $\Gamma_f^\dagger$ given in (4.10), the η can vary freely in $T^*(X_2)_{f(x)}$. So the issue of clean or transverse inersection comes down to the behavior of the first component. So, for example, we have the following theorem:

Theorem 16 *Let $f : X_1 \to X_2$ be a smooth map and Λ a Lagrangian submanifold of T^*X_2. If the maps f, and the restriction of the projection $\pi : T^*X_2 \to X_2$ to Λ are transverse, then $df^*\Lambda$ is a Lagrangian submanifold of T^*X_1.*

Here are two examples of the theorem:

- Suppose that Λ is a horizontal Lagrangian submanifold of T^*X_2. This means that restriction of the projection $\pi : T^*X_2 \to X_2$ to Λ is a diffeomorphism and so the transversality condition is satisfied for any f. Indeed, if $\Lambda = \Lambda_\phi$ for a smooth function ϕ on X_2 then

$$f^*(\Lambda_\phi) = \Lambda_{f^*\phi}.$$

- Suppose that $\Lambda = N^*(Y)$ is the normal bundle to a submanifold Y of X_2. The transversality condition becomes the condition that the map f is transversal to Y. Then $f^{-1}(Y)$ is a submanifold of X_1. If $x \in f^{-1}(Y)$ and $\xi = df_x^*\eta$ with $(f(x), \eta) \in N^*(Y)$ then ξ vanishes when restricted to $T(f^{-1}(Y))$, i.e. $(x, \xi) \in \mathcal{N}(f^{-1}(S))$. More precisely, the transversality asserts that at each $x \in f^{-1}(Y)$ we have $df_x(T(X_1)_x) + TY_{f(x)} = T(X_2)_{f(x)}$ so

$$T(X_1)_x/T(f^{-1}(Y))_x \cong T(X_2)_{f(x))}/TY_{f(x)}$$

and so we have an isomorphism of the dual spaces

$$N_x^*(f^{-1}(Y)) \cong N^*f(x)(Y).$$

In short, the pullback of $N^*(Y)$ is $N^*(f^{-1}(Y))$.

Chapter 5

Generating functions.

In this chapter we continue the study of canonical relations between cotangent bundles. We begin by studying the canonical relation associated to a map in the special case when this map is a fibration. This will allow us to generalize the local description of a Lagrangian submanifold of T^*X that we studied in Chapter 1. In Chapter 1 we showed that a *horizontal* Lagrangian submanifold of T^*X is locally described as the set of all $d\phi(x)$ where $\phi \in C^\infty(X)$ and we called such a function a "generating function". The purpose of this chapter is to generalize this concept by introducing the notion of a generating function relative to a fibration.

5.1 Fibrations.

In this section we will study in more detail the canonical relation associated to a fibration. So let X and Z be manifolds and

$$\pi : Z \to X$$

a smooth fibration. Then

$$\Gamma_\pi \in \mathrm{Morph}(T^*Z, T^*X)$$

consists of all $(z, \xi, x, \eta) \in T^*Z \times T^*X$ such that

$$x = \pi(z) \quad \text{and} \qquad \xi = (d\pi_z)^*\eta.$$

Then

$$\mathrm{pr}_1 : \Gamma_\pi \to T^*Z, \qquad (z, \xi, x, \eta) \mapsto (z, \xi)$$

maps Γ_π bijectively onto the sub-bundle of T^*Z consisting of those covectors which vanish on tangents to the fibers. We will call this sub-bundle the **horizontal sub-bundle** and denote it by H^*Z. So at each $z \in Z$, the fiber of the horizontal sub-bundle is

$$H^*(Z)_z = \{(d\pi_z)^*\eta,\ \eta \in T^*_{\pi(z)}X\}.$$

Let Λ_Z be a Lagrangian submanifold of T^*Z which we can also think of as an element of Morph(pt., T^*Z). We want to study the condition that Γ_π and Λ_Z be composible so that we be able to form

$$\Gamma_\pi(\Lambda_Z) = \Gamma_\pi \circ \Lambda_Z$$

which would then be a Lagrangian submanifold of T^*X. If $\iota : \Lambda_Z \to T^*Z$ denotes the inclusion map then the clean intersection part of the composibility condition requires that ι and pr_1 intersect cleanly. This is the same as saying that Λ_Z and H^*Z intersect cleanly in which case the intersection

$$F := \Lambda_Z \cap H^*Z$$

is a smooth manifold and we get a smooth map $\kappa : F \to T^*X$. The remaining hypotheses of Theorem 10 require that this map be proper and have connected fibers.

A more restrictive condition is that intersection be transversal, i.e.that

$$\Lambda_Z \pitchfork H^*Z$$

in which case we always get a Lagrangian immersion

$$F \to T^*X, \quad (z, d\pi_z^*\eta) \mapsto (\pi(z), \eta).$$

The additional composibilty condition is that this be an embedding.

Let us specialize further to the case where Λ_Z is a horizontal Lagrangian submanifold of T^*Z. That is, we assume that

$$\Lambda_Z = \Lambda_\phi = \gamma_\phi(Z) = \{(z, d\phi(z))\}$$

as in Chapter 1. When is

$$\Lambda_\phi \pitchfork H^*Z?$$

Now H^*Z is a subbundle of T^*Z so we have the exact sequence of vector bundles

$$0 \to H^*Z \to T^*Z \to V^*Z \to 0 \tag{5.1}$$

where

$$(V^*Z)_z = T_z^*Z/(H^*Z)_z = T_z^*(\pi^{-1}(x)), \quad x = \pi(x)$$

is the cotangent space to the fiber through z.

Any section $d\phi$ of T^*Z gives a section $d_{vert}\phi$ of V^*Z by the above exact sequence, and $\Lambda_\phi \pitchfork H^*Z$ if and only if this section intersects the zero section of V^* transversally. If this happens,

$$C_\phi := \{z \in Z | (d_{vert}\phi)_z = 0\}$$

is a submanifold of Z whose dimension is $\dim X$. Furthermore, at any $z \in C_\phi$

$$d\phi_z = (d\pi_z)^*\eta \quad \text{for a unique} \quad \eta \in T^*_{\pi(z)}X.$$

Thus Λ_ϕ and Γ_π are transversally composible if and only if

$$C_\phi \to T^*X, \quad z \mapsto (\pi(z), \eta)$$

is a Lagrangian embedding in which case its image is a Lagrangian submanifold

$$\Lambda = \Gamma_\pi(\Lambda_\phi) = \Gamma_\pi \circ \Lambda_\phi$$

of T^*X. When this happens we say that ϕ is a **a transverse generating function of Λ with respect to the fibration** (Z, π).

If Λ_ϕ and Γ_π are merely cleanly composible, we say that ϕ is a **clean** generating function with respect to π.

If ϕ is a transverse generating function for Λ with respect to the fibration, π, and $\pi_1 : Z_1 \to Z$ is a fibration over Z, then its easy to see that $\phi_1 = \pi_1^*\phi$ is a clean generating function for Λ with respect to the fibration, $\pi \circ \pi_1$; and we will show in the next section that there is a converse result: Locally *every* clean generating can be obtained in this way from a transverse generating function. For this reason it will suffice, for most of the things we'll be doing in this chapter, to work with transverse generating functions; and to simplify notation, we will henceforth, unless otherwise stated, use the terms "generating function" and "transverse generating function" interchangeably.

5.1.1 Transverse vs. clean generating functions.

Locally we can assume that Z is the product, $X \times S$, of X with an open subset, S, of $\mathbb{R}^k$. Then H^*Z is defined by the equations, $\eta_1 = \cdots = \eta_k = 0$, where the η_i's are the standard cotangent coordinates on T^*S; so $\Lambda_\phi \cap H^*Z$ is defined by the equations

$$\frac{\partial \phi}{\partial s_i} = 0\,, \quad i = 1, \ldots, k\,.$$

Let C_ϕ be the subset of $X \times S$ defined by these equations. Then if Λ_ϕ intersects H^*Z cleanly, C_ϕ is a submanifold of $X \times S$ of codimension. $r \leq k$; and, at every point, $(x_0, s_0) \in C_\phi$, C_ϕ can be defined locally near (x_0, s_0) by r of these equations, i.e., modulo repagination, by the equations

$$\frac{\partial \phi}{\partial s_i} = 0\,, \quad i = 1, \ldots, r\,.$$

Moreover these equations have to be non-degenerate: the tangent space at (x_0, s_0) to C_ϕ has to be defined by the equations

$$d\left(\frac{\partial \phi}{\partial s_i}\right)_{(x_0 \xi_0)} = 0\,, \quad i = 1, \ldots, r\,.$$

Suppose $r < k$ (i.e., suppose this clean intersection is not transverse). Since $\partial\phi/\partial s_k$ vanishes on C_ϕ, there exist C^∞ functions, $g_i \in C^\infty(X \times S)$, $i =$

$1, \ldots, r$ such that

$$\frac{\partial \phi}{\partial s_k} = \sum_{i=1}^{r} g_i \frac{\partial \phi}{\partial s_i}\,.$$

In other words, if ν is the vertical vector field

$$\nu = \frac{\partial}{\partial s_k} - \sum_{i=1}^{r} g_i(x,s) \frac{\partial}{\partial s_i}$$

then $L_\nu \phi = 0$. Therefore if we make a change of vertical coordinates

$$(s_i)_{\mathrm{new}} = (s_i)_{\mathrm{new}}(x,s)$$

so that in these new coordinates

$$\nu = \frac{\partial}{\partial s_k}$$

this equation reduces to

$$\frac{\partial}{\partial s_k} \phi(x,s) = 0\,,$$

so, in these new coordinates,

$$\phi(x,s) = \phi(x, s_1, \ldots, s_{k-1})\,.$$

Iterating this argument we can reduce the number of vertical coordinates so that $k = r$, i.e., so that ϕ is a transverse generating function in these new coordinates. In other words, a clean generating function is just a transverse generating function to which a certain number of vertical "ghost variables" ("ghost" meaning that the function doesn't depend on these variables) have been added. The number of these ghost variables is called the *excess* of the generating function. (Thus for the generating function in the paragraph above, its excess is $k - r$.) More intrinsically the *excess is the difference between the dimension of the critical set C_ϕ of ϕ and the dimension of X.*

5.2 The generating function in local coordinates.

Suppose that X is an open subset of $\mathbb{R}^n$, that

$$Z = X \times \mathbb{R}^k$$

that π is projection onto the first factor, and that (x,s) are coordinates on Z so that $\phi = \phi(x,s)$. Then $C_\phi \subset Z$ is defined by the k equations

$$\frac{\partial \phi}{\partial s_i} = 0, \qquad i = 1, \ldots, k.$$

and the transversality condition is that these equations be functionally independent. This amounts to the hypothesis that their differentials

$$d\left(\frac{\partial\phi}{\partial s_i}\right) \qquad i=1,\dots k$$

be linearly independent. Then $\Lambda \subset T^*X$ is the image of the embedding

$$C_\phi \to T^*X, \quad (x,s) \mapsto \frac{\partial\phi}{\partial x} = d_X\phi(x,s).$$

Example - a generating function for a conormal bundle. Suppose that

$$Y \subset X$$

is a submanifold defined by the k functionally independent equations

$$f_1(x) = \cdots = f_k(x) = 0.$$

Let $\phi : X \times \mathbb{R}^k \to \mathbb{R}$ be the function

$$\phi(x,s) := \sum_i f_i(x)s_i. \tag{5.2}$$

We claim that

$$\Lambda = \Gamma_\pi \circ \Lambda_\phi = N^*Y, \tag{5.3}$$

the conormal bundle of Y. Indeed,

$$\frac{\partial\phi}{\partial s_i} = f_i$$

so

$$C_\phi = Y \times \mathbb{R}^k$$

and the map

$$C_\phi \to T^*X$$

is given by

$$(x,s) \mapsto \sum s_i d_X f_(x).$$

The differentials $d_X f_x$ span the conormal bundle to Y at each $x \in Y$ proving (5.3). As a special case of this example, suppose that

$$X = \mathbb{R}^n \times \mathbb{R}^n$$

and that Y is the diagonal

$$\mathrm{diag}(X) = \{(x,x)\} \subset X$$

which may be described as the set of all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$x_i - y_i = 0, \quad \imath = 1, \ldots, n.$$

We may then choose

$$\phi(x, y, s) = \sum_i (x_i - y_i) s_i. \tag{5.4}$$

Now $\mathrm{diag}(X)$ is just the graph of the identity transformation so by Section 4.7 we know that $\varsigma_1 \times \mathrm{id}(N^*(\mathrm{diag}(X))$ is the canonical relation giving the identity map on T^*X. By abuse of language we can speak of ϕ as the generating function of the identity canonical relation. (But we must remember the ς_1.)

5.3 Example. The generating function of a geodesic flow.

A special case of our generating functions with respect to a fibration is when the fibration is trivial, i.e. π is a diffeomorphism. Then the vertical bundle is trivial and we have no "auxiliary variables". Such a generating function is just a generating function in the sense of Chapter 1. For example, let $X = \mathbb{R}^n$ and let $\phi_t \in C^\infty(X \times X)$ be defined by

$$\phi_t(x, y) := \frac{1}{2t} d(x, y)^2, \tag{5.5}$$

where

$$t \neq 0.$$

Let us compute Λ_ϕ and $(\varsigma_1 \times \mathrm{id})(\Lambda_\phi)$. We first do do this computation under the assumption that the metric occurring in (5.5) is the Euclidean metric so that

$$\begin{aligned}
\phi(x, y, t) &= \frac{1}{2t} \sum_i (x_i - y_y)^2 \\
\frac{\partial \phi}{\partial x_i} &= \frac{1}{t}(x_i - y_i) \\
\frac{\partial \phi}{\partial y_i} &= \frac{1}{t}(y_i - x_i) \quad \text{so} \\
\Lambda_\phi &= \{(x, \frac{1}{t}(x - y), y, \frac{1}{t}(y - x)\} \text{ and} \\
(\varsigma_1 \times \mathrm{id})(\Lambda_\phi) &= \{(x, \frac{1}{t}(y - x), y, \frac{1}{t}(y - x)\}.
\end{aligned}$$

In this last equation let us set $y - x = t\xi$, i.e.

$$\xi = \frac{1}{t}(y - x)$$

which is possible since $t \neq 0$. Then

$$(\varsigma_1 \times \mathrm{id})(\Lambda_\phi) = \{(x, \xi, x + t\xi, \xi)\}$$

which is the graph of the symplectic map

$$(x, \xi) \mapsto (x, x + t\xi).$$

If we identify cotangent vectors with tangent vectors (using the Eulidean metric) then $x+t\xi$ is the point along the line passing through x with tangent vector ξ a distance $t\|\xi\|$ out. The one parameter family of maps $(x, \xi) \mapsto (x, x + t\xi)$ is known as the geodesic flow. In the Euclidean space, the time t value of this flow is a diffeomorphism of T^*X with itself for every t. So long as $t \neq 0$ it has the generating function given by (5.5) with no need of auxiliary variables. When $t = 0$ the map is the identity and we need to introduce a fibration.

More generally, this same computation works on any geodesically convex Riemannian manifold:

A Riemannian manifold is called **geodesically convex** if, given any two points x and y in X there is a unique geodesic which joins them. We will show that the above computation of the generating function works for any geodesically convex Riemannian manifold. In fact, we will prove a more general result. Recall that geodesics on a Riemannian manifold can be described as follows: A Riemann metric on a manifold X is the same as a scalar on each tangent space T_xX which varies smoothly with X. This induces an identification of TX with T^*X an hence a scalar product $\langle\ ,\ \rangle_x$ on each T^*X. This in turn induces the "kinetic energy" Hamltonian

$$H(x, \xi) := \frac{1}{2}\langle \xi, \xi \rangle_x.$$

The principle of least action says that the solution curves of the corresponding vector field v_H project under $\pi : T^*X \to X$ to geodesics of X and every geodesic is the projection of such a trajectory. An important property of the kinetic energy Hamilonian is that it is quadratic of degree two in the fiber variables. We will prove a theorem (see Theorem **??** below) which generalizes the above computation and is valid for any Hamiltonian which is homogeneous of degree $k \neq 1$ in the fiber variables and which satisfies a condition analogous to the geodesic convexity theorem. We first recall some facts about homogeneous functions and Euler's theorem.

Consider the one parameter group of dilatations $t \mapsto \mathfrak{d}(t)$:

$$\mathfrak{d}(t) : T^*X \to T^*X : \qquad (x, \xi) \mapsto (x, e^t\xi).$$

A function f is homogenous of degree k in the fiber variables if and only if

$$\mathfrak{d}(t)^* f = e^{kt} f.$$

For example, the principal symbol of a k-th order linear partial differential operator on X is a function on T^*X with which is a polynomial in the fiber variables and is homogenous of degree k. Let $\mathcal{E}$ denote the vector field which is the infinitesimal generator of the one parameter group of dilatations. It is called the **Euler vector field**. Euler's theorem (which is a direct computation from the preceding equation) says that f is homogenous of degree k if and only if

$$\mathcal{E} f = kf.$$

Let $\alpha = \alpha_X$ be the canonical one form on T^*X. From its very definition (1.8) it follows that

$$\mathfrak{d}(t)^* \alpha = e^t \alpha$$

and hence that

$$D_{\mathcal{E}} \alpha = \alpha.$$

Since $\mathcal{E}$ is everywhere tangent to the fiber, it also follows from (1.8) that

$$i(\mathcal{E})\alpha = 0$$

and hence that

$$D_{\mathcal{E}} \alpha = i(\mathcal{E}) d\alpha = -i(\mathcal{E})\omega$$

where $\omega = \omega_X = -d\alpha$.

Now let H be a function on T^*X which is homogeneous of degree k in the fiber variables. Then

$$\begin{aligned}
kH = \mathcal{E}H &= i(\mathcal{E})dH \\
&= i(\mathcal{E})i(v_H)\omega \\
&= -i(v_H)i(\mathcal{E})\omega \\
&= i(v_H)\alpha \quad \text{so} \\
(\exp v_H)^*\alpha - \alpha &= \int_0^1 \frac{d}{dt}(\exp tv_H)^*\alpha dt \quad \text{and} \\
\frac{d}{dt}(\exp tv_H)^*\alpha &= (\exp tv_H)^*\left(i(v_H)d\alpha + di(v_H)\alpha\right) \\
&= (\exp tv_H)^*\left(-i(v_H)\omega + di(v_H)\alpha\right) \\
&= (\exp tv_H)^*\left(-dH + kdH\right) \\
&= (k-1)(\exp tv_H)^* dH \\
&= (k-1)d(\exp tv_H)^* H \\
&= (k-1)dH
\end{aligned}$$

since H is constant along the trajectories of v_H. So

$$(\exp tv_H)^*\alpha - \alpha = (k-1)dH. \tag{5.6}$$

Remark. In the above calculation we assumed that H was smooth on all of T^*X including the zero section, effectively implying that H is a polynomial in the fiber variables. But the same argument will go through (if $k > 0$) if all we assume is that H (and hence v_H) are defined on $T^*X\setminus$ the zero section, in which case H can be a more general homogeneous function on $T^*X\setminus$ the zero section.

Now $\exp v_H \ : T^*X \to T^*X$ is symplectic map, and let

$$\Gamma := \text{graph } (\exp v_H),$$

so $\Gamma \subset T^*X^- \times T^*X$ is a Lagrangian submanifold. Suppose that the projection $\pi_{X\times X}$ of Γ onto $X \times X$ is a diffeomorphism, i.e. suppose that Γ is horizontal. This says precisely that for every $(x, y) \in X \times X$ there is a unique $\xi \in T^*_x X$ such that

$$\pi \exp v_H(x, \xi) = y.$$

In the case of the geodesic flow, this is guaranteed by the condition of geodesic convexity.

Since Γ is horizontal, it has a generating function ϕ such that

$$d\phi = \mathrm{pr}_2^*\,\alpha - \mathrm{pr}_1^*\,\alpha$$

where $\mathrm{pr}_i, \quad i = 1, 2$ are the projections of $T^*(X \times X) = T^*X \times T^*X$ onto the first and second factors. On the other hand pr_1 is a diffeomorphism of Γ onto T^*X. So

$$\mathrm{pr}_1 \circ (\pi_{X\times X|\Lambda})^{-1}$$

is a diffeomorphism of $X \times X$ with T^*X.

Victor: Is this theorem vacuous when $k \neq 2$?. For example, if $k = 1$ then v_H is the lift of a vector field v on X and so the hypotheses are never satisfied.

Theorem 17 *Under the above hypotheses, then up to an additive constant we have*

$$\left(\mathrm{pr}_1 \circ (\pi_{X\times X|\Lambda})^{-1}\right)^* [(k-1)H] = \phi.$$

Indeed, this follows immediately from(5.6). An immediate corollary is that (5.5) is the generating function for the time t flow on a geodesically convex Riemmanian manifold.

As mentioned in the above remark, the same theorem will hold if H is only defined on $T^* \setminus \{0\}$ and the same hypotheses hold with $X \times X$ relpace by $X \times X \setminus \Delta$.

5.4 The generating function for the transpose.

Let

$$\Gamma \in \mathrm{Morph}(T^*X, T^*Y)$$

be a canonical relation, let

$$\pi : Z \to X \times Y$$

be a fibration and ϕ a generating function for Γ relative to this fibration. In local coordinates this says that $Z = X \times Y \times S$, that

$$C_\phi = \{(x, y, s) | \frac{\partial \phi}{\partial s} = 0\},$$

and that Γ is the image of C_ϕ under the map

$$(x, y, s) \mapsto (-d_X\phi, d_Y\phi).$$

Recall that

$$\Gamma^\dagger \in \mathrm{Morph}(T^*Y, T^*X)$$

is given by the set of all (γ_2, γ_1) such that $(\gamma_1, \gamma_2) \in \Gamma$. So if

$$\kappa : X \times Y \to Y \times X$$

denotes the transposition

$$\kappa(x, y) = (y, x)$$

then

$$\kappa \circ \pi : Z \to Y \times X$$

is a fibration and $-\phi$ is a generating function for $\Gamma^\dagger$ relative to $\kappa \circ \pi$. Put more succinctly, if $\phi(x, y, s)$ is a generating function for Γ then

$$\psi(y, x, s) = -\phi(x, y, s) \text{ is a generating function for } \Gamma^\dagger. \tag{5.7}$$

For example, if Γ is the graph of a symplectomorphism, then $\Gamma^\dagger$ is the graph of the inverse diffeomorphism. So (5.7) says that $-\phi(y, x, s)$ generates the inverse of the symplectomorphism generated by $\phi(x, y, s)$.

This suggests that there should be a simple formula which gives a generating function for the composition of two canonical relations in terms of the generating function of each. This was one of Hamilton's great achievements - that, in a suitable sense to be described in the next section - the generating function for the composition is the sum of the individual generating functions.

5.5 Transverse composition of canonical relations between cotangent bundles.

Let X_1, X_2 and X_3 be manifolds and

$$\Gamma_1 \in \mathrm{Morph}(T^*X_1, T^*X_2), \quad \Gamma_2 \in \mathrm{Morph}(T^*X_2, T^*X_3)$$

be canonical relations which are transversally composible. So we are assuming in particular that the maps

$$\Gamma_1 \to T^*X_2, \quad (p_1, p_2) \mapsto p_2 \quad \text{and} \quad \Gamma_2 \to T^*X_2, \quad (q_2, q_3) \mapsto q_2$$

are transverse.

Suppose that

$$\pi_1 : Z_1 \to X_1 \times X_2, \quad \pi : Z_2 \to X_2 \times X_3$$

are fibrations and that $\phi_i \in C^\infty(Z_i),\ i = 1, 2$ are generating functions for Γ_i with respect to π_i.

From π_1 and π_2 we get a map

$$\pi_1 \times \pi_2 : Z_1 \times Z_2 \to X_1 \times X_2 \times X_2 \times X_3.$$

Let

$$\Delta_2 \subset X_2 \times X_2$$

be the diagonal and let

$$Z := (\pi_1 \times \pi_2)^{-1}(X_1 \times \Delta_2 \times X_3).$$

Finally, let

$$\pi : Z \to X_1 \times X_3$$

be the fibration

$$Z \to Z_1 \times Z_2 \to X_1 \times X_2 \times X_2 \times X_3 \to X_1 \times X_3$$

where the first map is the inclusion map and the last map is projection onto the first and last components. Let

$$\phi : Z \to \mathbb{R}$$

be the restriction to Z of the function

$$(z_1, z_2) \mapsto \phi_1(z_1) + \phi_2(z_2).$$

Theorem 18 *ϕ is a generating function for $\Gamma_2 \circ \Gamma_1$ with respect to the fibration π.*

Proof. We may check this in local coordinates where the fibrations are trivial to that

$$Z_1 = X_1 \times X_2 \times S, \qquad Z_2 = X_2 \times X_3 \times T$$

so

$$Z = X_1 \times X_3 \times (X_2 \times S \times T)$$

and π is the projection of Z onto $X_1 \times X_3$. Notice that X_2 has now become a factor in the *parameter space*. The function ϕ is given by

$$\phi(x_1, x_3, x_2, s, t) = \phi_1(x_1, x_2, s) + \phi_2(x_2, x_3, t).$$

Then for $z = (x_1, x_3, x_2, s, t)$ to belong to C_ϕ the following three conditions must be satisfied:

1. $\frac{\partial \phi_1}{\partial s}(x_1, x_2, s) = 0$, i.e. $z_1 = (x_1, x_2, s) \in C_{\phi_1}$.
2. $\frac{\partial \phi_2}{\partial t} = 0$, i.e. $z_2 = (x_2, x_3, t) \in C_{\phi_2}$ and
3. $$\frac{\partial \phi_1}{\partial x_2}(x_1, x_2, s) + \frac{\partial \phi_2}{\partial x_2}(x_2, x_3, t) = 0.$$

Conditions 1) and 2) are clearly independent of one another and of 3). We are assuming that

$$\gamma_1 : C_{\phi_1} \to T^*(X_1 \times X_2) \quad z_1 \mapsto \left(-\frac{\partial \phi_1}{\partial x_1}, \frac{\partial \phi_1}{\partial x_2}\right)$$

is an embedding and a diffeomorphism of C_{ϕ_1} with Γ_1 and that

$$\gamma_2 : C_{\phi_1} \to T^*(X_2 \times X_3) \quad z_2 \mapsto \left(-\frac{\partial \phi_2}{\partial x_2}, \frac{\partial \phi_2}{\partial x_3}\right)$$

is an embedding and a diffeomorphism of C_{ϕ_2} with Γ_2. We are also assuming that the projection

$$\Gamma_1 \times \Gamma_2 \to T^*X_2 \times T^*X_2$$

intersects the diagonal transversally. So the map

$$\left(\frac{\partial \phi_1}{\partial x_2}, -\frac{\partial \phi_2}{\partial x_2}\right) : \; C_{\phi_1} \times C_{\phi_2} \to T^*X_2 \times T^*X_2$$

intersects the diagonal transversally and this is precisely the independence requirement on condition 3). The transverse composibility of Γ_2 and Γ_1 says that the map

$$C_\phi \to T^*X_1 \times T^*X_3, \quad z \mapsto \left(-\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_3}\right)$$

is a Lagrangian embedding which, by definition has its image $\Gamma_2 \circ \Gamma_1$. □

In the next section we will show that the arguments given above apply, essentially without change, to *clean* generating functions, since, as we saw in Section 5.1.1, clean generating functions are just transverse generating functions to which a number of vertical "ghost variables" have been added.

5.6 Clean composition of canonical relations between cotangent bundles.

Suppose that the canonical relation, Γ_1 and Γ_η are *cleanly* composible. Let $\phi_1 \in C^\infty(X_1 \times X_2 \times S)$ and $\phi_2 \in C^\infty(X_2 \times X_3 \times T)$ be *transverse* generating functions for Γ_1 and Γ_2 and as above let

$$\phi(x_1, x_3, x_2, s, t) = \phi_1(x_1, x_2, s) + \phi_2(x_2, x_3, t)\,.$$

We will prove below that ϕ is a clean generating function for $\Gamma_2 \circ \Gamma_1$ with respect to the fibration

$$X_1 \times X_3 x(X_2 \times S \times T) \to X_1 \times X_3 \,.$$

The argument is similar to that above: As above C_ϕ is defined by the three sets of equations:

1. $\frac{\partial \phi_1}{\partial s} = 0$
2. $\frac{\partial \phi_2}{\partial t} = 0$
3. $\frac{\partial \phi_1}{\partial x_2} + \frac{\partial \phi_2}{\partial x_2} = 0.$

Since ϕ_1 and ϕ_2 are transverse generating functions the equations 1 and 2 are an independent set of defining equations for $C_{\phi_1} \times C_{\phi_2}$. As for the equation 3, our assumption that Γ_1 and Γ_2 compose cleanly tells us that the mappings

$$\frac{\partial \phi_1}{\partial x_2} : C_{\phi_1} \to T^* X_2$$

and

$$\frac{\partial \phi_2}{\partial x_2} : C_{\phi_2} \to T^* X_2$$

intersect cleanly. In other words the subset, C_ϕ, of $C_{\phi_1} \times C_{\phi_2}$ defined by the equation, $\frac{\partial \phi}{\partial x_2} = 0$, is a submanifold of $C\phi_1 \times C\phi_2$, and its tangent space at each point is defined by the linear equation, $d\frac{\partial \phi}{\partial x_2} = 0$. Thus the set of equations, 1–3, are a *clean* set of defining equations for C_ϕ as a submanifold of $X_1 \times X_3 \times (X_2 \times S \times T)$. In other words ϕ is a clean generating function for $\Gamma_2 \circ \Gamma_1$.

The *excess*, ϵ, of this generating function is equal to the dimension of C_ϕ minus the dimension of $X_1 \times X_3$. One also gets a more intrinsic description of ϵ in terms of the projections of T_1 and T_2 onto $T^* X_2$. From these projections one gets a map

$$\Gamma_1 \times \Gamma_2 \to T^*(X_2 \times X_2)$$

which, by the cleanness assumption, intersects the conormal bundle of the diagonal cleanly; so its pre-image is a submanifold, $\Gamma_2 \star \Gamma_1$, of $\Gamma_1 \times \Gamma_2$. It's easy to see that

$$\epsilon = \dim \Gamma_2 \star \Gamma_1 - \dim \Gamma_2 \circ \Gamma_1 \,.$$

5.7 Reducing the number of fiber variables.

Let $\Lambda \subset T^* X$ be a Lagrangian manifold and let $\Phi \in C^\infty(Z)$ be a generating function for Λ relative to a fibration $\pi : Z \to X$. Let

$$x_0 \in X,$$

let

$$Z_0 := \pi^{-1}(x_0)$$

and let

$$\iota_0 : Z_0 \to Z$$

be the inclusion of the fiber Z_0 into Z. By definition, a point $z_0 \in Z_0$ belongs to C_ϕ if and only if z_0 is a critical point of the restriction $\iota_0^*\phi$ of ϕ to Z_0.

Theorem 19 *If z_0 is a non-degenerate critical point of $\iota_0^*\phi$ then Λ is horizontal at*

$$p_0 = (x_0, \xi_0) = \frac{\partial \phi}{\partial x}(z_0).$$

Moreover, there exists an neighborhood U of x_0 in X and a function $\psi \in C^\infty(U)$ such that

$$\Lambda = \Lambda_\psi$$

on a neighborhood of p_0 and

$$\pi^*\psi = \phi$$

on a neighborhood U' of z_0 in C_ϕ.

Proof. (In local coordinates.) So $Z = X \times \mathbb{R}^k$, $\phi = \phi(x, s)$ and C_ϕ is defined by the k independent equations

$$\frac{\partial \phi}{\partial s_i} = 0, \qquad i = 1, \dots k. \tag{5.8}$$

Let $z_0 = (x_0, s_0)$ so that s_0 is a non-degenerate critical point of $\iota_0^*\phi$ which is the function

$$s \mapsto \phi(x_0, s)$$

if and only if the Hessian matrix

$$\left(\frac{\partial^2 \phi}{\partial s_i \partial s_j} \right)$$

is of rank k. By the implicit function theorem we can solve equations (5.8) for s in terms of x near (x_0, s_0). This says that we can find a neighborhood U of x_0 in X and a C^∞ map

$$g : U \to \mathbb{R}^k$$

such that

$$g(x) = s \Leftrightarrow \frac{\partial \phi}{\partial s_i} = 0, \ i = 1, \dots, k$$

if (x, s) is in a neighborhood of $(x_0, s_0$ in Z. So the map

$$\gamma : U \to U \times \mathbb{R}^k, \qquad \gamma(x) = (x, g(x))$$

maps U diffeomorphically onto a neighborhood of (x_0, s_0) in C_ϕ. Consider the commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\gamma} & C_\phi \\ \downarrow & & \downarrow{\scriptstyle d_X\phi} \\ X & \xleftarrow[\pi_X]{} & \Lambda \end{array}$$

where the left vertical arrow is inclusion and π_X is the restriction to Λ of the projection $T^*X \to X$. From this diagram it is clear that the restriction of π to the image of U in C_ϕ is a diffeomorphism and that Λ is horizontal at p_0. Also

$$\mu := d\phi \circ \gamma$$

is a section of Λ over U. Let

$$\psi := \gamma^*\phi.$$

Then

$$\mu = d_X\phi \circ \gamma = d_X\phi \circ \gamma + d_S\phi \circ \gamma = d\phi \circ \gamma$$

since $d_S\phi \circ \gamma \equiv 0$. Also, if $v \in T_xX$ for $x \in U$, then

$$d\psi_x(v) = d\phi_{\gamma(x)}(d\gamma_x(v)) = d\phi_{\gamma(x)}(v, dg_x(v)) = d_X\phi \circ \gamma(v)$$

so

$$\langle \mu(x), v\rangle = \langle d\psi_x, v\rangle$$

so

$$\Lambda = \Lambda_\psi$$

over U and from $\pi : Z \to X$ and $\gamma \circ \pi = \mathrm{id}$ on $\gamma(U) \subset C_\phi$ we have

$$\pi^*\psi = \pi^*\gamma^*\phi = (\gamma \circ \pi)^*\phi = \phi$$

on $\gamma(U)$. $\square$

We can apply the proof of this theorem to the following situation: Suppose that the fibration

$$\pi : Z \to X$$

can be factored as a succession of fibrations

$$\pi = \pi_1 \circ \pi_0$$

where

$$\pi_0 : Z \to Z_1 \quad \text{and} \quad \pi_1 : Z_1 \to X$$

are fibrations. Moreover, suppose that the restriction of ϕ to each fiber

$$\pi_0^{-1}(z_1)$$

has a unique non-degenerate critical point $\gamma(z_1)$. The map

$$z_1 \mapsto \gamma(z_1)$$

defines a smooth section

$$\gamma : Z_1 \to Z$$

of π_0. Let

$$\phi_1 := \gamma^* \phi.$$

Theorem 20 *ϕ_1 is a generating function for Λ with respect to π_1.*

Proof. (Again in local coordinates.) We may assume that

$$Z = X \times S \times T$$

and

$$\pi(x, s, t) = x, \quad \pi_0(x, s, t) = (x, s), \quad \pi_1(x, s) = x.$$

The condition for (x, s, t) to belong to C_ϕ is that

$$\frac{\partial \phi}{\partial s} = 0$$

and

$$\frac{\partial \phi}{\partial t} = 0.$$

This last condition has a unique solution giving t as a smooth function of (x, s) by our non-degeneracy condition, and from the definition of ϕ_1 it follows that $(x, s) \in C_{\phi_1}$ if and only if $\gamma(x, s) \in C_\phi$. Furthermore

$$d_X \phi_1(x, s) = d_X \phi(x, s, t)$$

along $\gamma(C_{\phi_1})$. $\square$

For instance, suppose that $Z = X \times \mathbb{R}^k$ and $\phi = \phi(x, s)$ so that $z_0 = (x_0, s_0) \in C_\phi$ if and only if

$$\frac{\partial \phi}{\partial s_i}(x_0, s_0) = 0, \qquad i = 1, \ldots, k.$$

Suppose that the matrix

$$\left(\frac{\partial^2 \phi}{\partial s_i \partial s_j} \right)$$

is of rank r, for some $0 < r \leq k$. By a linear change of coordinates we can arrange that the upper left hand corner

$$\left(\frac{\partial^2 \phi}{\partial s_i \partial s_j} \right), \quad 1 \leq i, j, \leq r$$

is non-degenerate. We can apply Theorem 20 to the fibration

$$X \times \mathbb{R}^k \to X \times \mathbb{R}^\ell, \quad \ell = k - r$$

$$(x, s_1, \dots s_k) \mapsto (x, t_1, \dots, t_\ell), \quad t_i = s_{i+r}$$

to obtain a generating function $\phi_1(x,t)$ for Λ relative to the fibration

$$X \times \mathbb{R}^\ell \to X.$$

Thus by reducing the number of variables we can assume that at $z_0 = (x_0, t_0)$

$$\frac{\partial^2 \phi}{\partial t_i \partial t_j}(x_0, t_0) = 0, \quad i, j = 1, \dots, \ell. \tag{5.9}$$

A generating function satisfying this condition will be said to be **reduced** at (x_0, t_0).

5.8 The existence of generating functions.

In this section we will show that every Lagrangian submanifold of T^*X can be described locally by a generating function ϕ relative to some fibration $Z \to X$.

So let $\Lambda \subset T^*X$ be a Lagrangian submanifold and let $p_0 = (x_0, \xi_0) \in \Lambda$. To simplify the discussion let us temporarily make the assumption that

$$\xi_0 \neq 0. \tag{5.10}$$

If Λ is horizontal at p_0 then we know from Chapter 1 that there is a generating function for Λ near p_0 with the trivial (i.e. no) fibration. If Λ is not horizontal at p_0, we can find a Lagrangian subspace

$$V_1 \subset T_{p_0}(T^*X)$$

which is horizontal and transverse to $T_{p_0}(\Lambda)$. Let Λ_1 be a Lagrangian submanifold passing throuigh p_0 and whose tangent space at p_0 is V_1. So Λ_1 is a horizontal Lagrangian submanifold and

$$\Lambda_1 \pitchfork \Lambda = \{p_0\}.$$

In words, Λ_1 intersects Λ transversally at p_0. Since Λ_1 is horizontal, we can find a neighborhood U of x_0 and a function $\phi_1 \in C^\infty(U)$ such that $\Lambda_1 = \Lambda_{\phi_1}$. By our assumption (5.10)

$$(d\phi_1)_{x_0} = \xi_0 \neq 0.$$

So we can find a system of coordinates $x_1 \dots, x_n$ on U (or on a smaller neighborhood) so that

$$\phi_1 = x_1.$$

Let $\xi_1 \ldots, \xi_n$ be the dual coordinates so that in the coordinate system

$$x_1 \ldots, x_n, \xi_1 \ldots, \xi_n$$

on T^*X the Lagrangian submanifold Λ_1 is described by the equations

$$\xi_1 = 1, \xi_2 = \cdots = \xi_n = 0.$$

Consider the canonical transformation generated by the function

$$\tau : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad \tau(x, y) = x \cdot y.$$

The Lagrangian submanifold in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ generated by ψ is

$$\{(x, y, y, x)\}$$

so the canonical relation is

$$\{x, \xi, -\xi, x)\}.$$

In other words, it is the graph of the linear symplectic transformation

$$\gamma : (x, \xi) \mapsto (-\xi, x).$$

So $\gamma(\Lambda_1)$ is the cotangent space at $y_0 = (-1, 0, \ldots, 0)$. Since $\gamma(\Lambda)$ is transverse to this cotangent fiber, it follows that $\Gamma(\Lambda)$ is horizontal. So in some neighborhood W of y_0 there is a function ψ such that

$$\gamma(\Lambda) = \Lambda_\psi$$

over W. By equation (5.7) we know that

$$\tau^*(x, y) = -\tau(y, x) = -y \cdot x$$

is the generating function for γ^{-1}. Furthermore, near p_0,

$$\Lambda = \gamma^{-1}(\Lambda_\psi).$$

Hence, by Theorem 18 the function

$$\psi_1(x, y); = -y \cdot x + \psi(y) \tag{5.11}$$

is a generating function for Λ relative to the fibration

$$(x, y) \mapsto y.$$

We have proved the existence of a generating function under the auxiliary hypothesis (5.10). However it is easy to deal with the case $\xi_0 = 0$ as well. Namely, suppose that $\xi_0 = 0$. Let $f \in C^\infty(X)$ be such that $df(x_0) \neq 0$. Then

$$\gamma_f : T^*X \to T^*X, \quad (x, \xi) \mapsto (x, \xi + df)$$

is a symplectomorphism and $\gamma_f(p_0)$ satisfies (5.10). We can then form

$$\gamma \circ \gamma_f(\Lambda)$$

which is horizontal. Notice that $\gamma_\circ \gamma_f$ is given by

$$(x, \xi) \mapsto (x, \xi + df) \mapsto (-\xi - df, x).$$

If we consider the generating function on $\mathbb{R}^n \times \mathbb{R}^n$ given by

$$g(x, z) = x \cdot z + f(x)$$

then the corresponding Lagrangian submanifold is

$$\{(x, z + df, z, x)\}$$

so the canonical relation is

$$\{(x, -z - df, z, x)\}$$

or, setting $\xi = -z - df$ so $z = -\xi - df$ we get

$$\{(x, \xi, -\xi - df, x)\}$$

which is the graph of $\gamma \circ \gamma_f$. We can now repeat the previous argument. So we have proved:

Theorem 21 *Every Lagrangian submanifold of T^*X can be locally represented by a generating function relative to a fibration.*

5.9 Generating functions for canonical relations.

In this section we will give a slight refinement of Theorem 21 for the case of a canonical relation.

So let X and Y be manifolds and

$$\Gamma \subset T^*X \times T^*Y$$

a canonical relation. Let $(p_0, q_0) = (x_0, \xi_0, y_0, \eta_0) \in \Gamma$ and assume now that

$$\xi_0 \neq 0, \quad \eta_0 \neq 0. \tag{5.12}$$

We claim that the following theorem holds

Theorem 22 *There exist coordinate systems $(U, x_1, \ldots, x_n)$ about x_0 and $(V, y_1 \ldots, y_k)$ about y_0 such that if*

$$\gamma_U : \ T^*U \to T^*\mathbb{R}^n$$

is the transform

$$\gamma_U(x,\xi) = (-\xi, x)$$

and

$$\gamma_V : T^*V \to T^* * \mathbb{R}^k$$

is the transform

$$\gamma_V(y,\eta) = (-\eta, y)$$

then locally near

$$p_0' := \gamma_U^{-1}(p_0) \quad \text{and} \quad q_0' := \gamma_V(q_0)$$

the canonical relation

$$\gamma_V^{-1} \circ \Gamma \circ \gamma_U \tag{5.13}$$

is of the form

$$\Gamma_\phi, \quad \phi = \phi(x,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^k).$$

Proof. Let

$$M_1 := T^*X, \qquad M_2 = T^*Y$$

and

$$V_1 := T_{p_0}M_1, \qquad V_2 := T_{q_0}M_2, \qquad \Sigma := T_{(p_0,q_0)}\Gamma$$

so that Σ is a Lagrangian subspace of

$$V_1^- \times V_2.$$

Let W_1 be a Lagrangian subspace of V_1 so that (in the linear symplectic category)

$$\Sigma(W_1) = \Sigma \circ W_1$$

is a Lagrangian subspace of V_2. Let W_2 be another Lagrangian subspace of V_2 which is transverse to $\Sigma(W_1)$. We may choose W_1 and W_2 to be horizontal subspaces of $T_{p_0}M_1$ and $T_{q_0}M_2$. Then $W_1 \times W_2$ is transverse to Σ in $V_1 \times V_2$ and we may choose a Lagrangian submanifold passing through p_0 and tangent to W_1 and similarly a Lagrangian submanifold passing through q_0 and tangent to W_2. As in the proof of Theorem 21 we can arrange local coordinates $(x_1 \ldots, x_n)$ on X and hence dual coordinates $(x_1, \ldots x_n, \xi_1, \ldots, \xi_n)$ around p_0 such that the Lagrangian manifold tangent to W_1 is given by

$$\xi_1 = -1, \quad \xi_2 = \cdots \xi_n = 0$$

and similarly dual coordinates on $M_2 = T^*Y$ such that the second Lagrangian submanifold (the one tangent to W_2) is given by

$$\eta_1 = -1, \quad \eta_2 = \cdots = \eta_k = 0.$$

It follows that the Lagrangian submanifold corresponding to the canonical relation (5.13) is horizontal and hence is locally of the form Γ_ϕ. $\square$

5.10 The Legendre transformation.

Coming back to our proof of the existence of a generating function for Lagrangian manifolds, let's look a little more carefully at the details of this proof. Let $X = \mathbb{R}^n$ and let $\Lambda \subset T^*X$ be the Lagrangian manifold defined by the fibration, $Z = X \times \mathbb{R}^n \xrightarrow{\pi} X$ and the generating function

$$\phi(x,y) = -x \cdot y + \psi(y) \tag{5.14}$$

where $\psi \in C^\infty(\mathbb{R}^n)$. Then

$$(x,y) \in C_\phi \Leftrightarrow x = \frac{\partial \psi}{\partial y}(y)\,.$$

Recall also that $(x_0, y_0) \in C_\phi \Leftrightarrow$ the function $\phi(x_0, y)$ has a critical point at y_0. Let us suppose this is a *non-degenerate* critical point, i.e., that the matrix

$$\frac{\partial^2 \phi}{\partial y_i \partial y_j}(x_0, y_0) = \frac{\partial \psi}{\partial y_i \partial y_j}(y_0) \tag{5.15}$$

is of rank n. Then there exists a neighborhood $U \ni x_0$ and a function $\psi^* \in C^\infty(U)$ such that

$$\begin{aligned} \psi^*(x) &= \phi(x,y) \text{ at } (x,y) \in C_\phi & (5.16)\\ \Lambda &= \Lambda_\psi & (5.17) \end{aligned}$$

locally near the image $p_0 = (x_0, \xi_0)$ of the map $\frac{\partial \phi}{\partial x} : C_\phi \to \Lambda$. What do these three assertions say? Assertion(5.15) simply says that the map

$$y \to \frac{\partial \psi}{\partial y} \tag{5.18}$$

is a diffeomorphism at y_0. Assertion(5.16) says that

$$\psi^*(x) = -xy + \psi(x) \tag{5.19}$$

at $x = \frac{\partial \psi}{\partial y}$, and assertion(5.17) says that

$$x = \frac{\partial \psi}{\partial y} \Leftrightarrow y = -\frac{\partial \psi^*}{\partial x} \tag{5.20}$$

i.e., the map

$$x \to -\frac{\partial \psi^*}{\partial x} \tag{5.21}$$

is the inverse of the mapping (5.15). The mapping (5.15) is known as the Legendre transform associated with ψ and the formulas (5.19)– (5.21) are the famous inversion formula for the Legendre transform. Notice also that in the course of our proof that (5.19) is a generating function for Λ we proved that ψ is a generating function for $\gamma(\Lambda)$, i.e., locally near $\gamma(p_0)$

$$\gamma(\Lambda) = \Lambda_\psi\,.$$

Thus we've proved that locally near p_0

$$\Lambda_{\psi^*} = \gamma^{-1}(\Lambda_\psi)$$

where

$$\gamma^{-1} : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$$

is the transform $(y,\eta) \to (x,\xi)$ where

$$y = \xi \quad \text{and} \quad x = -\eta\,.$$

This identity will come up later when we try to compute the semi-classical Fourier transform of the rapidly oscillating function

$$a(y)e^{i\frac{\psi(y)}{\hbar}}\,,\; a(y) \in C_0^\infty(\mathbb{R}^n)\,.$$

5.11 The Hörmander-Morse lemma.

In this section we will describe some relations between different generating functions for the same Lagrangian submanifold.

Let Λ be a Lagrangian submanifold of T^*X, and let

$$Z_0 \stackrel{\pi_0}{\to} X, \quad Z_1 \stackrel{\pi_1}{\to} X$$

be two fibrations over X. Let ϕ_1 be a generating function for Λ with respect to $\pi_1 : Z_1 \to X$.

Proposition 10 *If*

$$f : Z_0 \to Z_1$$

is a diffeomorphism satisfying

$$\pi_1 \circ f = \pi_0$$

then

$$\phi_0 = f^*\phi_1$$

is a generating function for Λ with respect to π_0.

Proof. We have $d(\phi_1 \circ f) = d\phi_0$. Since f is fiber preserving,

$$d(\phi_\circ f)_{vert} = (d\phi_0)_|$$

so f maps C_{ϕ_0} diffeomorphically onto C_{ϕ_1}. Furthermore, on C_{ϕ_0} we have

$$d\phi_1 \circ f = (d\phi_1 \circ f)_{hor} = (d\phi_0)_{hor}$$

so f conjugates the maps $d_X\phi_i : C_{\phi_i}, \quad i = 0, 1$. Since $d_X\phi_1$ is a diffeomorphism of C_{ϕ_1} with Λ we conclude that $d_X\phi_0$ is a diffeomorphism of C_{ϕ_0} with Λ, i.e. ϕ_0 is a generating function. $\square$

Our goal is to prove a result in the opposite direction. So as above let $\pi_i : Z_i \to X,\ i = 0,1$ be fibrations and suppose that ϕ_0 and ϕ_1 are generating functions for Λ with respect to π_i. Let

$$p_0 \in \Lambda$$

and $z_i \in C_{\phi_i},\ i = 0,1$ be the preimages of p_0 under the diffeomorphism $d\phi_i$ of C_{ϕ_i} with Λ. So

$$d_X\phi_i(z_i) = p_0, \qquad i = 0,1$$

Finally let $x_0 \in X$ be given by

$$x_0 = \pi_0(z_0) = \pi_1(z_1)$$

and let $\psi_i,\ i = 0,1$ be the restriction of ϕ_i to the fiber $\pi_i^{-1}(x_0)$. Since $z_i \in C_{\phi_i}$ we know that z_i is a critical point for ψ_i. Let

$$d^2\psi_i(z_i)$$

be the Hessian of ψ_i at z_i.

Theorem 23 The Hörmander Morse lemma. *If $d^2\psi_0(z_0)$ and $d^2\psi_1(z_1)$ have the same rank and signature, then there exists neighborhood U_0 of z_0 in Z_0 and U_1 of z_1 in Z_1 and a diffeomorphism*

$$f : U_0 \to U_1$$

such that

$$\pi_1 \circ f = \pi_0$$

and

$$f^*\phi_1 = \phi_0 + \text{ const. } .$$

Proof. We will prove this theorem in a number of steps. We will first prove the theorem under the additional assumption that Λ is horizontal at p_0. Then we will reduce the general case to this special case.

Assume that Λ is horizontal at $p_0 = (x_0, \xi_0)$. Let S be an open subset of $\mathbb{R}^k$ and

$$\pi : X \times S \to X$$

projection onto the first factor. Suppose that $\phi \in C^\infty(X \times S)$ is a generating function for Λ with respect to π so that

$$d_X\phi : \quad C_\phi \to \Lambda$$

is a diffeomorphism, and let $z_0 \in C_\phi$ be the pre-image of p_0 under this diffeomorphism.

Since Λ is horizontal at p_0 there is a neighborhood U of x_0 and a $\psi \in C^\infty(U)$ such that

$$d\psi : U \to T^*X$$

maps U diffeomorphically onto a neighborhood of p_0 in Λ. So

$$(d\psi)^{-1} \circ d_X\phi : \; C_\phi \to U$$

is a diffeomorhism. But $d\psi^{-1}$ is just the restriction to Λ of the projection $\pi_X : T^*X \to X$. So $\pi_X \circ d_X\phi : C_\phi \to X$ is a diffeomorphism (when restricted to $\pi^{-1}(U)$). But

$$\pi_X \circ d_X\phi = \pi$$

so the restriction of π to C_ϕ is a diffeomorphism. So C_ϕ is horizontal at z_0, in the sense that

$$T_{z_0}C_\phi \cap T_{z_0}S = \{0\}.$$

So we have a smooth map

$$\mathbf{s} : U \to S$$

such that $x \mapsto (x, \mathbf{s}(x))$ is a smooth section of C_ϕ over U. We have

$$d_X\phi = d\phi \quad \text{at all points} \;\; (x, \mathbf{s}(x))$$

by the definition of C_ϕ and $d\psi(x) = d_X\phi(x, \mathbf{s}(x))$ so

$$\psi(x) = \phi(x, \mathbf{s}(x)) + \text{const.} \; . \tag{5.22}$$

The submanifold $C_\phi \subset Z = X \times S$ is defined by the k- equations

$$\frac{\partial \phi}{\partial s_i} = 0, \quad i = 1, \ldots, k$$

and hence $T_{z_0}C_\phi$ is defined by the k independent linear equations

$$d\left(\frac{\partial \phi}{\partial s_i}\right) = 0, \quad i = 1, \ldots, k.$$

A tangent vector to S at Z_0, i.e. a tangent vector of the form

$$(0, v), \quad v = (v^1, \ldots v^k)$$

will satisfy these equations if and only if

$$\sum_j \frac{\partial^2 \phi}{\partial s_i \partial s_j} v^j = 0, \quad i = 1, \ldots, k.$$

But we know that these equations have only the zero solution as no non-zero tangent vector to S lies in the tangent space to C_ϕ at z_0. We conclude that the vertical Hessian matrix

$$d_S^2\phi = \left(\frac{\partial^2 \phi}{\partial s_i \partial s_j}\right)$$

is non-degenerate.

We can apply the above considerations to each of the generating functions ϕ_0 and ϕ_1. So we get a section $\mathfrak{s}_0$ of $\pi_0 : Z_0 \to X$ over U with with $\mathfrak{s}_0(U) = C_{\phi_0}$ (over U) and similarly a section $\mathfrak{s}_1$ of $\pi_1 : Z_1 \to X$. By (5.22) we know that (up to adjusting an overall additive constant) we can arrange that

$$\phi_0(\mathfrak{s}_0(x)) = \phi_1(\mathfrak{s}_1(x)) \tag{5.23}$$

over U.

Let us again revert to local coordinates: We know that the vertical Hessians occurring in the statement of the theorem are both non-degenerate, and we are assuming that they are of the sake rank. So the fiber dimensions of π_0 and π_1 are the same. So we may assume that $Z_0 = X \times S$ and $Z_1 = X \times S$ where S is an open subset of $\mathbb{R}^k$ and that coordinates have been chosen so that the coordinates of z_0 are $(0,0)$ as are the coordinates of z_1. We write

$$\mathfrak{s}_0(x) = (x, \mathbf{s}_0(x)), \quad \mathfrak{s}_1(x) = (x, \mathbf{s}_1(x)),$$

where $\mathbf{s}_0$ and $\mathbf{s}_1$ are smooth maps $X \to \mathbb{R}^k$ with

$$\mathbf{s}_0(0) = \mathbf{s}_1(0) = 0.$$

Let us now take into account that the signatures of the vertical Hessians are the same at z_0. By continuity they must be the same at the points $(x, \mathbf{s}_0(x))$ and $(x, \mathbf{s}_1(x))$ for each $x \in U$. So for each fixed $x \in U$ we can make an affine change of coordinates in S to arrange that

1. $\mathbf{s}_0(x) = \mathbf{s}_1(x) = 0.$
2. $\frac{\partial \phi_0}{\partial s_i}(x,0) = \frac{\partial \phi_1}{\partial s_i}(x,0),\ i = 1.\dots,k.$
3. $\phi_0(x,0) = \phi_1(x,0).$
4. $d_S^2\phi_0(x,0) = d_S^2\phi_1(x,0).$

We can now apply Morse's lemma with parameters to conclude that there exists a fiber preserving diffeomorhism $f : U \times S \to U \times S$ with

$$f^*\phi_1 = \phi_0.$$

This completes the proof of Theorem 23 under the additional hypothesis that Lagrangian manifold Λ is horizontal.

Reduction of the number of fiber variables. Our next step in the proof of Theorem 23 will be an application of Theorem 20. Let $\pi : Z \to X$ be a fibration and ϕ a generating function for Λ with respect to π. Suppose we are in the setup of Theorem 20 which we recall with some minor changes in notation: We suppose that the fibration

$$\pi : Z \to X$$

can be factored as a succession of fibrations

$$\pi = \rho \circ \varrho$$

where

$$\rho : Z \to W \quad \text{and} \quad \varrho : W \to X$$

are fibrations. Moreover, suppose that the restriction of ϕ to each fiber

$$\rho^{-1}(w)$$

has a unique non-degenerate critical point $\gamma(w)$. The map

$$w \mapsto \gamma(w)$$

defines a smooth section

$$\gamma : W \to Z$$

of ρ. Let

$$\chi := \gamma^*\phi.$$

Theorem 20 asserts that χ is a generating function of Λ with respect to ϱ. Consider the Lagrangian submanifold

$$\Lambda_\chi \subset T^*W.$$

This is horizontal as a Lagrangian submanifold on T^*W and ϕ is a generating function for Λ_χ relative to the fibration $\rho : Z \to W$.

Now suppose that we had two fibrations and generating functions as in the hypotheses of Theorem 23 and suppose that they both factored as above with *the same* $\varrho : W \to X$ and the same χ. So we get fibrations $\varrho_O : Z_0 \to W$ and $\varrho_1 : Z_1 \to W$ We could then apply the above (horizontal) version of Theorem 23 to conclude the truth of the theorem.

Since the ranks of $d^2\psi_1$ and $d^2\psi$ at z_0 and z_1 are the same, we can apply the reduction leading equation (5.9) to each. So Theorem 23 will be proved once we prove it for the reduced case.

Some normalizations in the reduced case. We now examine a fibration $Z = X \times S \to S$ and generating function ϕ and assume that ϕ is reduced at $z_0 = (x_0, s_0)$ so all the second partial derivatives of ϕ in the S direction vanish, i. e.

$$\frac{\partial^2 \phi}{\partial s_i \partial s_j}(x_o, s_0) = 0 \quad \forall i, j.$$

This implies that

$$T_{s_0}S \cap T_{(x_0,s_0)}C_\phi = T_{s_0}S.$$

i.e. that

$$T_{s_0}S \subset T_{(x_0,s_0)}C_\phi. \tag{5.24}$$

Consider the map

$$d_X\phi : X \times S \to T^*X, \qquad (x,s) \mapsto d_X\phi(x,s).$$

The restriction of this map to C_ϕ is just our diffeomorphism of C_ϕ with Λ. So the restriction of the differential of this map to any subspace of any tangent space to C_ϕ is injective. By (5.24) the restriction of the differential of this map to $T_{s_0}S$ at (x_0,s_0) is injective. In other words, we have an embedding

$$\begin{array}{ccc} X \times S & \xrightarrow{d_X\phi} & W \subset T^*X \\ \pi\Big\downarrow & & \Big\downarrow\pi_X \\ X & \xrightarrow[\mathrm{id}]{} & X \end{array}$$

of $X \times S$ onto a subbundle W of T^*X.

Now let us return to the proof of our theorem. Suppose that we have two generating functions $\phi_i,\ i = 0,1\ X \times S_i \to X$ and both are reduced at the points z_i of C_{ϕ_1} corresponding to $p_0 \in \Lambda$. So we have two embeddings

$$\begin{array}{ccc} X \times S_i & \xrightarrow{d_X\phi_i} & W_i \subset T^*X \\ \pi\Big\downarrow & & \Big\downarrow\pi_X \\ X & \xrightarrow[\mathrm{id}]{} & X \end{array}$$

of $X \times S_i$ onto subbundle W_i of T^*X for $i = 0,1$. Each of these maps the corresponding C_{ϕ_i} diffeomorphically onto Λ.

Let V be a tubular neighborhood of W_1 in T^*X and $\tau : V \to W_1$ a projection of V onto W_1 so we have the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\tau} & W_1 \\ \pi_X\Big\downarrow & & \Big\downarrow\pi_X \\ X & \xrightarrow[\mathrm{id}]{} & X \end{array}$$

Let

$$\gamma := (d_X\phi_1)^{-1} \circ \tau.$$

So we have the diagram

$$\begin{array}{ccc} V & \xrightarrow{\gamma} & X \times S_1 \\ \pi\Big\downarrow & & \Big\downarrow\pi_X \\ X & \xrightarrow[\mathrm{id}]{} & X \end{array}$$

and

$$\gamma \circ d_X\phi_1 = \mathrm{id}\,.$$

We may assume that $W_0 \subset V$ so we gat a fiber map

$$g := \gamma \circ d_X\phi_0 \quad g : X \times S_0 \to X \times S_1.$$

When we restrict g to C_{ϕ_0} we get a diffeomorphism of C_{ϕ_0} onto C_{ϕ_1}. By (5.24) we know that

$$T_{s_i}S_i \subset T_{z_i}C_{\phi_i}$$

and so dg_{z_0} maps $T_{s_0}S_0$ bijectively onto $T_{s_1}S_1$. Hence g is locally a diffeomorphism at z_0. So by shrinking X and S_i we may assume that

$$g : X \times S_0 \to X \times S_1$$

is a fiber preserving diffeomorphism.

We now apply Proposition 10. So we replace ϕ_1 by $g^*\phi_1$. Then the two fibrations Z_0 and Z_1 are the same and $C_{\phi_0} = C_{\phi_1}$. Call this common submanifold C. Also $d_X\phi_0 = d_X\phi_1$ when restricted to C, and by definition the vertical derivatives vanish. So $d\phi_0 = d\phi_1$ on C, and so by adjusting an additive constant we can arrange that $\phi_0 = \phi_1$ on C.

Completion of the proof. We need to prove the theorem in the following situation:

- $Z_0 = Z_1 = X \times S$ and $\pi_0 = \pi_1$ is projection onto the first factor.
- The two generating functions ϕ_0 and ϕ_1 have the same critical set:
$$C_{\phi_0} = C_{\phi_1} = C.$$
- $\phi_0 = \phi_1$ on C.
- $d_S\phi_0 = 0,\ i = 0, 1$ on C and $d_X\phi_0 = d_X\phi_1$ on C.
- $$d\left(\frac{\partial\phi_0}{\partial s_i}\right) = d\left(\frac{\partial\phi_1}{\partial s_i}\right) \text{ at } z_0.$$

We will apply the Moser trick: Let

$$\phi_t := (1-t)\phi_0 + t\phi_1.$$

From the above we know that

- $\phi_t = \phi_0 = \phi_1$ on C.
- $d_S\phi_t = 0$ on C and $d_X\phi_t = d_X\phi_0 = d_X\phi_1$ on C.
- $$d\left(\frac{\partial\phi_t}{\partial s_i}\right) = d\left(\frac{\partial\phi_0}{\partial s_i}\right) = d\left(\frac{\partial\phi_1}{\partial s_i}\right) \text{ at } z_0.$$

So in a sufficiently small neighborhood of Z_0 the submanifold C is defined by the k independent equations

$$\frac{\partial \phi_t}{\partial s_i} = 0, \quad i = 1, \dots k.$$

We look for a vertical (time dependent) vector field

$$v_1 = \sum_i v_i(x,s,t)\frac{\partial}{\partial s_i}$$

on $X \times S$ such that

1. $D_{v_t}\phi_t = -\dot{\phi_1} = \phi_0 - \phi_1$ and
2. $v = 0$ on C.

Suppose we find such a v_t. Then solving the differential equations

$$\frac{d}{dt}f_t(m) = v_t(f_t(m)), \quad f_0(m) = m$$

will give a family of fiber preserving diffeomorphsms (since v_t is vertical) and

$$f_1^*\phi_1 - \phi_0 = \int_0^1 \frac{d}{dt}(f_t^*\phi_t)dt = \int_0^1 f_t^*[D_{v_t}\phi_t + \dot{\phi}_t]dt = 0.$$

So finding a vector field v_t satisfying 1) and 2) will complete the proof of the theorem. Now $\phi_0 - \phi_1$ vanishes to second order on C which is defined by the independent equations $\partial\phi_t/\partial s_i = 0$. So we can find functions

$$w_{ij}(x,s,t)$$

defined and smooth in some neighborhood of C such that

$$\phi_0 - \phi_1 = \sum_{ij} w_{ij}(x,s,t)\frac{\partial \phi_t}{\partial s_i}\frac{\partial \phi_t}{\partial s_j}$$

in this neighborhood. Set

$$v_i(x,s,t) = \sum_i w_{ij}(x,s,t)\frac{\partial \phi_t}{\partial s_j}.$$

Then condition 2) is clearly satisfied and

$$D_{v_t}\phi_t = \sum_{ij} w_{ij}(x,s,t)\frac{\partial \phi_t}{\partial si}\frac{\partial \phi_t}{\partial s_j} = \phi_0 - \phi_1 = -\dot{\phi}$$

as required. $\square$

5.12 Changing the generating function.

We summarize the results of the preceding section as follows: Suppose that $(\pi_1 : Z_1 \to X, \phi_1)$ and $(\pi_2 : Z_2 \to X, \phi_2)$ are two descriptions of the same Lagrangian submaniifold Λ of T^*X. Then locally one description can be obtained from the other by applying sequentially "moves" of the following two types:

Victor: We need more arg mentation here. Why do this follow from H-M?

1. **Equivalence.** There exists a diffeomorphism $g : Z_1 \to Z_2$ with

$$\pi_2 \circ g = \pi_1 \qquad \text{and} \qquad \text{and} \qquad \phi_2 \circ g = \phi_1.$$

2. **Increasing (or deceasing) the number of fiber variables.** Here $Z_2 = Z_1 \times \mathbb{R}^d$ and

$$\phi_2(s, s) = \phi_1(z) + \frac{1}{2}\langle As, s\rangle$$

where A is a non-degenerate $d \times d$ matrix.

5.13 The phase bundle of a Lagrangian submanifold of T^*X.

In this section and the next we introduce two important flat line bundles associated with a Lagrangian submanifold of a cotangent bundle. For a review of the basic facts about line bundles with connections, especially line bundles with flat connections, see Appendix III.

Let $\Lambda \subset T^*X$ be a Lagrangian submanifold. Let α_Λ denote the restriction of the canonical one form α_X to Λ. Since Λ is Lagrangian, α_Λ is closed.

The line bundle $\mathbb{L}_{\text{phase}}$ with flat connection is defined as follows: As a line bundle, $\mathbb{L}_{\text{phase}}$ is the trivial bundle

$$\mathbb{L}_{\text{phase}} = \Lambda \times \mathbb{C}.$$

The connection on $\mathbb{L}_{\text{phase}}$ is given by setting

$$\frac{\nabla s_0}{s_0} == \frac{i}{\hbar}\alpha_\Lambda$$

where s_0 is the trivial section

$$s_0(p) = (p, 1).$$

In order for a section $s = f s_0$ to be flat, we must have $\nabla s = 0$ which translates into

$$df - \frac{i}{\hbar}\alpha_\Lambda = 0.$$

If ψ is a function on an open subset of Λ which satisfies $d\psi = \alpha_\Lambda$ then this says that
$$f = ce^{i\psi/\hbar}, \quad c \in \mathbb{C}.$$
Suppose that $\pi : Z \to X$ is a fibration and that ϕ is a generating function for Λ relative to this fibration. So $d_X\phi$ gives a diffeomorphism
$$\lambda_\phi : C_\phi \to \Lambda.$$
Let
$$\psi := \phi \circ \lambda_\phi^{-1}.$$
Then
$$d\phi = \alpha_\Lambda.$$
So if we have a generating function, we get flat section of $\mathbb{L}$.

5.14 The Maslov bundle.

We first define the Maslov line bundle $\mathbb{L}_{\text{Maslov}} \to \Lambda$ first in terms of a global generating function, and then show that the definition is invariant under change of generating function. We then use the local existence of generating functions to patch the line bundle together globally. Here are the details:

Suppose that ϕ is a generating function for Λ relative to a fibration $\pi : Z \to X$. For each z be a point of the critical set C_ϕ, let $x = \pi(z)$ and let $F = \pi^{-1}(x)$ be the fiber contiaining z. The restriction of ϕ to the fiber F has a critical point at z. Let $\text{sgn}^\#(z)$ be the signature of the Hessian at z of ϕ restricted to F. This gives an integer valued function on C_ϕ:
$$\text{sgn}^\# : C_\phi \to \mathbb{Z}, \quad z \mapsto \text{sgn}^\#(z).$$
From the diffeomorphism $\lambda_\phi = d_X\phi$
$$\lambda_\phi : C_\phi \to \Lambda$$
we get a $\mathbb{Z}$ valued function
$$\text{sgn}_\phi := \text{sgn}^\sharp \circ \lambda_\phi^{-1}.$$
Let
$$s_\phi := e^{\frac{\pi i}{4}\,\text{sgn}_\phi}.$$
So
$$s_\phi : \Lambda \to \mathbb{C}^*$$
taking values in the eighth roots of unity.

We define the Maslov bundle $\mathbb{L}_{\text{Maslov}} \to \Lambda$ to be the trivial flat bundle having s_ϕ as its defining flat section. For this definition to make sense we have to show that if (Z_i, π_i, ϕ_i), $\quad i = 1, 2$ are two descriptions of Λ by generating functions, then

$$s_{\phi_1} = c_{1,2} s_{\phi_2} \tag{5.25}$$

for some constant $c_{1,2} \in \mathbb{C}^*$. So we need to check this for the two types of move of Section 5.12. For moves of type 1), i.e. equivalences this is obvious.

For a move of type 2) the $\mathrm{sgn}_1^\#$ and $\mathrm{sgn}_2^\#$ are related by

$$\mathrm{sgn}_1^\# = \mathrm{sgn}_2^\# + \text{signature of} A.$$

This proves (5.25), and defines the Maslov bundle when a global generating function exists.

Now consider a general Lagrangian submanifold $\Lambda \subset T^*X$. Cover Λ by open sets U_i such that each U_i *is* defined by a generating function. We get function $s_{\phi_i} : U_i \to \mathbb{C}$ such that on every overlap $U_i \cap U_j$

$$s_{\phi_i} = c_{ij} s_{\phi_j}$$

with constants c_{ij} with $|c_[ij]| = 1$. In other words we get a Cech cocycle on the one skeleton of the nerve of this cover and 'hence a line bundle.

We will study the geometry of the Maslov bundle in more detail in Chapter **??**

5.15 Examples.

5.15.1 The image of a Lagrangian submanifold under geodesic flow.

Let X be a geodesically convex Riemannian manifold, for example $X = \mathbb{R}^n$. Let f_t denote geodesic flow on X. We know that for $t \neq 0$ a generating function for the symplectomorphism f_t is

$$\psi_t(x, y) = \frac{1}{2t} d(x, y)^2.$$

Let Λ be a Lagrangian submanifold of T^*X. Even if Λ is horizontal, there is no reason to expect that $f_t(\Lambda)$ be horizontal - caustics can develop. But our theorem about the generating function of the composition of two canonical relasions will give a generating function for $f_t(\Lambda)$. Indeed, suppose that ϕ is a generating function for Λ relative to a fibration

$$\pi : X \times S \to X.$$

Then

$$\frac{1}{2} d(x, y)^2 + \psi(y, s)$$

is a generating function for $f_t(\Lambda)$ relative to the fibration

$$X \times X \times S \to X, \quad (x, y, s) \mapsto x.$$

5.15.2 The billiard map and its iterates.

Definition of the billiard map.

Let Ω be a bounded open convex domain in $\mathbb{R}^n$ with smooth boundary X. We may identify the tangent space to any point of $\mathbb{R}^n$ with $\mathbb{R}^n$ using the vector space structure, and identify $\mathbb{R}^n$ with $(\mathbb{R}^n)^*$ using the standard inner product. Then at any $x \in X$ we have the identifications

$$T_xX \cong T_xX^*$$

using the euclidean scalar product on T_xX and

$$T_xX = \{v \in \mathbb{R}^n |\ v \cdot n(x) = 0\} \tag{5.26}$$

where $n(x)$ denotes the inward pointing unit normal to X at x. Let $U \subset TX$ denote the open subset consisting of all tangent vectors (under the above identification) satisfying

$$\|v\| < 1.$$

For each $x \in X$ and $v \in T_xX$ satisfying $\|v\| < 1$ let

$$u := v + an(x) \ \text{ where } a := (1 - \|v\|^2)^{\frac{1}{2}}.$$

So u is the unique inward pointing unit vector at x whose orthogonal projection onto T_xX is v.

Consider the ray through x in the direction of u, i.e. the ray

$$x + tu, \quad t > 0.$$

Since Ω is convex and bounded, this ray will intersect X at a unique point y. Let w be the orthogonal projection of u on T_yX. So we have defined a map

$$\mathcal{B} : U \to U, \qquad (x, v) \mapsto (y, w)$$

which is known as the **billiard map**.

The generating function of the billiard map.

We shall show that the billiard map is a symplectomorphism by writing down a function ϕ which is its generating function.

Consider the function

$$\psi :\ \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},\ \psi(x, y) = \|x - y\|.$$

This is smooth at all points $(x, y),\ x \neq y$. Let us compute $d_x\psi(v)$ at such a point (x, y) where $v \in T_xX$.

$$\frac{d}{dt}\psi(x + tv, y)_{|t=0} = \left(\frac{x - y}{\|y - x\|}, v\right)$$

where (,) denotes the scalar product on $\mathbb{R}^n$. Identifying $T\mathbb{R}^n$ with $T^*\mathbb{R}^n$ using this scalar product, we can write that for all $x \neq y$

$$d_x\psi(x,y) = -\frac{y-x}{\|x-y\|}, \qquad d_y\psi(x,y) = \frac{y-x}{\|x-y\|}.$$

If we set

$$u = \frac{y-x}{\|x-y\|}, \quad t = \|x-y\|$$

we have

$$\|u\| = 1$$

and

$$y = x + tu.$$

Let ϕ be the restriction of ψ to $X \times X \subset \mathbb{R}^n \times \mathbb{R}^n$. Let

$$\iota : X \to \mathbb{R}^n$$

denote the embedding of X into $\mathbb{R}^n$. Under the identifications

$$T_x\mathbb{R}^n \cong T_x^*\mathbb{R}^n, \quad T_xX \cong T_x^*X$$

the orthogonal projection

$$T_x^*\mathbb{R}^n \cong T_x\mathbb{R}^n \ni u \mapsto v \in T_xX \cong T_x^*X$$

is just the map

$$d\iota_x^* : T_x^*\mathbb{R}^n \to T_x^*X, \quad u \mapsto v.$$

So

$$v = d\iota_x^* u = d\iota_x^* d_x\psi(x,y) = d_x\phi(x,y).$$

So we have verified the conditions

$$v = -d_x\phi(x,y), \qquad w = d_y\phi(x,y)$$

which say that ϕ is a generating function for the billiard map $\mathcal{B}$.

Iteration of the billiard map.

Our general prescription for the composite of two canonical relations says that a generating function for the composite is given by the sum of generating functions for each (where the intermediate variable is regarded as a fiber variable over the initial and final variables). Therefore a generating function for $\mathcal{B}^n$ is given by the function

$$\phi(x_0, x_1, \ldots, x_n) = \|x_1 - x_0\| + \|x_2 - x_1\| + \cdots + \|x_n - x_{n-1}\|.$$

5.15.3 The classical analogue of the Fourier transform.

We repeat a previous computation: Let $X = \mathbb{R}^n$ and consider the map

$$\mathfrak{F} : T^*X \to T^*X, \quad (x, \xi) \mapsto (-\xi, x).$$

The generating function for this symplectomorphism is

$$x \cdot y.$$

Since the transpose of the graph of a symplectomorphism is the graph of the inverse, the generating function for the inverse is

$$-y \cdot x.$$

So a generating function for the identity is

$$\phi \in C^\infty(X \times X, \times \mathbb{R}^n)$$

$$\phi(x, z, y) = (x - z) \cdot y.$$

Chapter 6

The calculus of $\frac{1}{2}$ densities.

An essential ingredient in our symbol calculus will be the notion of a $\frac{1}{2}$-density on a canonical relation. We begin this chapter with a description of densities of arbitrary order on a manifold, and then specialize to the study of canonical relations.

6.1 The linear algebra of densities.

6.1.1 The definition of a density on a vector space.

Let V be an n-dimensional vector space over the real numbers. A basis $\mathbf{e} = e_1, \ldots, e_n$ of V is the same as an isomorphism $\ell_{\mathbf{e}}$ of $\mathbb{R}^n$ with V according to the rule

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto x_1 e_1 + \cdots + x_n e_n.$$

We can write this as

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto (e_1, \ldots e_n) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

or even more succinctly as

$$\ell_{\mathbf{e}} : \quad \mathbf{x} \mapsto \mathbf{e} \cdot \mathbf{x}$$

where

$$\mathbf{x} := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{e} = (e_1, \ldots, e_n).$$

The group $Gl(n) = Gl(n, \mathbb{R})$ acts on the set $\mathcal{F}(V)$ of all bases of V according to the rule

$$\ell_{\mathbf{e}} \mapsto \ell_{\mathbf{e}} \circ A^{-1}, \qquad A \in Gl(n)$$

which is the same as the "matrix multiplication"

$$\mathbf{e} \mapsto \mathbf{e} \cdot A^{-1}.$$

This action is effective and transitive:

- If $\mathbf{e} = \mathbf{e} \cdot A^{-1}$ for some basis $\mathbf{e}$ then $A = I$, the identity matrix, and
- Given any two bases $\mathbf{e}$ and $\mathbf{f}$ these exists a (unique) A such that $\mathbf{e} = \mathbf{f} \cdot A$.

Let $\alpha \in \mathbb{C}$ be any complex number. A **density of order** α on V is a function

$$\rho \ \mathcal{F}(V) \to \mathbb{C}$$

satisfying

$$\rho(\mathbf{e} \cdot A) = \rho(\mathbf{e})|\det a|^{\alpha} \qquad \forall A \in Gl(n). \tag{6.1}$$

We will denote the space of all densities of order α on V by

$$|V|^{\alpha}.$$

This is a one dimensional vector space over the complex numbers.

Let $L : V \to V$ be a linear map. If L is invertible and $\mathbf{e} \in \mathcal{F}(V)$ then $L\mathbf{e} = (Le_1, \ldots, Le_n)$ is again a basis of V. If we write

$$Le_j = \sum_i L_{ij} e_i$$

then

$$L\mathbf{e} = \mathbf{e}\mathrm{L}$$

where L is the matrix

$$\mathrm{L} := (L_{ij})$$

so if $\rho \in |V|^{\alpha}$ then

$$\rho(L\mathbf{e}) = (\det L)\rho(\mathbf{e}).$$

We can extend this to all L, non necessarily invertible, where the right hand side is 0. So here is an equivalent definition of a density on an n-dimensional real vector space:

A density ρ of order α is a rule which assigns a number $\rho(v_1, \ldots, v_n)$ to every n-tuplet of vectors and which satisfies

$$\rho(Lv_1, \ldots, Lv_n) = |\det L|^{\alpha} \rho(v_1, \ldots, v_n) \tag{6.2}$$

for any linear transformation $L : V \to V$. Of course, if the $v_1, \ldots, v_n$ are not linearly independent then

$$\rho(v_1, \ldots, v_n) = 0.$$

6.1.2 Multiplication.

If $\rho \in |V|^\alpha$ and $\tau \in |V|^\beta$ then we get a density $\rho \cdot \tau$ of order $\alpha + \beta$ given by

$$\rho \cdot \tau(\mathbf{e}) = \rho(\mathbf{e})\tau(\mathbf{e}).$$

In other words we have an isomorphism:

$$|V|^\alpha \otimes |V|^\beta \cong |V|^{\alpha+\beta} \tag{6.3}$$

6.1.3 Complex conjugation.

If $\rho \in |V|^\alpha$ then $\overline{\rho}$ defined by

$$\overline{\rho}(\mathbf{e}) = \overline{\rho(\mathbf{e})}$$

is a density of order $\overline{\alpha}$ on V. In other words we have an anti-linear map

$$|V|^\alpha \to |V|^{\overline{\alpha}}, \qquad \rho \mapsto \overline{\rho}.$$

This map is clearly an anti-linear isomorphism. Combined with (6.3) we get a sesquilinear map

$$|V|^\alpha \otimes |V|^\beta \to |V|^{\alpha+\overline{\beta}}.$$

We will especially want to use this for the case $\alpha = \beta = \frac{1}{2} + is$ where s is a real number. In this case we get a sesquilinear map

$$|V|^{\frac{1}{2}+is} \otimes |V|^{\frac{1}{2}+is} \to |V|^1. \tag{6.4}$$

6.1.4 Elementary consequences of the definition.

There are two obvious but very useful facts that we will use repeatedly:

1. An element of $|V|^\alpha$ is completely determined by its value on a single basis $\mathbf{e}$.

2. More generally, suppose we are given a subset S of the set of bases on which a subgroup $H \subset Gl(n)$ acts transitively and a function $\rho : S \to \mathbb{C}$ such (6.1) holds for all $A \in H$. Then ρ extends uniquely to a density of order α on V.

Here are some typical ways that we will use these facts:

Orthonormal frames: Suppose that V is equipped with a scalar product. This picks out a subset $\mathcal{O}(V) \subset \mathcal{F}(V)$ consisting of the orthonormal frames. The corresponding subgroup of $Gl(n)$ is $O(n)$ and every element of $O(n)$ has determinant ± 1. So any density of any order must take on a constant value on orthonormal frames, and item 2 above implies that any constant then determines a density of any order. We have trivialized the space $|V|^\alpha$ for all α. Another way of saying the same thing is that V has a preferred density of order α, namely the density which assigns the value one to any orthonormal frame. The same applies if V has any non-degenerate quadratic form, not necessarily positive definite.

Symplectic frames: Suppose that V is a symplectic vector space, so $n = \dim V = 2d$ is even. This picks out a collection of preferred bases, namely those of the form $e_1, \ldots, e_d,\ f_1, \ldots f_d$ where

$$\omega(e_i, e_j) = 0, \quad \omega(f_i, f_j) = 0. \quad \omega(e_i, f_j) = \delta_{ij}$$

where ω denotes the symplectic form. These are known as the symplectic frames. In this case $H = Sp(n)$ and every element of $Sp(n)$ has determinant one. So again $|V|^\alpha$ is trivialized. Again, another way of saying this is that a symplectic vector space has a preferred density of any order - the density which assigns the value one to any symplectic frame.

Transverse Lagrangian subspaces: Suppose that V is a symplectic vector space and that M and N are Lagrangian subspaces of V with $M \cap N = \{0\}$. *Any* basis $e_1, \ldots e_d$ of N determines a dual basis $f_1, \ldots f_d$ of N according to the requirement that

$$\omega(e_i, f_j) = \delta_{ij}$$

and then $e_1, \ldots e_d, f_1 \ldots f_d$ is a symplectic basis of V. If $C \in Gl(d)$ and we make the replacement

$$\mathbf{e} \mapsto \mathbf{e} \cdot C$$

then we must make the replacement

$$\mathbf{f} \mapsto \mathbf{f} \cdot (C^t)^{-1}.$$

So if ρ is a density of order α on M and τ is a density of order α on N they fit together to get a density of order zero (i.e. a constant) on V according to the rule

$$(\mathbf{e}, \mathbf{f}) = (e_1, \ldots, e_d,\ f_1, \ldots, f_d) \mapsto \rho(\mathbf{e})\tau(\mathbf{f})$$

on frames of the above dual type. The corresponding subgroup of $Gl(n)$ is a subgroup of $Sp(n)$ isomorphic to $Gl(d)$. So we have a canonical isomorphism

$$|M|^\alpha \otimes |N|^\alpha \cong \mathbb{C}. \tag{6.5}$$

Using (6.3) we can rewrite this as

$$|M|^\alpha \cong |N|^{-\alpha}.$$

Dual spaces: If we start with a vector space M we can make $M \oplus M^*$ into a symplectic vector space with M and M^* transverse Lagrangian subspaces and the pairing B between M and M^* just the standard pairing of a vector space with its dual space. So making a change in notation we have

$$|V|^\alpha \cong |V^*|^{-\alpha}. \tag{6.6}$$

Short exact sequences: Let

$$0 \to V' \to V \to V'' \to 0$$

be an exact sequence of linear maps of vector spaces. We can choose a preferred set of bases of V as follows : Let $(e_1, \dots, e_k)$ be a basis of V' and extend it to a basis $(e_1, \dots, e_k, e_{k+1}, \dots e_n$ of V. then the images of $e_i,\ i = k+1, \dots n$ form a basis of V''. Any two bases of this type differ by the action of an $A \in Gl(n)$ of the form

$$A = \begin{pmatrix} A' & * \\ 0 & A'' \end{pmatrix}$$

so

$$\det A = \det A' \cdot \det A''.$$

This shows that we have an isomorphism

$$|V|^\alpha \cong |V'|^\alpha \otimes |V''|^\alpha \tag{6.7}$$

for any α.

Long exact sequences Let

$$0 \to V_1 \to V_2 \to \cdots V_k \to 0$$

be an exact sequence of vector spaces. Then using (6.7) inductively we get

$$\bigotimes_{j \text{ even}} |V_j|^\alpha \cong \bigotimes_{j \text{ odd}} |V_j|^\alpha \tag{6.8}$$

for any α.

6.1.5 Pullback and pushforward under isomorphism.

Let

$$L : V \to W$$

be an isomorphism of n- dimensional vector spaces. If

$$\mathbf{e} = (e_1, \dots, e_n)$$

is a basis of V then

$$L\mathbf{e} := (Le_1, \dots, Le_n)$$

is a basis of W and

$$L(\mathbf{e} \cdot A) = (L\mathbf{e}) \cdot A \quad \forall a \in Gl(n).$$

So if $\rho \in |W|^\alpha$ then $L^*\rho$ defined by

$$(L^*\rho)(\mathbf{e}) := \rho(L\mathbf{e})$$

is an element of $|V|^\alpha$. In other words we have a pullback isomorphism

$$L^* : |W|^\alpha \to |V|^\alpha, \quad \rho \mapsto L^*\rho.$$

Applied to L^{-1} gives a pushforward map

$$L_* : |V|^\alpha \to |W|^\alpha, \quad L_* = (L^{-1})^*.$$

6.1.6 Lefschetz symplectic linear transformations.

There is a special case of (6.5) which we will use a lot in our applications, so we will work out the details here. A linear map $L : V \to V$ on a vector space is called **Lefschetz** if it has no eigenvalue equal to 1. Another way of saying this is that $I - L$ is invertible. Yet another way of saying this is the following: Let

$$\operatorname{graph} L \subset V \oplus V$$

be the graph of L so

$$\operatorname{graph} L = \{(v, Lv) \quad v \in V\}.$$

Let

$$\Delta \subset V \oplus V$$

be the diagonal, i.e. the graph of the identity transformation. Then

$$\operatorname{graph} L \cap \Delta = \{0\}. \tag{6.9}$$

Now suppose that V is a symplectic vector space and we consider $V^- \oplus V$ as a symplectic vector space. Suppose also that L is a (linear) symplectic transformation so that $\operatorname{graph} L$ is a Lagrangian subspace of $V^- \oplus V$ as is Δ. Suppose that L is also Lefschetz so that (6.9) holds.

The isomorphism

$$V \to \operatorname{graph} L : \qquad v \mapsto (v, Lv)$$

pushes the canonical α-density on V to an α-density on $\operatorname{graph} L$, namely, if $v_1, \ldots, v_n$ is a symplectic basis of V, then this pushforward α density assigns the value one to the basis

$$((v_1, Lv_1), \ldots, (v_n, Lv_n)) \quad \text{of } \operatorname{graph} L.$$

Let us call this α-density ρ_L. Similarly, we can use the map

$$\mathrm{diag}: V \to \Delta, \qquad v \mapsto (v,v)$$

to push the canonical α density to an α-density ρ_Δ on Δ. So ρ_Δ assigns the value one to the basis

$$((v_1,v_1),\ldots,(v_1,v_1)) \qquad \text{of } \Delta.$$

According to (6.5)

$$|\,\mathrm{graph}\, L|^\alpha \otimes |\Delta|^\alpha \cong \mathbb{C}.$$

So we get a number $\langle \rho_L, \rho_\Delta\rangle$ attached to these two α-densities. We claim that

$$\langle \rho_L, \rho_\Delta\rangle = |\det(I-L)|^{-\alpha}. \tag{6.10}$$

Before proving this formula, let us give another derivation of (6.5). Let M and N be subspaces of a symplectic vector space W. (The letter V is currently overworked.) Suppose that $M\cap N = \{0\}$ so that $W = M\oplus N$ as a vector space and so by (6.7) we have

$$|W|^\alpha = |M|^\alpha \otimes |N|^\alpha.$$

We have an identification of $|W|^\alpha$ with $\mathbb{C}$ given by sending

$$|W|^\alpha \ni \rho_W \mapsto \rho_W(\mathbf{w})$$

where $\mathbf{w}$ is any symplectic basis of W. Combing the last two equations gives an identification of $|M|^\alpha \otimes |N|^\alpha$ with $\mathbb{C}$ which coincides with (6.5) in case M and N are Lagrangian subspaces. Put another way, let $\mathbf{w}$ be a symplectic basis of W and suppose that $A\in Gl(\dim W)$ is such that

$$\mathbf{w}\cdot A = (\mathbf{m},\mathbf{n})$$

where $\mathbf{m}$ is a basis of M and $\mathbf{n}$ is a basis of N. Then the pairing of of $\rho_M\in |M|^\alpha$ with $\rho_N \in |N|^\alpha$ is given by

$$\langle \rho_M, \rho_N\rangle = |\det A|^{-\alpha}\rho_M(\mathbf{m})\rho_N(\mathbf{n}). \tag{6.11}$$

Now let us go back to the proof of (6.10). If $\mathbf{e},\mathbf{f} = e_1,\ldots,e_d, f_1\ldots,f_d$ is a symplectic basis of V then

$$((\mathbf{e},0)(0,\mathbf{e}),(\mathbf{f},0),(0,-\mathbf{f}))$$

is a symplectic basis of $V^-\oplus V$. We have

$$((\mathbf{e},0)(0,\mathbf{e}),(\mathbf{f},0),(0,-\mathbf{f}))\begin{pmatrix} I_d & 0 & 0 & 0\\ 0 & 0 & I_d & 0\\ 0 & I_d & 0 & 0\\ 0 & 0 & 0 & -I_d\end{pmatrix} = ((\mathbf{e},0),\mathbf{f},0),(0,\mathbf{e}),(0,\mathbf{f}))$$

and

$$\det \begin{pmatrix} I_d & 0 & 0 & 0 \\ 0 & 0 & I_d & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & -I_d \end{pmatrix} = 1.$$

Let $\mathbf{v}$ denote the symplectic basis $\mathbf{e}, \mathbf{f}$ of V so that

$$((\mathbf{e}, 0), \mathbf{f}, 0), (0, \mathbf{e}), (0, \mathbf{f})) = ((\mathbf{v}, 0), (0, \mathbf{v}))\,.$$

Write

$$Lv_j = \sum_i L_{ij} v_j, \quad \mathrm{L} = (L_{ij}).$$

Then

$$((\mathbf{v}, 0), (0, \mathbf{v})) \begin{pmatrix} I_n & I_n \\ \mathrm{L} & I_n \end{pmatrix} = ((\mathbf{v}, L\mathbf{v}), (\mathbf{v}, \mathbf{v}))\,.$$

So taking

$$A = \begin{pmatrix} I_d & 0 & 0 & 0 \\ 0 & 0 & I_d & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & -I_d \end{pmatrix} \begin{pmatrix} I_n & I_n \\ \mathrm{L} & I_n \end{pmatrix} = ((\mathbf{v}, L\mathbf{v}), (\mathbf{v}, \mathbf{v}))$$

in (6.11) proves (6.10) since

$$\det A = \det \begin{pmatrix} I_d & 0 & 0 & 0 \\ 0 & 0 & I_d & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & -I_d \end{pmatrix} \det \begin{pmatrix} I_n & I_n \\ \mathrm{L} & I_n \end{pmatrix} = \det(I_n - L).$$

6.2 Densities on manifolds.

If $E \to X$ be a real vector bundle. We can then conisder the complex line bundle

$$|E|^\alpha \to X$$

whose fiber over $x \in X$ is $|E_x|^\alpha$. The formulas of the preceding section apply pointwise.

We will be primarily interested in the tangent bundle TX. So $|TX|^\alpha$ a line bundle which we will call the α-density bundle and a smooth section of $|TX|^\alpha$ will be called an **α-density** or a **density of order** α.

Examples.

- . Let $X = \mathbb{R}^n$ with its standard coordinates and hence the standard vector fields

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}.$$

This means that at each point $p \in \mathbb{R}^n$ we have a preferred basis

$$\left(\frac{\partial}{\partial x_1},\right)_p, \dots, \left(\frac{\partial}{\partial x_n},\right)_p.$$

We let

$$dx^\alpha$$

denote the α-density which assigns, at each point p, the value 1 to the above basis. So the most general α-density on $\mathbb{R}^n$ can be written as

$$u \cdot dx^\alpha$$

or simply as

$$udx^\alpha$$

where u is a smooth function.

- Let X be an n-dimensional Riemannian manifold. At each point p we have a preferred family of bases of the tangent space - the orthonormal bases. We thus get a preferred density of order α - the density which assigns the value one to eacu orthonormal basis at each point.

- Let X be an orientable manifold and Ω a nowhere vanishing n-form on X. Then we get an α-density according to the rule: At each $p \in X$ assign to each basis $e_1, \dots, e_n$ of T_pX the value

  $$|\Omega(e_1, \dots, e_n)|^\alpha.$$

  We will denote this density by

  $$|\Omega|^\alpha.$$

- As a special case of the preceding example, if M is a symplectic manifold of dimension $2d$ with symplectic form ω, take

  $$\Omega = \omega \wedge \cdots \omega \qquad d \text{ factors.}$$

  So every symplectic manifold has a preferred α-density for any α.

If μ is an α density and ν is a β density the we can multiply them (pointwise) to obtain an $\alpha + \beta$ density $\mu\nu$. Similarly, we can take the complex conjugate of an α density to obtain an $\overline{\alpha}$ density.

Since a density is a section of a line bundle, it makes sense to say that a density *is* or *is not* zero at a point. The **support** of a density is defined to be the closure of the set of points where it is *not* zero.

6.3 Pull-back of a density under a diffeomorphism.

If
$$f : X \to Y$$
is a diffeomorphism, then we get, at each $x \in X$, a linear isomorphism
$$df_x T_x X \to T_{f(x)} Y.$$
A density ν of order α on Y assigns a density of order α (in the sense of vector spaces) to each $T_y Y$ which we can then pull back using df_x to obtain a density of order α on X. We denote this pulled back density by $f^*\nu$. For example, suppose that
$$\nu = |\Omega|^\alpha$$
for an n-form Ω on Y (where $n = \dim Y$). Then
$$f^*|\Omega|^\alpha = |f^*\Omega|^\alpha \tag{6.12}$$
where the $f^*\Omega$ occurring on right hand side of this equation is the usual pull-back of forms.

As an example, suppose that X and Y are open subsets of $\mathbb{R}^n$, then
$$dx^\alpha = |dx_1 \wedge \cdots dx_n|^\alpha, \qquad |dy|^\alpha = |dy_1 \wedge \cdots \wedge dy_n|^\alpha$$
and
$$f^*(dy1 \wedge \cdots \wedge dy_n) = \det J(f) dx_1 \wedge \cdots \wedge dx_n$$
where $J(f)$ is the Jacobian matrix of f. So
$$f^* dy^\alpha = |\det J(f)|^\alpha dx^\alpha. \tag{6.13}$$

Here is a second application of (6.12). Let $f_t : X \to X$ be a one parameter group of diffeomorphisms generated by a vector field v, and let ν be a density of order α on X. As usual, we define the Lie derivative $D_v \nu$ by
$$D_v \nu := \frac{d}{dt} f_t^* \nu_{|t=0}.$$
If $\nu = |\Omega|^\alpha$ then
$$D_v \nu = \alpha D_v |\Omega| \cdot |\Omega|^{\alpha - 1}$$
and if X is oriented, then we can identify $|\Omega|$ with Ω on oriented bases, so
$$D_v |\Omega| = D_v \Omega = d i(v) \Omega$$
on oriented bases. For example,
$$D_v dx^{\frac{1}{2}} = \frac{1}{2} (\mathrm{div}\ v) dx^{\frac{1}{2}} \tag{6.14}$$
where
$$\mathrm{div}\ v = \frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n} \quad \text{if} \quad v = v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n}.$$

6.4 Densities of order 1.

If we set $\alpha = 1$ in (6.13) we get

$$f^* dy = |\det J(f)| dx$$

or, more generally,

$$f^*(udy) = (u \circ f)|\det J(f)|dx$$

which is the change of variables formula for a multiple integral. So if ν is a density of order one of compact support which is supported on a coordinate patch $(U, x_1, \ldots, x_n)$, and we write

$$\nu = gdx$$

then

$$\int \nu := \int_U gdx$$

is independent of the choice of coordinates. If ν is a density of order one of compact support we can use a partition of unity to break it into a finite sum of densities of order one and of compact support contained in coordinate patches

$$\nu = \nu_1 + \cdots + \nu_r$$

and $\int_X \nu$ defined as

$$\int_X \nu := \int \nu_1 + \cdots + \int \nu_r$$

is independent of all choices. In other words densities of order one (usually just called densities) are objects which can be integrated (if of compact support). Furthermore, if

$$f : X \to Y$$

is a diffeomorphism, and ν is a density of order one of compact support on Y, we have the genral "change of variables formula"

$$\int_X f^* \nu = \int_Y \nu. \tag{6.15}$$

Suppose that α and β are complex numbers with

$$\alpha + \overline{\beta} = 1.$$

Suppose that μ is a density of order α and ν is a density of order β on X and that one of them has compact support. Then $\mu \cdot \overline{\nu}$ is a density of order one and we can form

$$\langle \mu, \nu \rangle := \int_X \mu \overline{\nu}.$$

So we get an intrinsic sesquilinear pairing between the densities of order α of compact support and the densities of order $1 - \overline{\alpha}$.

6.5 The principal series representation of Diff(X).

So if $s \in \mathbb{R}$, we get a pre-Hilbert space structure on the space of densities of order $\frac{1}{2} + is$ given by

$$(\mu, \nu) := \int_X mu\overline{\nu}.$$

If $f \in \text{Diff}(X)$, i.e. if $f : X \to X$ is a diffeomorphism, then

$$(f^{\mu}, f^*\nu) = (\mu, \nu)$$

and

$$(f \circ g)^* = g^* \circ f^*.$$

Let $\mathfrak{H}_s$ denote the completion of the pre-Hilbert space of densities of order $\frac{1}{2} + is$. The Hilbert space $\mathfrak{H}_s$ is known as the bf intrinsic Hilbert space of order s. The map

$$f \mapsto (f^{-1})^*$$

is a representation of Diff (X) on the space of densities or order $\frac{1}{2} + is$ which extends by completion to a unitary representation of Diff (X on $\mathfrak{H}_s$. This collection of representations (parametrized by s) is known as the principal series of representations.

If we take $S = S^1 = \mathbb{PR}^1$ and restrict the above representations of Diff(X) to $G = PL(2, \mathbb{R})$ we get the principal series of representations of G.

We will concentrate on the case $s = 0$, i.e. we will deal primarily with densities of order $\frac{1}{2}$.

6.6 The push-forward of a density of order one by a fibration.

There is an important generalization of the notion of the integral of a density of compact support: Let

$$\pi : Z \to X$$

be a *proper* fibration. Let μ be a density of order one on Z. We are going to define

$$\pi_*\mu$$

which will be a density of order one on X. We proceed as follows: for $x \in X$, let

$$F = F_x := \pi^{-1}(x)$$

be the fiber over x. Let $z \in F$. We have the exact sequence

$$0 \to T_zF \to T_zZ \stackrel{d\pi_z}{\to} T_xX \to 0$$

which gives rise to the isomorphism

$$|T_zF| \otimes |T_xX| \cong |T_zZ|.$$

The density μ thus assigns, to each z in the manifold F an element of $|T_Z| \otimes |T_x X|$. In other words, on the manifold F it is a density of order one with values in the fixed one dimensional vector space $|T_x X|$. Since F is compact, we can integrate this density over F to obtain an element of $|T_x X|$. As we do this for all x, we have obtained a density of order one on X.

Let us see what the operation $\mu \mapsto \pi_* \mu$ looks like in local coordinates. Let us choose local ccordinates $(U, x_1,$
$dots, x_n . s_1 \ldots, s_d)$ on Z and coordinates $y_1, \ldots, y_n$ on X so that

$$\pi : (x_1, \ldots, x_n, s_1, \ldots, s_d) \mapsto (x_1, \ldots, x_n).$$

Suppose that μ is supported on U and we write

$$\mu = u dx ds = u(x_1, \ldots, x_n, s_1 dots, s_d) dx_1 \ldots dx_n ds_1 \ldots ds_d.$$

Then

$$\pi_* \mu = \left(\int u(x_1, \ldots, x_n, s_1, \ldots, s_d) ds_1 \ldots ds_d \right) dx_1 \ldots, dx_n. \tag{6.16}$$

In the special case that X is a point, $\pi_* \mu = \int_Z \mu$. Also, Fubini's theorem says that if

$$W \xrightarrow{\rho} Z \xrightarrow{\pi} X$$

are fibrations with compact fibers then

$$(\pi \circ \rho)_* = \pi_* \circ \rho_*. \tag{6.17}$$

In particular, if μ is a density of compact support on Z with $Z \to X$ a fibration then $\pi_* \mu$ is defined and

$$\int_X \pi_* \mu = \int_Z \mu. \tag{6.18}$$

If f is a C^∞ function on X of compact support and $\pi : Z \to X$ is a proper fibration then $\pi^* f$ is constant along fibers and (6.18) says that

$$\int_Z \pi^* f \mu = \int_X f \mu. \tag{6.19}$$

In other words, the operations

$$\pi^* : C_0^\infty(X) \to^* : C_0^\infty(Z)$$

and

$$\pi_* : C^\infty(|TZ|) \to C^\infty(|TX|)$$

are transposes of one another.

Chapter 7

The enhanced symplectic "category".

Suppose that M_1, M_2, and M_3 are symplectic manifolds, and that

$$\Gamma_2 \in \operatorname{Morph}(M_2, M_3) \quad \text{and} \qquad \Gamma_1 \in \operatorname{Morph}(M_1, M_2)$$

are canonical relations which can be composed in the sense of Chapter 4. Let ρ_1 be a $\frac{1}{2}$-density on Γ_1 and ρ_2 a $\frac{1}{2}$-density on Λ_2. The purpose of this chapter is to define a $\frac{1}{2}$-density $\rho_2 \circ \rho_1$ on $\Gamma_2 \circ \Gamma_1$ and to study the properties of this composition. In particular we will show that the composition

$$(\Gamma_2, \rho_2) \times (\Gamma_1, \rho_1) \mapsto (\Gamma_2 \circ \Gamma_1, \rho_2 \circ \rho_1)$$

is associative when defined, and that the axioms for a "category" are satisfied.

7.1 The underlying linear algebra.

Let V_1 and V_2 be symplectic vector spaces and let $\Gamma \subset V_1^- \times V_2$ be a linear canonical relation. Let

$$\pi : \Gamma \to V_2$$

be the projection onto the second factor. Define

- $\operatorname{Ker} \Gamma \subset V_1$ by $\operatorname{Ker} \Gamma = \{v \in V_1 | (v, 0) \in \Gamma\}$.
- $\operatorname{Im} \Gamma \subset V_2 = \Gamma(V_1)$.

So $\Gamma^\dagger \subset V_2^- \oplus V_1$ and hence both $\ker \Gamma^\dagger$ and $\operatorname{Im} \Gamma$ are linear subspaces of the symplectic vector space V_2. We claim that

$$(\ker \Gamma^\dagger)^\perp = \operatorname{Im} \Gamma. \tag{7.1}$$

Here $\perp$ means perpendicular relative to the symplectic structure on V_2.

Proof. Let B_1 and B_2 be the symplectic bilinear forms on V_1 and V_2 so that $\tilde{B} = -B_1 \oplus (B_2)$ is the symplectic form on $V_1^- \oplus V_2$. So $v \in V_2$ is in $\operatorname{Ker}\Gamma^\dagger$ if and only if $(0,v) \in \Gamma$. Since Γ is Lagrangian,

$$(0,v) \in \Gamma^\perp \Leftrightarrow 0 = -B_1(0,v_1) + B_2(v,v_2) = -B_2(v,v_2)\ \forall\ (v_1,v_2) \in \Gamma.$$

But this is precisely the condition that $v \in (\text{Im }\Gamma)^\perp$. $\square$

Now let V_1, V_2, V_3 be symplectic vectors spaces and $\Gamma_1 \subset V_1^- \times V_2$ and $\Gamma_2 \subset V_2^- \times V_3$ be linear canonical relations. Let

$$\pi : \Gamma_2 \to V_2, \qquad \pi(v_1,v_2) = v_2$$

and

$$\rho : \Gamma_2 \to V_2, \qquad \rho(v_2,v_3) = v_2$$

so that the fiber product of π and ρ is given by

$$F := \{(v_1,v_2,v_3) | (v_1,v_2) \in \Gamma_1,\ \text{and}\ \ (v_2,v_3) \in \Gamma_2\}.$$

Let

$$\alpha : F \to V_1 \times V_3, \qquad \alpha(v_1,v_2,v_3) := (v_1,v_3).$$

The image of α is, by definition, $\Gamma_2 \circ \Gamma_1$. The kernel of α consists of those $(0,v,0) \in F$. So if we let

$$\mu : F \to V_2, \qquad \mu(v_1,v_2,v_3) = v_2$$

denote the projection of F onto the middle factor we have

$$\mu(\operatorname{Ker}\alpha) = \operatorname{Ker}\Gamma_1^\dagger \cap \operatorname{Ker}\Gamma_2. \tag{7.2}$$

Let

$$\tau :\ \Gamma_1 \times \Gamma_2 \to V_2$$

be defined by

$$\tau(\gamma_1,\gamma_2) := \pi(\gamma_1) - \rho(\gamma_2)$$

so that the definition of the fiber product F gives the exact sequence

$$0 \to F \to \Gamma_1 \times \Gamma_2 \xrightarrow{\tau} V_2 \to \text{Coker }\tau \to 0. \tag{7.3}$$

We have

$$\text{Im }\tau = \text{Im }\Gamma_1 + \text{Im }\Gamma_2^*$$

so that

$$(\text{Im }\tau)^\perp = \operatorname{Ker}\Gamma_1^\dagger \cap \operatorname{Ker}\Gamma_2 = \mu(\text{Ker }\ \alpha).$$

From the definition of $\operatorname{Ker}\Gamma$ we see that it is always isotropic so $\operatorname{Ker}\Gamma_1^\dagger \cap \operatorname{Ker}\Gamma_2$ is isotropic and hence $\operatorname{Im}\,\tau$ is a co-isotropic subspace of V_2 and the symplectic bilinear form B_2 on V_2 induces a non-singular bilinear pairing

$$(V_2/\operatorname{Im}\,\tau) \times \mu(\operatorname{Ker}\ \alpha) \to \mathbb{R}.$$

Furthermore, the map μ restricted to $\operatorname{Ker}\alpha$ is an isomorphism. So we have produced a canonical isomorphism

$$(\operatorname{Coker}\,\tau)^* \cong \operatorname{Ker}\alpha. \tag{7.4}$$

From the exact sequence (7.3) we get the isomorphism

$$|F|^{\frac{1}{2}} \otimes |V_2|^{\frac{1}{2}} \cong |\Gamma_1|^{\frac{1}{2}} \otimes |\Gamma_2|^{\frac{1}{2}} \otimes |\operatorname{Coker}\,\tau|^{\frac{1}{2}}.$$

The symplectic form on V_2 gives a canonical trivialization $|V_2|^{\frac{1}{2}} \cong \mathbb{C}$. Also we have $|F|^{\frac{1}{2}} \cong |\operatorname{Ker}\alpha|^{\frac{1}{2}} \otimes |\operatorname{Im}\,\alpha|^{\frac{1}{2}}$. From (7.4) we have

$$|\operatorname{Coker}\,\tau|^{\frac{1}{2}} \cong |\operatorname{Ker}\ \alpha|^{-\frac{1}{2}}.$$

Substituting these into the preceding isomorphism we get

$$|\operatorname{Ker}\alpha| \otimes |\operatorname{Im}\,\alpha|^{\frac{1}{2}} \cong |\Gamma_1|^{\frac{1}{2}} \otimes |\Gamma_2|^{\frac{1}{2}}.$$

But since $\operatorname{Im}\,\alpha = \Gamma_2 \circ \Gamma_1$ we get the key formula

$$|\operatorname{Ker}\ \alpha| \otimes |\Gamma_2 \circ \Gamma_1|^{\frac{1}{2}} \cong |\Gamma_1|^{\frac{1}{2}} \otimes |\Gamma_2|^{\frac{1}{2}}. \tag{7.5}$$

7.2 Half densities and clean canonical compositions.

Let M_1, M_2, M_3 be symplectic manifolds and let $\Gamma_1 \subset M_1^- \times M_2$ and $\Gamma_2 \subset M_2^- \times M_3$ be canonical relations. Let

$$\pi:\ \Gamma_1 \to M_2,\ \pi(m_1,m_2) = m_2, \qquad \rho:\ \Gamma_2 \to M_2,\ \rho(m_2,m_3) = m_2,$$

and $F \subset \Gamma_1 \times \Gamma_2$ the fiber product:

$$F = \{(m_1,m_2,m_3) | (m_1,m_2) \in \Gamma_1, \quad (m_2,m_3) \in \Gamma_2\}.$$

Let

$$\alpha : F \to M_1 \times M_3, \quad \alpha(m_1,m_2,m_3) = (m_1,m_3).$$

The image of α is the composition $\Gamma_2 \circ \Gamma_1$.

Recall that we say that Γ_1 and Γ_2 intersect cleanly if the maps ρ and π intersect cleanly. If π and ρ intersect cleanly then their fiber product F is a submanifold of $\Gamma_1 \times \Gamma_2$ and the arrows in the exact square

$$\begin{array}{ccc} F & \longrightarrow & \Gamma_1 \\ \big\downarrow & & \big\downarrow \pi \\ \Gamma_2 & \xrightarrow[\rho]{} & M_2 \end{array}$$

are smooth maps. Furthermore the differentials of these maps at any point give an exact square of the corresponding linear canonical relations. In particular, α is of constant rank and $\Gamma_2 \circ \Gamma_1$ is an immersed canonical relation. If we further assume that

1. α is proper and
2. the level sets of α are connected,

then $\Gamma_2 \circ \Gamma_1$ is an embedded Lagrangian submanifold of $M_1^- \times M_2$ and

$$\alpha : F \to \Gamma_2 \circ \Gamma_1$$

is a fiber map with proper fibers. So our key identity (7.5) holds at the tangent space level: If we let $m = (m_1, m_2, m_3) \in F$ and $q = \alpha(m) \in \Gamma_2 \circ \Gamma_1$ we get an isomorphism

$$|T_m F| \otimes |T_q(\Gamma_2 \circ \Gamma_1)|^{\frac{1}{2}} \cong |T_{m_1,m_2}\Gamma_1|^{\frac{1}{2}} \otimes |T_{(m_2,m_3)}\Gamma_2|^{\frac{1}{2}}.$$

This means that if we are given half densities ρ_1 on Γ_1 and ρ_2 on Γ_2 we get a half density on $\Gamma_2 \circ \Gamma_1$ by integrating the expression obtained from the left hand side of the above isomorphism over the fiber. This gives us the composition law for half densities. Once we establish the associative law and the existence of the identity we will have have *enhanced* our symplectic category so that now the morphisms consist of pairs (Γ, ρ) where Γ is a canonical relation and where ρ is a half density on Γ.

7.3 Rewriting the composition law.

We will rewrite the composition law in the spirit of Sections 3.3.2 and 4.4: If $\Gamma \subset M^- \times M$ is the graph of a symplectomorphism, then the projection of Γ onto the first factor is a diffeomorphism. The symplectic form on M determines a canonical $\frac{1}{2}$density on M, and hence on Γ. In particular, we can apply this fact to the identity map, so $\Delta \subset M^- \times M$ carries a canonical $\frac{1}{2}$-density. Hence, the submanifold

$$\tilde{\Delta}_{M_1,M_2,M_3} = \{(x, y, y, z, x, z)\} \subset M_1 \times M_2 \times M_2 \times M_3 \times M_1 \times M_3$$

as in (4.6) carries a canonical $\frac{1}{2}$-form $\tau_{1,2,3}$. Then we know that

$$\Gamma_2 \circ \Gamma_1 = \tilde{\Delta}_{M_1,M_2,M_3} \circ \Gamma_1 \times \Gamma_2$$

More details?

and it is easy to check that

$$\rho_2 \circ \rho_1 = \tau_{123} \circ (\rho_1 \times \rho_2.$$

Similarly,

$$(\Gamma_3 \circ \Gamma_2) \circ \Gamma_1) = \Gamma_3 \circ (\Gamma_2 \circ \Gamma_1) = \tilde{\Delta}_{M_1,M_2,M_3,M_4} \circ (\Gamma_1 \times \Gamma_2 \times \Gamma_3)$$

and $\tilde{\Delta}_{M_1,M_2,M_3,M_4}$ carries a canonical $\frac{1}{2}$-density $\tau_{1,2,3,4}$ with

$$(\rho_3 \circ \rho_2) \circ \rho_1 = \rho_3 \circ (\rho_2 \circ \rho_1) = \tau_{1.2.3.4} \circ (\rho_1 \times \rho_2 \times \rho_3.$$

This establishes the associative law.

7.4 Enhancing the category of smooth manifolds and maps.

Let X and Y be smooth manifolds and $E \to X$ and $F \to Y$ be vector bundles. According to Atiyah and Bott, a morphism from $E \to X$ to $F \to Y$ consists of a smooth map

$$f : X \to Y$$

and a section

$$r \in C^\infty(f^*F, E).$$

We described the finite set analogue of this concept in Section 3.3.5. If s is a smooth section of $F \to Y$ then we get a smooth section of $E \to X$ via

$$(f, r)^* s(x) := r(s(f(x))).$$

We want to specialize this construction of Atiyah-Bott to the case where E and F are the line bundles of $\frac{1}{2}$-forms. So we say that r is an enhancement of the smooth map $F : X \to Y$ or that (f, r) is an enhanced smooth map if r is a smooth section of the line bundle

$$\mathrm{Hom}(|f^*TY|, |TX|).$$

The composition of two enhanced maps

$$(f, r) : (E \to X) \to (F \to Y) \quad \text{ and } \quad (g, r') : (F \to Y) \to (G \to Z)$$

is $(g \circ f, r \circ r')$ where, for $\tau \in |T_{g(f(x))}Z)|^{\frac{1}{2}}$

$$(r \circ r')(\tau) = r(r'(\tau)).$$

We thus obtain a category whose objects are manifolds and whose morphisms are enhanced maps.

If ρ is a $\frac{1}{2}$ density on Y and (f, r) is an enhanced map then we get a $\frac{1}{2}$ density on X by the Atiyah-Bott rule

$$(f, r)^* \rho(x) = r(\rho(f(x)) \in |T_x X|^{\frac{1}{2}}.$$

Then we know that the assignment $(f, r) \mapsto (f, r)^*$ is functorial.

We now give some examples of enhancement of particular kinds of maps:

7.4.1 Enhancing an immersion.

Suppose $f : X \to Y$ is an immersion. We then get the conormal bundle $N_f^* X$ whose fiber at x consists of all covectors $\xi \in T^*_{f(x)} Y$ such that $df_x^* \xi = 0$. We have the exact sequence

$$0 \to T_x X \stackrel{df_x}{\to} T_{f(x)} Y \to N_x Y \to\to 0.$$

Here N_xY is *defined* as the quotient $T_{f(x))Y/(df_x(T_xX)}$. The fact that f is an immersion is the statement that df_x is injective. The space $(N_f^*X_x)$ is the dual space of N_xY. From this exact sequence we get the isomorphism

$$|T_{f(x)}Y|^{\frac{1}{2}} \cong N_xY|^{frac12} \otimes |T_xX|^{\frac{1}{2}}.$$

So

$$\mathrm{Hom}(|T_{f(x)}Y|^{\frac{1}{2}}, |T_xX|^{\frac{1}{2}}) \cong |T_xX|^{\frac{1}{2}} \otimes \mathrm{Hom}(|T_{f(x)}Y|^{-\frac{1}{2}} \cong |N_x|^{-\frac{1}{2}} \cong |(N_f^*X)_x|^{\frac{1}{2}}.$$

Conclusion. Enhancing an immersion is the same as giving a section of $|N_f^*X|^{\frac{1}{2}}$.

7.4.2 Enhancing a fibration.

Suppose that $\pi : Z \to X$ is a submersion. If $z \in Z$, let V_z denote the tangent space to to the fiber $\pi^{-1}(x)$ at z where $x = \pi(z)$. Thus V_z is the kernel of $d\pi_z : T_zZ \to T_{\pi(z)}X$. So we have an exact sequence

$$0 \to V_z \to T_zZ \to T_{\pi(z)}X \to 0$$

and hence the isomorphism

$$|T_zZ|^{\frac{1}{2}} \cong |V_z|^{\frac{1}{2}} \otimes |T_{\pi(z)}X|^{\frac{1}{2}}.$$

So

$$\mathrm{Hom}(|T_{\pi(z)}X|^{\frac{1}{2}}, |T_zZ|^{\frac{1}{2}}) \cong |T_{\pi(z)}X|^{-\frac{1}{2}} \otimes |T_zZ|^{\frac{1}{2}} \cong |V_z|^{\frac{1}{2}}. \tag{7.6}$$

Conclusion. Enhancing a fibration is the same as giving a section of $|V|^{\frac{1}{2}}$ where V denote the vertical sub-bundle of the tangent bundle, i.e. the sub-bundle tangent to the fibers of the fibration.

7.4.3 The pushforward via an enhanced fibration.

Suppose that $\pi : Z \to X$ is a fibration with compact fibers and r is an enhancement of π so that r is given by a section of the line-bundle $V|^{\frac{1}{2}}$ as we have just seen. Let ρ be a $\frac{1}{2}$-form on Z. From the isomorphism

$$|T_zZ|^{\frac{1}{2}} \cong |V_z|^{\frac{1}{2}} \otimes |T_{\pi(z)}X|^{\frac{1}{2}}$$

we can regard ρ as section of $|V_z|^{\frac{1}{2}} \otimes \pi^*|TX|^{\frac{1}{2}}$ and hence

$$r \cdot \rho$$

is a section of $|V| \otimes \pi^*|TX|^{\frac{1}{2}}$. Put another way, for each $x \in X$ $r \cdot \rho$ gives a density (of order one)) on $\pi^{-1}(x)$ with values in the fixed vector space $|T_xX|^{\frac{1}{2}}$. So we can integrate this density of order one over the fiber to obtain

$$\pi_*(r \cdot \rho)$$

which is a $\frac{1}{2}$-density on X. If the enhancement r of π is understood, we will denote the push-forward of the $\frac{1}{2}$-density ρ simply by

$$\pi_*\rho.$$

We have the obvious variants on this construction if π is not proper. We can construct $\pi_*(r\cdot\rho)$ if either r or ρ are compactly supported in the fiber direction.

An enhanced fibration $\pi = (\pi, r)$ gives a pull-back operation π^* from half densities on X to $\frac{1}{2}$-densities on Z. So if μ is a $\frac{1}{2}$-density on X and ν is a $\frac{1}{2}$-density on Z then

$$\nu \cdot \pi^*\mu$$

is a density on Z. If μ is of compact support and if ν is compactly supported in the fiber direction, then $\nu\cdot\pi^*\mu$ is a density (of order one) of compact support on Z which we can integrate over Z. We can also form

$$(\pi_*\nu)\cdot\mu.$$

which is a density (of order one) which is of compact support on X. It follows form Fubini's theorem that

$$\int_Z \nu\cdot\pi^*\mu = \int_X (\pi_*\nu)\cdot\mu.$$

7.5 Enhancing a map enhances the corresponding canonical relation.

Let $f : X \to Y$ be a smooth map. We can enhance this map by giving a section r of $\mathrm{Hom}(|TY|^{\frac{1}{2}}, |TX|^{\frac{1}{2}})$. On the other hand, we can construct the canonical relation

$$\Gamma_f \in \mathrm{Morph}(T^*X, T^*Y)$$

as described in Section 4.7. Enhancing this canonical relation amounts to giving a $\frac{1}{2}$-form ρ on Γ_f. In this section we show how the enhancement r of the map f gives rise to a $\frac{1}{2}$-form on Γ_f.

Recall (4.9) which says that

$$\Gamma_f = \{(x_1,\xi_1,x_2,\xi_2) | x_2 = f(x_1), \quad \xi_1 = df^*_{x_1}\xi_2\}.$$

From this description we see that Γ_f is a vector bundle over X whose fiber over $x \in X$ is $T^*_{f(x)}Y$. So at each point $z = (x,\xi_1,y,\eta) \in \Gamma_f$ we have the isomorphism

$$|T_z\Gamma_f|^{\frac{1}{2}} \cong |T_xX|^{\frac{1}{2}} \otimes |T_\eta(T^*_{f(x)}Y)|^{\frac{1}{2}}.$$

But $(T^*_{f(x)}Y)$ is a vector space, and at any point η in a vector space W we have a canonical idenitication of $T_\eta W$ with W. So at each $z \in \Gamma_f$ we have an isomorphism

$$|T_z\Gamma_f|^{\frac{1}{2}} \cong |T_xX|^{\frac{1}{2}} \otimes |T_\eta(T^*_{f(x)}Y)|^{\frac{1}{2}} = \mathrm{Hom}(|T_{f(x)}Y|^{\frac{1}{2}}, |T_xX|^{\frac{1}{2}})$$

and at each x, $r(x)$ is an element of $\mathrm{Hom}(|T_{f(x)}Y|^{\frac{1}{2}}, |T_xX|^{\frac{1}{2}})$. So r gives rise to a $\frac{1}{2}$-density on Γ_f

I still need to write up the functoriality of this relation.

7.6 The involutive structure of the enhanced symplectic "category".

Recall that if $\Gamma \in \mathrm{Morph}(M_1, M_2)$ then we defined $\Gamma^\dagger \in (M_2, M_1)$ be

$$\Gamma^\dagger = \{(y,x)|(x,y) \in \Gamma\}.$$

We have the switching diffeomorphism

$$s : \Gamma^\dagger \to \Gamma, \qquad (y,x) \mapsto (x,y),$$

and so if ρ is a $\frac{1}{2}$-density on Γ then $s^*\rho$ is a $\frac{1}{2}$-density on $\Gamma^\dagger$. We define

$$\rho^\dagger = \overline{s^*\rho}. \tag{7.7}$$

Starting with an enhanced morphism (Γ, ρ) we define

$$(\Gamma, \rho)^\dagger = (\Gamma^\dagger, \rho^\dagger).$$

We show that $\dagger : (\Gamma, \rho) \mapsto (\Gamma, \rho)^\dagger$ satisfies the conditions for a involutive structure. Since $s^2 = \mathrm{id}$ it is clear that $\dagger^2 = \mathrm{id}$. If $\Gamma_2 \in \mathrm{Morph}(M_2, M_1)$ and $\Gamma_1 \in \mathrm{Morph}(M_1, M_2)$ are composible morphsims, we know that the composition of (Γ_2, ρ_2) with (Γ_1, ρ_1) is given by

$$(\tilde{\Delta}_{M_1,M_2,M_3}, \tau_{123}) \circ (\Gamma_1 \times \Gamma_2, \rho_1 \times \rho_2).$$

where

$$\tilde{\Delta}_{M_1,M_2,M_3} = \{(x,y,y,z,x,z)|x \in M_1, y \in M_2, z \in M_3\}$$

and τ_{123} is the canonical (real) $\frac{1}{2}$-density arising from the symplectic structures on M_1, M_2 and M_3. So

$$s : (\Gamma_1 \circ \Gamma_2)^\dagger = \Gamma_1^\dagger \circ \Gamma_2^\dagger \to \Gamma_2 \circ \Gamma_1$$

is given by applying the operator S switching x and z

$$S : \tilde{\Delta}_{M_3,M_2,M_1} \to \tilde{\Delta}_{M_1,M_2,M_3},$$

applying the switching operators $s_1 : \Gamma_1^\dagger \to \Gamma_1$ and $s_2 : \Gamma_2^\dagger \to \Gamma_2$ and also switching the order of Γ_1 and Γ_2. Pull-back under switching the order of Γ_1 and Γ_2 sends $\rho_1 \times \rho_2$ to $\rho_2 \times \rho_1$, applying the individual s_1^* and s_2^* and taking complex conjugates sends $\rho_2 \times \rho_1$ to $\rho_2^\dagger \times \rho_1^\dagger$. Also

$$S^*\tau_{123} = \tau_{321}$$

and τ_{321} is real. Putting all these facts together shows that

$$((\Gamma_2,\rho_2)\circ(\Gamma_1,\rho_1))^\dagger = (\Gamma_1,\rho_1)^\dagger \circ (\Gamma_2,\rho_2)^\dagger$$

proving that $\dagger$ satisfies the conditions for a involutive structure.

Let M be an object in our “category”, i.e. a symplectic manifold. A “point” of M in our enchanced “category” will consist of a Lagrangian submanifold $\Lambda \subset M$ thought of as an element of $\mathrm{Morph}(\mathrm{pt.}, M)$ (in $\mathcal{S}$) together with a $\frac{1}{2}$-density on Λ. If (Λ,ρ) is such a point, then $(\Lambda,\rho)^\dagger = (\Lambda^\dagger,\rho^\dagger)$ where we now think of the Lagrangian submanifold $\Lambda^\dagger$ as an element of $\mathrm{Morph}(M,\mathrm{pt.})$.

Suppose that (Λ_1,ρ_1) and (Λ_2,ρ_2) are “points” of M and that $\Lambda_2^\dagger$ and Λ_1 are composible. Then $\Lambda_2^\dagger\circ\Lambda_1$ in $\mathcal{S}$ is an element of $\mathrm{Morph}(\mathrm{pt.},\mathrm{pt.})$ which consists of a (single) point. So in our enhanced “category” $\tilde{\mathcal{S}}$

$$(\Lambda_2,\rho_2)^\dagger(\Lambda_1,\rho_1)$$

is a $\frac{1}{2}$-density on a point, i.e. a complex number. We will denote this number by

$$\langle(\Lambda_1,\rho_1),(\Lambda_2,\rho_2)\rangle\,.$$

7.6.1 Computing the pairing $\langle(\Lambda_1,\rho_1),(\Lambda_2,\rho_2)\rangle$.

This is, of course, a special case of the computation of Section 7.2.

The first condition that $\Lambda_2^\dagger$ and Λ_1 be composible is that $F=\Lambda_1$ and Λ_2 intersect cleanly as submanifolds of M. Here F is a special case of the fiber product of Section 7.2 and the argument there show that we have an isomorphism

$$|T_pF| = |T_p(\Lambda_1\cap\Lambda_2| \cong |T_p\Lambda_1|^{\frac{1}{2}}\otimes|T_p\Lambda_2|^{\frac{1}{2}}$$

and so ρ_1 and $\overline{\rho_2}$ multiply together to give a density $\rho_1\overline{\rho_2}$ on $\Lambda_1\cap\Lambda_2$. A second condition on composibility requires that $\Lambda_1\cap\Lambda_2$ be compact and then

$$\langle(\Lambda_1,\rho_1),(\Lambda_2,\rho_2)\rangle = \int_{\Lambda_1\cap\Lambda_2}\rho_1\overline{\rho_2}.$$

7.6.2 $\dagger$ and the adjoint under the pairing.

In the category of whose objects are Hilbert spaces and whose morphisms are bounded operators, the adjoint $A^\dagger$ of a operator $A: H_1\to H_2$ is defined by

$$\langle Av,w\rangle_2 = \langle v, A^\dagger w\rangle_1, \tag{7.8}$$

for all $v\in H_1, w\in H_2$ where $\langle\ ,\ \rangle_i$ denotes the scalar product on H_i, $i=1,2$. This can be given a more categorical interpretation as follows: A vector u in a Hilbert space H determines and is determined by a bounded linear map from $\mathbb{C}$ to H,

$$z \mapsto zu.$$

In other words, if we regard $\mathbb{C}$ as the pt. in the category of Hilbert spaces, then we can regard $u \in H$ as an element of of $\mathrm{Morph}(\mathrm{pt.}, H)$. So if $v \in H$ we can regard $v^\dagger$ as an element of $\mathrm{Morph}(H, \mathrm{pt.})$ where

$$v^\dagger(u) = \langle u, v\rangle.$$

So if we regard $\dagger$ as the primary operation, then the scalar product on each Hilbert space is determined by the preceding equation - the right hand side is *defined* as being equal to the left hand side. Then equation (7.8) is a consequence of the associative law and the laws $(A \circ B)^\dagger = B^\dagger \circ A^\dagger$ and $\dagger^2 = \mathrm{id}.$. Indeed

$$\langle Av, w\rangle_2 := w^\dagger \circ A \circ v = (A^\dagger \circ w)^\dagger \circ v =: \langle v, A^\dagger w\rangle_1.$$

So once we agree that a $\frac{1}{2}$-density is just a complex number, we can conclude that the analogue of (7.8) holds in our enhanced category $\tilde{\mathcal{S}}$: If (Λ_1, ρ_1) is a “point ” of M_1 in our enhanced category, and if (Λ_2, ρ_2) is a “point ” of M_2 and if $(\Gamma, \tau) \in \mathrm{Morph}(M_1, M_2)$ then (assuming that the various morphisms are composible) we have

$$\langle ((\Gamma,\tau) \circ (\Lambda_1, \rho_1), (\Lambda_2, \rho_2)\rangle_2 = \left\langle (\Lambda_1, \rho_1), ((\Gamma,\tau)^\dagger \circ (\Lambda_2, \rho_2)\right\rangle_1. \tag{7.9}$$

7.7 The moment Lagrangian.

Let (M, ω) be a symplectic manifold. Let Z, X and S be manifolds and suppose that

$$\pi : Z \to S$$

is a fibration with fibers diffeomorphic to X. Let

$$G : Z \to M$$

$$g_s : Z_s \to M, \quad Z_s := \pi^{-1}(s)$$

denote the restriction of G to Z_s. We assume that

$$g_s \text{ is a Lagrangian embedding} \tag{7.10}$$

and let

$$\Lambda_s := g_s(Z_s) \tag{7.11}$$

denote the image of g_s. So we have a family of Lagrangian submanifolds $\Lambda_s \subset M$ parametrized by the points of S.

Let $s \in S$ and $\xi \in T_sS$. For $z \in Z_s$ and $w \in T_zZ_s$ tangent to the fiber Z_s

$$dG_z w = (dg_s)_z w \in T_{G(z)}\Lambda_s$$

so dG_z induces a map, which by abuse of language we will continue to denote by dG_z

$$dG_z : T_zZ/T_zZ_s \to T_mM/T_m\Lambda, \quad m = G(z). \tag{7.12}$$

But $d\pi_z$ induces an identification

$$T_zZ/T_z(Z_s) = T_sS. \tag{7.13}$$

Furthermore, we have an identification

$$T_mM/T_m(\Lambda_s) = T_m^*\Lambda_s \tag{7.14}$$

given by

$$T_mM \ni u \mapsto i(u)\omega_m(\cdot) = \omega_m(u,\cdot).$$

Thus (7.12) shows that each $\xi \in T_sS$ gives rise to a one form on Λ_s and hence by pull-back a one form on Z_s. To be explicit, let us choose a trivialization of our bundle around Z_s so we have an identification

$$H : Z_s \times U \to \pi^{-1}(U)$$

where U is a neighborhood of s in S. Then if $t \mapsto s(t)$ is any curve on S with $s(0) = s,\ s'(0) = \xi$ we get a curve of maps $h_{s(t)}$ of $Z_s \to M$ where

$$h_{s(t)} = g_{s(t)} \circ H.$$

We thus get a vector field v^ξ along the map h_s

$$v^\xi : Z_s \to TM, \qquad v^\xi(z) = \frac{d}{dt}h_{s(t)}(z)_{|t=0}.$$

Then the one form in question is

$$\tau^\xi = h_s^*(i(v^\xi)\omega).$$

A direct check shows that this one form is exactly the one form described above (and hence is independent of all the choices). We claim that

$$d\tau^\xi = 0. \tag{7.15}$$

Indeed, the general form of the Weil formula (See Chapter **??**) and the fact that $d\omega = 0$ gives

$$\frac{d}{dt}h_{s(t)}^*\omega_{|t=0} = dh_s^*i(v^\xi)\omega$$

and the fact that Γ_s is Lagrangian for all s implies that the left hand side and hence the right hand side is zero.

Assume that $H^1(X) = \{0\}$. Since the fiber Z_s is diffeomorphic to X, this implies that

$$\tau^\xi = d\phi^\xi$$

for some C^∞ function ϕ^ξ on Z_s. The function ϕ^ξ is uniquely determined up to an additive constant (if X is connected) which we can fix (in various ways) so that it depends smoothly on s and linearly on ξ. For example, if we have a cross-section $c : S \to Z$ we can demand that $\phi(c(s))^\xi \equiv 0$ for all s and ξ. Alternatively, if each Z_s is compact and equipped with a positive density dz_s we can demand that $\int_{Z_s} \phi^\xi dz_s = 0$ for all ξ and s.

Victor: Do we want to be more explicit about hypotheses here?

Suppose that we have made such choice. Then for fixed $z \in Z_s$ the number $\phi^\xi(z)$ depends linearly on ξ,. Hence we get a map

$$\Phi_0 : Z \to T^*S, \quad \Phi_0(z) = \lambda \Leftrightarrow \lambda(\xi) = \phi^\xi(z). \tag{7.16}$$

As we shall see in the next section, Φ_0 can be considered as a generalization of the moment map for a Hamiltonian group action.

Our choice determines ϕ^ξ up to an additive constant (if X is connected) $\mu(s, \xi)$ which we can assume to be smooth in s and linear in ξ. Replacing ϕ^ξ by $\phi^\xi + \mu(s, \xi)$ has the effect of making the replacement

$$\Phi_0 \mapsto \Phi_0 + \mu \circ \pi$$

where $\mu : S \to T^*S$ is the one form $\langle \mu_s, \xi\rangle = \mu(s, \xi)$.

Let ω_S denote the canocial two form on T^*S.

Theorem 24 *Assume that $H^2(S) = \{0\}$. Then there exists a $\nu \in \Omega^1(S)$ such that if we set*

$$\Phi = \Phi_0 + \nu \circ \pi$$

then

$$G^*\omega + \Phi^*\omega_S = 0. \tag{7.17}$$

As a consequence, the map

$$\tilde{G} : Z \to M \times T^*S, \qquad z \mapsto (G(z), \Phi(z)) \tag{7.18}$$

is a Lagrangian embedding.

Proof.

We first prove a local version of the theorem. Locally, we may assume that $Z = X \times S$ and by the Weinstein tubular neighborhood theorem we may assume (locally) that $M = T^*X$ and that for a fixed $s_0 \in S$ the Lagrangian submanifold Λ_{s_0} is the zero section of T^*X and that the map

$$G : X \times S \to T^*X$$

is given by

$$G(x, s) = d_X\psi(x, s)$$

where $\psi \in C^\infty(X \times S)$. So in terms of these choices, the maps $h_{s(t)}$ used above are given by

$$h_{s(t)}(x) = d_X\psi(x, s(t))$$

and hence the one form τ^ξ is given by

$$d_S d_X \psi(x,\xi) = \langle d_X d_S \psi, \xi\rangle$$

so we may choose

$$\Phi(x,s) = d_S\psi(x,s).$$

Thus

$$G^*\alpha_X = d_X\psi, \qquad \Phi^*\alpha_S = d_S\psi$$

and hence

$$G^*\omega_X + \Phi^*\omega_S = -dd\psi = 0.$$

This proves a local version of the theorem. We now pass from the local to the global:

By uniqueness, our global Φ_0 must agree with our local Φ up to the replacement $\Phi \mapsto \Phi + \mu\circ\pi$. So we know that

$$G^*\omega + \Phi_0^*\omega_S = (\mu\circ\pi)^*\omega_S = \pi^*\mu^*\omega_S.$$

Here μ is a one form on S regarded as a map $S \to T^*S$. But

$$d\pi^*\mu^*\omega_S = \pi^*\mu^* d\omega_S = 0.$$

So we know that $G^*\omega + \Phi_0^*\omega_S$ is a closed two form which is locally and hence globally of the form $\pi^*\tau$ where $d\tau = 0$. Now we make use of our assumption that $H^2(S) = \{0\}$ to write $\tau = d\nu$. Replacing Φ_0 by $\Phi_0 + \nu$ replaces τ by $\tau + \nu^*\omega_S$. But

$$\nu^*\omega_S = -\nu^* d\alpha_S = -d\nu = -\tau. \qquad \square$$

Remarks 1. If $H^2(S) \neq \{0\}$ then we can not succeed by modifying Φ. But we can modify the symplectic form on T^*S replacing ω_S by $\omega_S - \pi_S^*\sigma$ where π_S denotes the projection $T^*S \to S$.

2. Suppose that a compact Lie group K acts as fiber bundle automorphisms of $\pi : Z \to S$ and acts as symplectomorphisms of M. Suppose further that the fibers of Z are compact and equipped with a density along the fiber which is invariant under the group action. Finally suppose that the map G is equivariant for the group actions of K on Z and on M. Then the map $\tilde{G}$ can be chosen to be equivariant for the actions of K on Z and the induced action of K on $M \times T^*G$.

Victor; I think this is right but please check.

3. More generally we want to consider situations where a Lie group K acts on Z as fiber bundle automorphisms and on M and where we know by explicit construction that the map $\tilde{G}$ can be chosen to be equivariant . This will be the case for the classical moment map for a Hamiltonian group action as we shall see in the next section.

4. Let $\Gamma \subset M \otimes T^*S$ denote the image of $\tilde{G}$. If we are given a $\frac{1}{2}$-density μ on Z. then we get its image $\tilde{G}_*\mu$, a $\frac{1}{2}$-density on Γ.

7.7.1 The derivative of the moment map.

We continue the current notation. So we have the moment map

$$\Phi : Z \to T^*S.$$

Fix $s \in S$. The restriction of Φ to the fiber Z_s maps $Z_s \to T_s^*S$. since T_s^*S is a vector space, we may identify its tangent space at any point with T_s^*S itself. Hence for $z \in Z_s$ we may regard $d\Phi_z$ as a linear map from T_zZ to T_s^*S. So we write

$$d\Phi_z : T_zZ_s \to T_s^*S. \tag{7.19}$$

On the other hand, recall that using the identifications (7.13) and (7.14) we got a map

$$dG_z : T_sS \to T_m^*\Lambda, \quad m = G(z)$$

and hence composing with $d(g_s)_z^* : T_m^*\Lambda \to T_z^*Z_s$ a linear map

$$\chi_z := d(g_s)_z^* \circ dG_z : T_sS \to T_z^*Z. \tag{7.20}$$

Theorem 25 *The maps $d\Phi_z$ given by (7.19) and χ_z given by (7.20) are transposes of one another.*

Proof. Each $\xi \in T_sS$ gives rise to a one form τ^ξ on Z_s and by definition, the value of this one form at $z \in Z_s$ is exactly $\chi_z(\xi)$. The function ϕ^ξ was defined on $Z_s$so as to satisfy $d\phi^\xi = \tau^\xi$. In other words, for $v \in T_zZ$

$$\langle \chi_z(\xi), v \rangle = \langle d\Phi_z(v), \xi \rangle. \quad \square$$

Corollary 26 *The kernel of χ_z is the annihilator of the image of the map (7.19). In particular z is a regular point of the map $\Phi : Z_s \to T_s^*S$ if the map χ_z is injective.*

Corollary 27 *The kernel of the map (7.19) is the annihilator of the image of χ_z.*

7.8 Families of symplectomorphisms.

Let us now specialize to the case of a parametrized family of symplectomorphisms. So let $(M\omega)$ be a symplectic manifold, S a manifold and

$$F : M \times S \to M$$

a smooth map such that

$$f_s : M \to M$$

is a symplectomophism for each s, where $f_s(m) = F(m, s)$. We can apply the results of the preceding section where now $\Lambda_s \subset M \times M^-$ is the graph of f_s (and the M of the preceding section is replaced by$M \times M^-$) and so

$$G : M \times S \to M \times M^-, \qquad G(m, s) = (m, F(m, s)). \tag{7.21}$$

Theorem 24 says that get a map

$$\Phi : M \times S \to T^*S$$

and a moment Lagrangian

$$\Gamma_\Phi \subset M \times M^- \times T^*S.$$

7.8.1 Hamiltonian group actions.

Let us specialize further by assuming that S is a Lie group K and that $F : M \times K \to M$ is a Hamiltonian group action. So we have a map

$$G : M \times K \to M \times M^-, \qquad (m, a) \mapsto (m, am).$$

Let K act on $Z = M \times K$ via its left action on K so $a \in K$ acts on Z as

$$a : (m, b) \mapsto (m, ab).$$

We expect to be able to construct $\tilde{G} : M \times K \to T^*K$ so as to be equivariant for the action of K on $Z = M \times K$ and the induced action of K on T^*K.

To say that the action is Hamiltonian with moment map $\Psi : M \to \mathfrak{k}^*$ is to say that

$$i(\xi_M)\omega = -d\langle \Psi, \xi \rangle.$$

Thus under the left invariant identification of T^*K with $K \times \mathfrak{k}^*$ we see that Ψ determines a map

$$\Phi : M \times K \to T^*K, \qquad \Phi(m, a) = (a, \Psi(m)).$$

So our Φ of (7.16) is indeed a generalization of the moment map for Hamiltonian group actions.

7.8.2 The derivative of the moment map.

This section will be rewritten in view of the more general version given above.

In this section we will generalize an basic result about moment maps for Hamiltonian group actions to parametrized families of symplectomorphisms. We recall our notation: $(M\omega)$ is a symplectic manifold, S a manifold and

$$F : M \times S \to M$$

a smooth map such that

$$f_s : M \to M$$

is a symplectomophism for each s, where $f_s(m) = F(m, s)$.

For $p \in M$ and $s_0 \in S$ define

$$\gamma_0 : \quad S \to M \quad \text{ by } \ \gamma_0(s) = f_s\left(f_{s_0}^{-1}(p)\right).$$

Differentiating this map at s_0 gives linear map

$$(d\gamma_0)_{s_0} : \; T_{s_0} Z \to T_p M. \tag{7.22}$$

On the other hand, restricting the moment map $\Phi : M \times S \to T^*S$ to $M \times \{s_0\}$ gives a map

$$\Phi_0 : \; M \to T^*_{s_0} S$$

and differentiating Φ_0 at p, and using the fact that $T^*_{s_0} M$ is a vector space, gives a map

$$(d\Phi_0)_p : \; T_p M \to T^*_{s_0} S. \tag{7.23}$$

The bilinear form ω_p on $T_p M$ gives a bijective linear map

$$T_p M \to T^*_p m$$

with an inverse

$$T^*_p M \to T_p M. \tag{7.24}$$

Composing (7.23) and (7.23) gives a map

$$(d\tilde{\phi_0})_p : \quad T^*_p M \to T_{s_0} S. \tag{7.25}$$

I haven't had time to think about how this works for the general case of families of canonical relations and so did not put in details here.

Theorem 28 *The maps (7.22) and (7.24) are transposes of one another and hence*

- *The kernel of the map (7.23) is the symplectic orthocomplement of the image of the map (7.22) and*
- *The image of the map (7.23) is the annihilator in $T^*_{s_0} S$ of the kernel of the map (7.22).*

7.9 The symbolic distributional trace.

We continue with the notation of the precding section.

7.9.1 The $\frac{1}{2}$-density on Γ.

Since M is symplectic it has a canonical $\frac{1}{2}$ density. So if we equip S with a half density ρ_S we get a $\frac{1}{2}$ density on $M \times S$ and hence a $\frac{1}{2}$ density ρ_Γ making Γ into a morphism

$$(\Gamma, \rho_\Gamma) \in \mathrm{Morph}(M^- \times M, T^*S)$$

in our enhanced symplectic category.

Let $\Delta \subset M^- \times M$ be the diagonal. The map

$$M \to M^- \times M \qquad m \mapsto (m, m)$$

carries the canonical $\frac{1}{2}$-density on M to a $\frac{1}{2}$-density, call it ρ_Δ on Δ making Δ into a morphism

$$(\Delta, \rho_\Delta) \in \mathrm{Morph}(\mathrm{pt.}.M^- \times M).$$

The generalized trace in our enhanced symplectic "category".

Suppose that Γ and Δ are composable. Then we get a Lagrangian submanifold

$$\Lambda = \Gamma \circ \Delta$$

and a $\frac{1}{2}$-density

$$\rho_\lambda := \rho_\Gamma \circ \rho_\Delta$$

on Λ. the operation of passing from F to (Λ, ρ_Λ) can be regarded as the symbolic version of the distributional trace operation in operator theory.

7.9.2 Example: The symbolic trace.

Suppose that we have a single symplectomorphism $f : M \to M$ so that S is a point as is T^*S. Let

$$\Gamma = \Gamma_f = \text{graph } f = \{(m, f(m)),\ m \in M\}$$

considered as a morphism from $M \times M^-$ to a point. Suppose that Γ and Δ intersect transversally so that $\Gamma \cap \Delta$ is discrete. Suppose in fact that it is finite. We have the $\frac{1}{2}$-densities ρ_Δ on $T_m\Delta$ and $T_m\Gamma$ at each point m of of $\Gamma \cap \Delta$. Hence, by (6.10) The result is

This is equation (6.10)

$$\sum_{m \in \Delta \cap \Gamma} |\det(I - df_m)|^{-\frac{1}{2}}\,. \tag{7.26}$$

7.9.3 General transverse trace.

Let S be arbitrary. We examine the meaning of the hypothesis that that the inclusion $\iota : \Delta \to M \times M$ and the projection $\Gamma \to M \times M$ be transverse.

Since Γ is the image of $(G, \Phi) : M \times S \to M \times M \times T^*$, the projection of Γ onto $M \times M$ is just the image of the map G given in (7.21). So the transverse compobility condition is

$$G \pitchfork \Delta. \tag{7.27}$$

The fiber product of Γ and Δ can thus be identified with the "fixed point submanifold" of $M \times S$:

$$\mathfrak{F} := \{(m, s) | f_s(m) = m\}.$$

The transversality assumption guarantees that this is a submanifold of $M \times S$ whose dimension is equal to $\dim S$. The transversal version of our composition law for morphisms in the category $\mathcal{S}$ assert that

$$\Phi : \mathfrak{F} \to T * S$$

is a Lagrangian immersion whose image is

$$\Lambda = \Gamma \circ \Delta.$$

Let us assume that $\mathfrak{F}$ is connected and that Φ is a Lagrangian imbedding. (More generally we might want to assume that $\mathfrak{F}$ has a finite number of connected components and that Φ restricted to each of these components is an imbedding. Then the discussion below would apply separately to each component of $\mathfrak{F}$.)

Let us derive some consequences of the transversality hypothesis $G \pitchfork \Delta$. By the Thom transverslity theorem, there exists an open subset

$$S_O \subset S$$

such that for every $s \in S_O$, the map

$$g_s : M \to M \times M, \quad g_s(m) = G(m,s) = (mf_s(m))$$

is transverse to Δ. So for $s \in S_O$,

$$g_s^{-1}(\Delta) = \{m_i(s), i = 1, \ldots, r\}$$

is a finite subset of M and the m_i depend smoothly on $s \in S_O$. For each i, $\Phi(m_i(s)) \in T_s^*S$ then depends smoothly on $s \in S_O$. So we get one forms

$$\mu_i := \Phi(m_i(s)) \tag{7.28}$$

parametrizing open subsets Λ_i of Λ. Since Λ is Lagrangian, these one forms are closed. So if we assume taht $H^1(S_O) = \{0\}$, we cad write

$$\mu_i = d\psi_i$$

for $\psi_i \in C^\infty(S_O)$ and

$$\Lambda_i = \Lambda_{\psi_i}.$$

The maps

$$S_O \to \Lambda_i, \quad s \mapsto (s, d\psi_i(s))$$

map S_O diffeomorphically onto Λ_i. The pull-backs of the $\frac{1}{2}$-density $\rho_\Lambda = \rho_\Gamma \circ \rho_\Delta$ under these maps can be written as

$$h_i \rho_S$$

where ρ_S is the $\frac{1}{2}$-density we started with on S and where the h_i are the smooth functions

Victor: details here?

$$h_i(s) = \left|\det(I - df_{m_i})\right|^{-\frac{1}{2}}. \tag{7.29}$$

In other words, on the generic set S_O where g_s is transverse to Δ, we can compute the symbolic trace $h(s)$ of g_s as in the preceding section. At points not in S_O, the "fixed points coalesce" so that g_s is no longer transverse to Δ and the individual g_s no longer have a trace as individual maps. Nevertheless, the parametrized family of maps have a trace as a $\frac{1}{2}$-form on Λ which need not be horizontal over points of S which are not in S_O.

7.9.4 Example: Periodic Hamiltonian trajectories.

Let (M, ω) be a symplectic manifold and

$$H : M \to \mathbb{R}$$

a proper smooth function with no critical points. Let $v = v_H$ be the corresponding Hamiltonian vector field, so that

$$i(v)\omega = -dH.$$

The fact that H is proper implies that v generates a global one parameter group of transformations, so we get a Hamiltonian action of $\mathbb{R}$ on M with Hamiltonian H, so we know that the function Φ of (7.16) (determined up to a constant) can be taken to be

$$\Phi : M \times \mathbb{R} \to T^*\mathbb{R} = \mathbb{R} \times \mathbb{R}, \qquad \Phi(m, t) = (t, H(m)).$$

The fact that $dH_m \neq 0$ for any m implies that the vector field v has no zeros.

Notice that in this case the transversality hypothesis of the previous example is never satisfied. For if it were, we could find a dense set of t for which $\exp tv : M \to M$ has isolated fixed points. But if m is fixed under $\exp tv$ then every point on the orbit $(\exp sv)m$ of m is also fixed under $\exp tv$ and we know that this orbit is a curve since v has no zeros.

So the best we can do is assume clean intersection: Our Γ in this case is

$$\Gamma = \{m, (\exp sv)m, s, H(m))\}.$$

If we set $f_s = \exp sv$ we write this as

$$\Gamma = \{(m, f_s(m), s, H(m))\}.$$

The assumption that the maps $\Gamma \to M \times M$ and

$$\iota : \Delta \to M \times M$$

intersect cleanly means that the fiber product

$$X = \{(m, s) \in M \times \mathbb{R} | f_s(m) = m\}$$

and that its tangent space at (m, s) is

$$\{(v, c) \in T_m M \times \mathbb{R} | v = (df_s)_m(v) + cv(m) \tag{7.30}$$

since

$$dF_{(m,s)}\left(v, c\frac{\partial}{\partial t}\right) = (df_s)_m(v) + cv(m).$$

The enery-period relation.

As we know, a consequence of our clean intersection assumption is that the map Φ restricted to X is of constant rank, and its image is an immersed Lagrangian submanifold of $T^*\mathbb{R}$. So if t is the standard coordinate of $\mathbb{R}$ and (t,τ) the corresponding coordinates of $T^*\mathbb{R} = \mathbb{R} \times \mathbb{R}$, we know that

$$dH \wedge dt = -\Phi^*(dt \wedge d\tau)$$

vanishes when restricted to X. Now the function t when restricted to X gives the value at (m,s) for which $f_t(m) = m$. It is a period of the trajectory through m. So if c is a regular value H, so $dH(m) \neq 0$ at all $m \in H^{-1}(c)$, then dt must be a multiple of $dH(m)$ at such points of X. We conclude:

Proposition 11 *If c is a regurlar value of H, then on every connected component of $H^{-1}(c) \cap X$ all trajectories of v have the sameperiod.*

The trace and the Poincaré map.

Victor: I have not been able to figure out all the details of this section.

Chapter 8

Oscillatory $\frac{1}{2}$-densities.

Let $\Lambda \subset T^*X$ be a Lagrangian submanifold. Let

$$\infty < k < \infty.$$

The plan of this chapter is to associate to Λ and to k a space

$$I^k(X, \Lambda)$$

of rapidly oscillating $\frac{1}{2}$-densities on X and to study the properties of these spaces. If

$$\Lambda = \Lambda_\psi, \quad \psi \in C^\infty(X),$$

this space will consist of $\frac{1}{2}$-densities of the form

$$e^{i\frac{c}{\hbar}} \hbar^k a(x, \hbar) e^{i\frac{\psi(x)}{\hbar}} \rho_0$$

where $c \in \mathbb{R}$, where ρ_0 is a fixed non-vanishing $\frac{1}{2}$-density on X and where

$$a \in C^\infty(X \times \mathbb{R}).$$

In other words, so long as Λ is horizontal, our space will consist of the $\frac{1}{2}$-densities we studied in Chapter 1.

As we saw in Chapter 1, one must take into account, when solving hyperbolic partial differential equations, the fact that caustics develop as a result of the Hamiltonian flow applied to initial conditions. So we will need a more general definition. We will make a more general definition in terms of a general generating function relative to a fibration, and then show that the class of oscillating $\frac{1}{2}$-densities on X that we obtain this way is independent of the choice of generating functions.

8.1 Definition of $I^k(X,\Lambda)$ in terms of a generating function.

Let $\pi : Z \to X$ be a fibration which is enhanced in the sense of Section 7.4.2. Recall that this means that we are given a smooth section r of $|V|^{\frac{1}{2}}$ where V is the vertical. We will assume that r vanishes nowhere. If ν is a $\frac{1}{2}$-density on Z which is of compact support in the vertical direction, then recall from Section 7.4.3 that we get from this data a push-forward $\frac{1}{2}$-density $\pi_*\nu$ on X.

Now suppose that ϕ is a global generating function for Λ with respect to π. Let

$$d := \dim Z - \dim X.$$

We define $I_0^k(X,\Lambda,\phi)$ to be the space of all compactly supported $\frac{1}{2}$-densities on X of the form

$$\mu = \hbar^{k-\frac{d}{2}}\pi_*\left(ae^{i\frac{\phi}{\hbar}}\tau\right)e^{i\frac{c}{\hbar}} \tag{8.1}$$

where

$$a \in C_0^\infty(Z\times\mathbb{R})$$

and where τ is a nowhere vanishing $\frac{1}{2}$-density on Z. Then define $I^k(X,\Lambda,\phi)$ to consist of those $\frac{1}{2}$-densities μ such that $\rho\mu \in I_0^k(X,\Lambda,\phi)$ for every $\rho \in C_0^\infty(X)$.

It is clear that $I^k(X,\Lambda,\phi)$ does not depend on the choice of the enhancement r of π or on the choice of τ.

8.1.1 Local description of $I^k(X,\Lambda,\phi)$.

Suppose that $Z = X\times S$ where S is an open subset of $\mathbb{R}^d$ and π is projection onto the first factor. We may choose our fiber $\frac{1}{2}$-density to be the Euclidean $\frac{1}{2}$-density $ds^{\frac{1}{2}}$ and τ to be $\tau_0\otimes ds^{\frac{1}{2}}$ where τ_0 is a nowhere vanishing $\frac{1}{2}$-density on X. Then $\phi = \phi(x,s)$ and (8.1) becomes the oscillating integral

$$e^{i\frac{c}{\hbar}}\left(\int_S a(x,s,\hbar)e^{i\frac{\phi}{\hbar}}ds\right)\tau_0. \tag{8.2}$$

8.1.2 Independence of the generating function.

Let $\pi_i : Z_i \to X,\ \phi_i$ be two fibrations and and generating functions for the same Lagrangian submanifold $\Lambda \subset T^*X$. We wish to show that $I^k(X,\Lambda,\phi_1) = I^k(X,\Lambda,\phi_2)$. By a partition of unity, it is enough to prove this locally. According to Section 5.12, it is enough to check this for two types of change of generating functions, 1) equivalence and 2)increasing the number of fiber variables. Let us examine each of the two cases:

Equivalence.

There exists a diffeomorphism $g:\ Z_1 \to Z_2$ with

$$\pi_2 \circ g = \pi_1 \qquad \text{and} \qquad \text{and} \qquad \phi_2 \circ g = \phi_1 .$$

Let us fix a non-vanishing section r of the vertical $\frac{1}{2}$-density bundle $|V_1|^{\frac{1}{2}}$ of Z_1 and a $\frac{1}{2}$-density τ_1 on Z_1. Since g is a fiber map, these determine vertical $\frac{1}{2}$-densities and $\frac{1}{2}$-densities $g_* r$ and $g_*\tau$ on Z_2. If $a \in C_0^\infty(Z_2 \times \mathbb{R})$ then the change of variables formula for an integral implies that

$$\pi_{2,*} a e^{i\frac{\phi_2}{\hbar}} g_*\tau = \pi_{1,*} g_* a e^{\frac{\phi_1}{\hbar}}$$

where the push forward $\pi_{2,*}$ on the left is relative to $g_* r$ and the push forward on the right is relative to r. $\square$

Increasing the number of fiber variables.

We may assume that $Z_2 = Z_2 \times S$ where S is an open subset of $\mathbb{R}^m$ and

$$\phi_2(z,s) = \phi_1(z) + \frac{1}{2}\langle As, s\rangle$$

where A is a symmetric non-degenerate $m \times m$ matrix. We write Z for Z_1. If d is the fiber dimension of Z then $d+m$ is the fiber dimension of Z_2. Let r be a vertical $\frac{1}{2}$-density on Z so that $r \otimes ds^{\frac{1}{2}}$ is a vertical $\frac{1}{2}$ density on Z_2. Let τ be a $\frac{1}{2}$ density on Z so that $\tau \otimes ds^{\frac{1}{2}}$ is a $\frac{1}{2}$-density on Z_2. We want to consider the expression

$$\hbar^{k-\frac{d+m}{2}} \pi_{2*} a_2(z,s,\hbar) e^{i\frac{\phi_1(z,s)}{\hbar}} (\tau \otimes ds^{\frac{1}{2}}).$$

Let $\pi_{2,1} : Z \times S \to Z$ be projection onto the first factor so that

$$\pi_{2*} = \pi_{1*} \circ \pi_{2,1*}$$

and the operation $\pi_{2,1*}$ sends

$$a_2(z,s,\hbar) e^{i\frac{\phi_2}{\hbar}} \tau \otimes ds^{\frac{1}{2}} \mapsto b(z,\hbar) e^{i\frac{\phi_1}{\hbar}} \tau$$

where

$$b(z,\hbar) = \int a_2(z,s,\hbar) e^{\frac{\langle As,s\rangle}{2\hbar}} ds.$$

We now apply the Lemma of Stationary Phase (see Chapter **??**) to conclude that

$$b(z,\hbar) = \hbar^m a_1(z,\hbar)$$

and in fact $a_1(z,\hbar) = a_2(z,0,\hbar) + O(\hbar)$. $\square$

8.1.3 The global definition of $I^k(X,\Lambda)$.

Let Λ be a Lagrangian submanifold of T^*X. We can find a locally finite open cover of Λ by open sets Λ_i such that each Λ_i is defined by a generating function ϕ_i relative to a fibration $\pi_i : Z_i \to U_i$ where the U_i are open subsets of X. We let $I_0^k(X,\Lambda)$ consist of those $\frac{1}{2}$-densities which can be written as a finite sum of the form

$$\mu = \sum_{j=1}^{N} \mu_{i_j}, \qquad \mu_{i_j} \in I_0^k(X, \Lambda_{i_j}).$$

By the results of the preceding section we know that this definition is independent of the choice of open cover and of the local descriptions by generating functions.

We then define the space $I^k(X,\Lambda)$ to consist of those $\frac{1}{2}$-densities μ on X such that $\rho\mu \in I_0^k(X,\Lambda)$ for every C^∞ function ρ on X of compact support.

8.2 Semi-classical Fourier integral operators.

Let X_1 and X_2 be manifolds, let

$$X = X_1 \times X_2$$

and let

$$M_i = T^*X_i, \qquad i = 1, 2.$$

Finally, let

$$\Gamma \in \mathrm{Morph}(M_1, M_2)$$

be a canonical relation, so

$$\Gamma \subset M_1^- \times M_2.$$

Let

$$\varsigma_1 : M_1^- \to M_1, \qquad \varsigma_1(x_1,\xi_1) = (x_1, -\xi_1)$$

so that

$$\Lambda := (\varsigma_1 \times \mathrm{id})(\Gamma)$$

is a Lagrangian submanifold of

$$T^*X = T^*X_1 \times T^*X_2.$$

Associated with Λ we have the space of compactly supported oscillatory $\frac{1}{2}$-densities $I_0^k(X,\Lambda)$. Choose a nowhere vanishing density on X_1 which we will denote (with some abuse of language) as dx_1 and similarly choose a nowhere vanishing density dx_2 on X_2. We can then write a typical element of $I_0^k(X,\Lambda)$ as

$$u(x_1, x_2, \hbar) dx_1^{\frac{1}{2}} dx_2^{\frac{1}{2}}$$

where u is a smooth function of compact support in all three "variables".

Recall that $L^2(X_i)$ is the intrinsic Hilbert space of L^2 half densities on X_i. Since u is compactly supported, we can define the integral operator

$$F_\mu = F_{\mu,\hbar} : \quad Ł^2(X_1) \to L^2(X_2)$$

by

$$F_\mu(f dx_1^{\frac{1}{2}}) = \left(\int_{X_1} f(x_1) u(x_1, x_2) dx_1 \right) dx_2^{\frac{1}{2}}. \tag{8.3}$$

We will denote the space of such operators by

$$\mathcal{F}_0^k(\Gamma)$$

and call them compactly supported **semi-classical Fourier integral operators**. We could, more generally, demand merely that $u(x_1, x_2, \hbar) dx_1^{\frac{1}{2}}$ be an element of $L_2(X_1)$ in this definition, in which case we would drop the subscript 0.

Let X_1, X_2 and X_3 be manifolds, let $M_i = T^* X_i, \ i = 1, 2, 3$ and let

$$\Gamma_1 \in \mathrm{Morph}(M_1, M_2), \qquad \Gamma_2 \in \mathrm{Morph}(M_2, M_3)$$

be canonical relations. Let

$$F_1 \in \mathcal{F}_0^{m_1}(\Gamma_1) \quad \text{and} \quad F_2 \in \mathcal{F}_0^{m_2}(\Gamma_2).$$

Finally, let

$$n = \dim X_2.$$

Theorem 29 *If* Γ_2 *and* Γ_1 *are transversally composible, then*

$$F_2 \circ F_1 \in \mathcal{F}_0^{m_1 + m_2 + \frac{n}{2}}(\Gamma_2 \circ \Gamma_1). \tag{8.4}$$

Proof. By partition of unity we may assume that we have fibrations

$$\pi_1 : X_1 \times X_2 \times S_1 \to X_1 \times X_2, \qquad \pi_2 : X_2 \times X_3 \times S_2 \to X_2 \times X_3$$

where S_1 and S_2 are open subsets of $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ and that ϕ_1 and ϕ_2 are generating functions for Γ_1 and Γ_2 with respect to these fibrations. We also fix nowhere vansihing $\frac{1}{2}$-densities $dx_i^{\frac{1}{2}}$ on $X_i, \ i = 1, 2, 3$. So F_1 is an integral operator with respect to a kernel of the form (8.3) where

$$u_1(x_1, x_2, \hbar) = e^{\frac{ic_1}{\hbar}} \hbar^{m_1 - \frac{d_1}{2}} \int a_1(x_1, x_2, s_1, \hbar) e^{i \frac{\phi_1(x_1, x_2, s_1)}{\hbar}} ds_1$$

and F_2 has a similar expression (under the change $1 \mapsto 2, \ 2 \mapsto 3$). So their composition is the integral operator

$$f dx_1^{\frac{1}{2}} \mapsto \left(\int_{X_1} f(x_1) u(x_1, x_3) dx_1 \right) dx_3^{\frac{1}{2}}$$

where

$$u(x_1,x_3)=e^{i\frac{c_1+c_2}{\hbar}}h^{m_1+m_2-\frac{d_1+d_2}{2}}\int a_1(x_1,x_2,s_1,\hbar)a_2(x_2,x_3,s_2,\hbar)e^{i\frac{\phi_1+\phi_2}{\hbar}}ds_1ds_2dx_2.$$

Victor: I think we only proved this above for transverse composition. In your notes you state a more general theorem here in which case we may have to go back and prove the general case.

By Theorem 18 $\phi_1(x_1,x_2,s_1)+\phi_2(x_2,x_3,s_2)$ is a generating function for $\Gamma_2\circ\Gamma_1$ with respect to the fibration

$$X_1\times X_3\times(X_2\times S_1\times S_2)\to X_1\times X_3.$$

Since the fiber dimension is d_1+d_2+n and the exponent of $\hbar$ in the above expression is $m_1+m_2-\frac{d_1+d_2}{2}$ we obtain (8.4). $\square$

8.3 The symbol of an element of $I^k(X,\Lambda)$.

Let Λ be a Lagrangian submanifold of T^*X. We have attached to Λ the space $I^k(X,\Lambda)$ of oscillating $\frac{1}{2}$-densities. The goal of this section is to give an intrinsic description of the quotient

$$I^k(X,\Lambda)/I^{k+1}(X,\Lambda)$$

as sections of line bundle $\mathbb{L}\to\Lambda$. This line bundle will locally look like the line bundle $|T\Lambda|^{\frac{1}{2}}$ whose sections are $\frac{1}{2}$-densities on Λ. However we will have to tensor this bundle with some flat line bundles in order to get a precise global description of $I^k(X,\Lambda)/I^{k+1}(X,\Lambda)$.

Victor: Should we mention Keller and Maslov here?

8.3.1 A local description of $I^k(X,\Lambda)/I^{k+1}(X,\Lambda)$.

Let S be an open subset of $\mathbb{R}^d$ and suppose that we have a generating function $\phi=\phi(x,s)$ for Λ with respect to the fibration

$$X\times S\to X,\quad (x,s)\mapsto x.$$

Fix a nowhere vanishing C^∞ $\frac{1}{2}$-density ν on X so that any other $\frac{1}{2}$-density μ on X can be written as

$$\mu=u\nu$$

where u is a C^∞ function on X.

The critical set C_ϕ is defined by the d independent equations

$$\frac{\partial\phi}{\partial s_i}=0,\qquad i=1,\dots d \tag{8.5}$$

That fact that ϕ is a generating function of Λ asserts that the map

$$\lambda_\phi:C_\phi\to T^*X,\qquad (x,s)\mapsto(x,d\phi_X(x,s)) \tag{8.6}$$

is a diffeomorphism of C_ϕ with Λ. To say that $\mu=u\nu$ belongs to $I^k_0(X,\Lambda)$ means that the function $u(x,\hbar)$ can be expressed as the oscillatory integral

$$u(x,\hbar)=\hbar^{k-\frac{d}{2}}\int a(x,s,\hbar)e^{i\frac{\phi(x,s)}{\hbar}}ds,\quad\text{where}\quad a\in C_0^\infty(X\times S\times\mathbb{R}). \tag{8.7}$$

Proposition 12 *If $a(x,s,0)=0$ on C_ϕ then $\mu \in I^{k+1}(X,\Lambda)$.*

Proof. $a(x,s,0)=0$ on C_ϕ then by the description (8.5) of C_ϕ we see that we can write

$$a=\sum_{j=1}^d a_j(x,s,\hbar)\frac{\partial\phi}{\partial s_j}+a_0(x,s,\hbar)\hbar.$$

We can then write the integral (8.7) as $v+u_0$ where

$$u_0(x,\hbar)=\hbar^{k+1-\frac{d}{2}}\int a_0(x,s,\hbar)e^{i\frac{\phi(x,s)}{\hbar}}ds$$

so

$$\mu_0=u_0\nu\in I^{k+1}(X,\Lambda)$$

and

$$\begin{aligned} v &= \hbar^{k-\frac{d}{2}}\sum_{j=1}^d\int a_h(x,s,\hbar)\frac{\partial\phi}{\partial s_j}e^{i\frac{\phi}{\hbar}}\\ &= -i\hbar^{k+1-\frac{d}{2}}\sum_{j=1}^d\int a_j(x,s,\hbar)\frac{\partial}{\partial s_j}e^{i\frac{\phi}{\hbar}}ds\\ &= i\hbar^{k+1-\frac{d}{2}}\sum_{j=1}^d\int\left(\frac{\partial}{\partial s_j}a_j(x,s,\hbar)\right)e^{i\frac{\phi}{\hbar}}ds \end{aligned}$$

so

$$v=i\hbar^{k+1-\frac{d}{2}}\int b(x,s,\hbar)e^{i\frac{\phi}{\hbar}}ds \qquad \text{where} \qquad b=i\sum_{j=1}^d\frac{\partial a_j}{\partial s_j}. \tag{8.8}$$

This completes the proof of Proposition 12. □

This proof can be applied inductively to conclude the following sharper result:

Proposition 13 *Suppose that for $i=0,\dots,\ell$*

$$\frac{\partial^i a}{(\partial\hbar)^i}(x,s,0)$$

vanishes to order $2(\ell-i)$ on C_ϕ. Then

$$\mu\in I_0^{k+\ell}(X,\Lambda).$$

As a corollary we obtain:

Proposition 14 *If a vanishes to infinite order on C_ϕ then $\mu\in I^\infty(X,\Lambda)$, i.e.*

$$\mu\in\bigcap_k I^k(X,\Lambda).$$

Victor: In the "hints" to your notes of Lect 32 theorem 2 you say that you lose 2 degrees when differentiating. Why don't you lose just one?

We will now use stationary phase to prove the following converse to Proposition 12:

Proposition 15 *If $\mu \in I_0^{k+1}(X,\Lambda)$ then the restriction of $a(x,s,0)$ to C_ϕ vanishes identically.*

Recall the following fact from the formula of stationary phase: Suppose that Y is a manifold with a nowhere vanishing density dy and that $\psi : Y \to \mathbb{R}$ is a a C^∞ function on Y with a single non-degenerate critical point p_0. Suppose that $f \in C_0^\infty(Y)$. The formula of stationary phase (see Chapter **??**) implies that

$$I(\hbar) := \int_Y f(y)e^{i\frac{\psi(y)}{\hbar}}\,dy$$

satisfies

$$I(\hbar) = \hbar^{\frac{\dim Y}{2}}\left(\gamma f(p_0) + O(\hbar)\right)$$

where γ is a non-zero constant. In particular, if we write $m = \dim Y$

$$I(\hbar) = O(\hbar^{\frac{m}{2}+1}) \quad \Leftrightarrow \quad f(p_0) = 0. \tag{8.9}$$

Proof of Propostion 15. As usual, we choose a nowhere vanishing $\frac{1}{2}$-density on X and write $\mu = u\nu$ where

$$u(x,\hbar) = \hbar^{=-\frac{d}{2}}\int a(x,s,\hbar)e^{i\frac{\phi(x,s)}{\hbar}}\,ds$$

where d is the fiber dimension. Let $p_0 = (x_0,s_0) \in C_\phi$ and let

$$(x_0,\xi_0) = \lambda_\phi(p_0) \in \Lambda.$$

Let Γ be a Lagrangian submanifold of T^*X which is horizontal and which intersects Λ transversally at (x_0,ξ_0). We will view Γ as a "point" of T^*X, that is as an element of

$$\mathrm{Morph}(\mathrm{pt.}, T^*X).$$

Since Γ is horizontal, it is defined by a generating function $\chi \in C^\infty(X)$. In other words, $(x,\xi) \in \Gamma$ if and only if $d\chi(x) = \xi$. Let b be any element of $C_0^\infty(X)$ with $b(x_0) \neq 0$. Let

$$v(x) = b(x)e^{-i\frac{\chi(x)}{\hbar}}.$$

This is is the integral kernel of a semi-classical Fourier integral operator

$$F_v \in I^0(\Gamma^\dagger)$$

associated to the canonical relation

$$\Gamma^\dagger \in \mathrm{Morph}(T^*X, \mathrm{pt.}).$$

Since

$$\Gamma^\dagger \pitchfork \Lambda$$

we can compose F_v with $\mu \in I_0^{k+1}(X.\Lambda)$ to get an element

$$\int_X v(x,\hbar)u(x,\hbar)dx \in I^{k+1+\frac{n}{2}}(\mathrm{pt.}).$$

This says that

$$\int_X v(x,\hbar)u(x,\hbar)dx = O(\hbar^{k+1+\frac{n}{2}}).$$

So

$$\hbar^{k-\frac{d}{2}}\int b(x)a(x,s,\hbar)e^{i\frac{-\chi(x)+\phi(x,s)}{\hbar}}dxds = O(\hbar^{k+1+\frac{n}{2}}).$$

So if we set

$$\psi(x,s) = -\chi(x)+\phi(x,s)$$

then

$$\int b(x)a(x,s,0)e^{i\frac{\psi(x,s)}{\hbar}}dxds = O(\hbar^{\frac{d+n}{2}+1}).$$

We want to apply (8.9) with $Y = X\times S$ and $f = ba$. First observe that (x_0,s_0) is a critical point of ψ. Indeed

$$\frac{\partial\psi}{\partial s_i} = \frac{\partial\phi}{\partial s_i} = 0$$

because $(x_0,s_0)\in C_\phi$ and

$$d_X\psi(x_0) = -d\chi(x_0)+d_X\phi(x_0,s_0) = -\xi_0+\xi_0 = 0.$$

We claim that (x_0,s_0) is a non-degenerate critical point of ψ. Indeed, we know that $\psi(x,s) = -\chi(x)+\phi(x,s)$ is a generating function for pt. $=\Gamma^\dagger\circ\Lambda$ with respect to the fibration $X\times S\to$ pt.. The condition for being such a generating function says that the differentials of all the partial derivatives of ψ be linearly independent at (x_0,s_0) which is the same as saying that (x_0,s_0) is a non-degenerate critical point. So $b(x_0)a(x_0,s_0,0)=0$ and since $b(x_0)\neq 0$ we must have $a(x_0,s_0,0)=0$. Since this is true at all points of C_ϕ we conclude that $a(x,s,0)\equiv 0$ on C_ϕ. □

We can now summarize the results of the last few propositions: Given $\mu\in I_0^k(X,\Lambda)$, suppose that we can write $\mu = udx^{\frac{1}{2}}$ where $dx^{\frac{1}{2}}$ is a nowhere vanishing $\frac{1}{2}$-density on X and suppose there is a generating function ϕ for Λ valid over an open set containing the support of μ such that u is of the form

$$u = \hbar^{k-\frac{d}{2}}\int a(x,s,\hbar)e^{i\frac{\phi(x,s)}{\hbar}}ds$$

where $a\in C_0^\infty(X\times S\times\mathbb{R})$. We know from Proposition 12 that the function $a(x,s,0)_{|C_\phi}$ depends only on the equivalence class of μ mod $I_0^{k+1}(X,\Lambda)$ (once ϕ is fixed) and from Proposition 14 that the map

$$\mu\mapsto a(x,s,0)_{|C_\phi}$$

is an isomorphism of $I_0^k(X,\Lambda)/I_0^{k+1}(X,\Lambda)$ with $C_0^\infty(C_\phi)$. Now the map

$$\lambda_\phi : C_\phi \to \Lambda; \qquad (x,s) \mapsto (x, d\phi_X(x,s))$$

is a diffeomorphism. So we have proved

Theorem 30 *Let Λ be a Lagrangian submanifold of $T^*(X)$ and ϕ a generating function for Λ relative to $\pi : X \times S \to X$ and let ν be a nowhere vanishing $\frac{1}{2}$-density on X so that every element of $I_0^k(X,\Lambda)$ has a representation as an oscillatory integral of the form (8.7). For each $\mu \in I_0^k(X,\Lambda)$ define the symbol*

$$\sigma_\phi(\mu) \in C_0^\infty(\Lambda)$$

by

$$\sigma_\phi(\mu)(x,\xi) = a(x,s,0) \textit{where } (x,s) \in C_\phi \quad \textit{and } \lambda_\phi(x,s) = (x,\xi) \tag{8.10}$$

for every $(x,\xi) \in \Lambda$. Then σ_ϕ defines an isomorphism

$$\sigma_\phi : \quad I_0^k(X,\Lambda)/I_0^{k+1}(X,\Lambda) \cong C_0^\infty(\Lambda).$$

The isomorphism σ_ϕ depends on the choice of the generating function ϕ. We shall remedy this by reinterpreting $\sigma_\phi(\mu)$ as a section of an appropriate line bundle. Recall from Sections 5.13 and 5.14 that the generating function ϕ gives a local flat trivialization of the line bundles $\mathbb{L}_{\mathrm{phase}}$ and $\mathbb{L}_{\mathrm{Maslov}}$. We shall show in the next section that if we use these trivializations and our choice of $\frac{1}{2}$-densities to identify $\sigma_\phi(\mu)$ as a section of

$$|T\Lambda|^{\frac{1}{2}} \otimes \mathbb{L}_{\mathrm{phase}} \otimes \mathbb{L}_{\mathrm{Maslov}}$$

then the resulting section is independent of all these choices and we will be able to define an isomorphism of $I_0^k(X,\Lambda)/I_0^{k+1}(X,\Lambda)$ with smooth sections of compact support of $|T\Lambda|^{\frac{1}{2}} \otimes \mathbb{L}_{\mathrm{phase}} \otimes \mathbb{L}_{\mathrm{Maslov}}$.

8.3.2 The global definition of the symbol.

Let $\pi : Z \to X$ be an enhanced fibration. This means that the fibers are equipped with a $\frac{1}{2}$-density and hence that the corresponding canonical relation

$$\Gamma_\pi \in \mathrm{Morph}(T^*Z, T^*X), \quad \Gamma_\pi = H^*(Z)$$

is equipped with a $\frac{1}{2}$-density. Recall that this defines a pushforward map on $\frac{1}{2}$-densities of compact support:

$$\pi_* C_0^\infty(|Z|^{\frac{1}{2}}) \to C_0^\infty(|X|^{\frac{1}{2}}.$$

Let $\upsilon = \upsilon(z,\hbar)$ be a smooth $\frac{1}{2}$-density of compact support on Z depending smoothly on $\hbar$. Then we can rewrite (8.1) as

$$\mu = \hbar^{k-\frac{d}{2}} \pi_* \left(\upsilon e^{i\frac{\phi}{\hbar}} \right). \tag{8.11}$$

By definition, an element of $I_0^k(X,\Lambda)$ is a $\frac{1}{2}$-density on X which can be written as a finite sum of such terms.

Recall that ϕ defines the horizontal Lagrangian submanifold $\Lambda_\phi \subset T^*Z$, and so a diffeomorphism

$$\gamma_\phi : Z \to \Lambda_\phi, \quad z \mapsto (z, d\phi(z))$$

and hence a pushforward isomorphism

$$\gamma_{\phi *}: \ C_0^\infty(|Z|^{\frac{1}{2}}) \to C_0^\infty(|\Lambda_\phi|^{\frac{1}{2}}).$$

By assumption,

$$\Gamma_\pi \overline{\pitchfork} \Lambda_\phi$$

and, locally,

$$\Lambda = \Gamma_\pi(\Lambda_\phi).$$

The enhancement of Γ_π defines a map

$$\Gamma_{\pi *} : C_0^\infty(\Lambda_\phi) \to C_0^\infty(\Lambda).$$

Hence

$$\Gamma_{\pi *} \circ \gamma_{\phi *}: \ C_0^\infty(|Z|^{\frac{1}{2}}) \to C_0^\infty(\Lambda).$$

We now define

$$\sigma_{\phi,\mathrm{new}}(\mu) := (2\pi)^{-\frac{d}{2}} \hbar^k e^{\frac{\pi i}{4}\sigma_\phi} (\Gamma_{\pi *} \circ \gamma_{\phi *}) \left(\upsilon(z,0) e^{i\frac{\phi(z)}{\hbar}} \right) \tag{8.12}$$

where

$$d = \dim Z - \dim X$$

and where σ_ϕ is defined in Section 5.14.

Let us see how this new definition of the symbol is related to the one given in Theorem 30. We begin by being more explicit about the map $\Gamma_{\pi *}$. Let

$$M = T^*Z.$$

The fact that $\Gamma_\pi \overline{\pitchfork} \Lambda_\phi$ says that at every $z \in C_\phi$ we have the exact sequence

$$0 \to T_z(C_\phi) \to T_q(\Lambda_\phi) \oplus T_q(\Gamma_\pi) \to T_q M \to 0 \tag{8.13}$$

where $q = \gamma_\phi(z)$. Since M is a symplectic manifold, it carries a canonical $\frac{1}{2}$-density. The enhancement of Γ_π means that Γ_π is equipped with a $\frac{1}{2}$-density, call it τ. If we are given a C^∞ $\frac{1}{2}$-density ρ on Λ_ϕ, the above exact sequence implies that from the $\frac{1}{2}$-density

$$\rho_q \otimes \tau_q \tag{8.14}$$

we get a $\frac{1}{2}$-density, call it $\rho_z^\sharp$ on $T_z(C_\phi)$. So we get a $\frac{1}{2}$-density $\rho^\sharp$ on C_ϕ. Then

$$\Gamma_{\pi *}\rho = (\lambda_\phi^{-1})^* \rho^\sharp \in C^\infty(\Lambda). \tag{8.15}$$

Fix a nowhere vanishing $\frac{1}{2}$-density τ_Z on Z and write

$$\upsilon(z,\hbar) = a(z,\hbar)\tau_Z, \qquad a \in C_0^\infty(Z \times \mathbb{R}).$$

Define the function $a^\sharp$ on C_ϕ by

$$a^\sharp(z) = a(z,0), \qquad z \in C_\phi$$

and define the function $\phi^\sharp$ on C_ϕ by

$$\phi^\sharp = \phi_{|C_\phi}.$$

Thus we can write the $\sigma_\phi(\mu)$ as given in equation (8.10) as

$$\sigma_\phi(\mu) = (\lambda_\phi^{-1})^* a^\sharp.$$

Let ψ be the function on Λ defined by

$$\psi := (\lambda_\phi^{-1})^* \phi^\sharp. \tag{8.16}$$

Then it follows directly from these definitions that

$$\sigma_{\phi,\mathrm{new}}(\mu) = \sigma_\phi(\mu)\kappa e^{i\left(\frac{\psi}{\hbar} + \frac{\pi}{4}\,\mathrm{sgn}_\phi\right)} \tag{8.17}$$

where

$$\kappa := (2\pi)^{-\frac{d}{2}}\hbar^k \Gamma_{\pi*}(\gamma_{\phi*}\tau_Z) \tag{8.18}$$

does not depend on μ.

From the above discussion it follows that

Proposition 16 *$\sigma_{\phi,\mathrm{new}}$ depends only on Γ_π but not on its enhancement.*

Proof. Indeed, if we replace τ by $f\tau$ where f is a nowhere vanishing function, then $\pi_*\beta$ is replaced by $\pi_*(f\beta)$ for any $\frac{1}{2}$-density β on Z. This means that in the description (8.11) of μ we must replace υ by $f^{-1}\upsilon$. So in (8.14), we replace τ by $f\tau$ and ρ by $f^{-1}\rho$. So these two changes cancel one another in in (8.14) and hence in (8.12). □

Let us now examine the meaning of of the factor

$$e^{i\frac{\psi}{\hbar}}$$

occurring in (8.17). Let α_Λ denote the restriction of the canonical one form $\alpha_X$of T^*X to Λ. We claim that the function ψ on Λ given by (8.16) satisfies

$$d\psi = \alpha_\Lambda \tag{8.19}$$

and so the factor $e^{i\frac{\psi}{\hbar}}$ is a flat section of the line bundle $\mathbb{L}_{\mathrm{phase}}$ as defined in Section 5.13. Since the factor $e^{\frac{\pi i}{4}\,\mathrm{sgn}_\phi}$ is a flat section of $\mathbb{L}_{\mathrm{Maslov}}$ and since κ

as given by (8.18) is a $\frac{1}{2}$-density on Λ, this allows us to interpret $\sigma_{\phi,\mathrm{new}}(\mu)$ as a section of the line bundle

$$|\Lambda|^{\frac{1}{2}} \otimes \mathbb{L}_{\mathrm{phase}} \otimes \mathbb{L}_{\mathrm{Maslov}}.$$

Proof of (8.19). Let α_Z denote the canonical one form of T^*Z so that the canonical one form of $T^*(Z \times X) = T^*Z \times T^*X$ is given by

$$\alpha_{Z\times X} = \mathrm{pr}_1^*\,\alpha_Z + \mathrm{pr}_2^*\,\alpha_X.$$

Recall that

$$(\varsigma_1 \times \mathrm{id})\Gamma_\pi = N^*(\mathrm{graph}\pi$$

is a Lagrangian submanifold. Hence

$$\mathrm{pr}_1^*\,\alpha_Z = \mathrm{pr}_2^*\,\alpha_X$$

on Γ_π. But $d\phi = \gamma_\phi^*\alpha_Z$ by the definition of $\Lambda_\phi = \gamma_\phi(Z)$ and hence the restriction of γ_ϕ to C_ϕ, which is a diffeomorphism of C_ϕ with $\Gamma_\pi \cap \Lambda_\phi$ satisfies

$$(\gamma_{\phi|C_\phi})^*\,\mathrm{pr}_2^*\,\alpha_X = d\phi^\sharp.$$

Applying $(\lambda_\phi^{-1})^*$ proves (8.19). $\square$

Theorem 31 *The definition of the map*

$$\sigma_{\phi,\mathrm{new}} : I_0^k \to C_0^\infty(|\Lambda|^{\frac{1}{2}} \otimes \mathbb{L}_{\mathrm{phase}} \otimes \mathbb{L}_{\mathrm{Maslov}})$$

is independent of the choice of generating function and fibration and hence defines (locally) an isomorphism

$$\sigma : I^k(X,\Lambda)/I^{k+1}(X,\Lambda) \to C^\infty(|\Lambda|^{\frac{1}{2}} \otimes \mathbb{L}_{\mathrm{phase}} \otimes \mathbb{L}_{\mathrm{Maslov}}).$$

Proof. The second assertion follows from what we proved in the preceding section. So we need to prove the first assertion. By Section 5.12, we need to prove independence under two kinds of moves - equivalence and increasing the number of fiber variables.

Invariance under equivalence.

So we have (Z_1,π_1,ϕ_1) and (Z_2,π_2,ϕ_2) and a diffeomorphism

$$g : Z_1 \to Z_2$$

with

$$\pi_1 = \pi_2 \circ g \quad \text{ and } \quad \phi_1 = \phi_2 \circ g.$$

Then g determines a symplectomorphsim

$$\Gamma_g \in \mathrm{Morph}(T^*Z_1, T^*Z_2)$$

with

$$\Gamma_{\pi_1} = \Gamma_{\pi_2} \circ \Gamma_g \quad \text{and} \quad \gamma_{\phi_2} = \Gamma_g \circ \gamma_{\phi_1}.$$

We may choose the nowhere vanishing $\frac{1}{2}$-densities on Z_1 and Z_2 to be consisitent as we did in Section8.1.2 By Proposition 16 we may also choose the enhancements consistently in the sense that

$$(\pi_1)_* = (\pi_2)_* \circ g_*.$$

We also know that the signatures entering into formula (8.17) are the same for ϕ_1 and ϕ_2. Thus (8.17) gives the same answer for ϕ_1 and ϕ_2.

Invariance under increasing the number of fiber variables.

So now

$$Z_2 = Z_1 \times \mathbb{R}^m$$

and

$$\phi_2(z,y) = \phi_1(z_1) + \frac{1}{2}\langle Ay, y\rangle$$

where A is a non-degenerate symmetric matrix and

$$\pi_2 = \pi_1 \circ \pi, \qquad \pi(z_1, y) = z_1.$$

We choose an enhancement r of $\pi_1 : Z_1 \to X$ and then pick the enhancement of $\pi_2 : Z_2 \to X$ to be $r \otimes dy^{\frac{1}{2}}$. This is legitimate by Proposition 16. So if we choose $dy^{\frac{1}{2}}$ to be the enhancement of π we have

$$\pi_{2*} = \pi_{1*} \circ \pi_* \tag{8.20}$$

as maps from $C_0^\infty(|Z_2|^{\frac{1}{2}}) \to C_0^\infty(|X|^{\frac{1}{2}})$ and

$$\Gamma_{\pi_2 *} = \Gamma_{\pi_1 *} \circ \Gamma_{\pi *} \tag{8.21}$$

as maps from $\frac{1}{2}$-densities on Λ_{ϕ_2} to $\frac{1}{2}$-densities on Λ.

Let us also choose a nowhere vanishing τ_{Z_1} on Z_1 and choose the nowhere vanishing $\frac{1}{2}$-density on Z_2 to be

$$\tau_{Z_2} = \tau_{Z_1} \otimes dy^{\frac{1}{2}}.$$

Let us now rewrite the definitions (8.1), (8.10) and (8.12) in terms of a general fibration $\pi : Z \to X$ and generating function ϕ as follows: First consider the manifold Z relative to the trivial fibration over itself, and the Lagrangian submanifold $\Lambda_\phi \subset T^*Z$ given by the the function ϕ so that $\Lambda_\phi = \gamma_\phi(Z)$. Let$\tau_Z$ be a nowhere vanishing $\frac{1}{2}$-density on Z. Definition (8.1) (relative to the trivial fibration of Z over itself) says that $I_0^{k-\frac{d}{2}}(Z, \Lambda_\phi)$ consists of all $\frac{1}{2}$ densities on Z of the form

$$\upsilon = \hbar^{k-\frac{d}{2}} a(z,\hbar)\tau_Z e^{i\frac{\phi(z)}{\hbar}}.$$

We may write

$$\upsilon = \upsilon_0 + O(\hbar^{k-\frac{d}{2}+1})$$

where

$$\upsilon_0 = \hbar^{k-\frac{d}{2}} a(z,0)\tau_Z e^{i\frac{\phi(z)}{\hbar}}.$$

The definition of the symbol for this trivial fibration then says that

$$\sigma_{\mathrm{new}}(\upsilon) = \gamma_{\phi_*}\upsilon.$$

If we set

$$\sigma_{\Lambda_\phi} := \gamma_{\phi*}\sigma_Z$$

and use the above representation of υ then

$$\sigma_{\mathrm{new}}(\upsilon) = \gamma_{\phi_*}\upsilon_0 = \hbar^{k-\frac{d}{2}}(\gamma_\phi^{-1})^*\left(a(z,0)e^{i\frac{\phi}{\hbar}}\right)\sigma_{\Lambda_\phi}. \tag{8.22}$$

Now (8.1) says that a general element of $I^k(X,\Lambda)$ can be written locally as

$$\mu = \pi_*\upsilon, \qquad \upsilon \in I^{k-\frac{d}{2}}(Z,\Lambda_\phi)$$

and then (8.12) says that

$$\sigma_{\phi,\mathrm{new}} = e^{i\frac{\pi}{4}\sigma_\phi}\left(\frac{\hbar}{2\pi}\right)^{\frac{d}{2}}\Gamma_{\pi*}\sigma(\upsilon). \tag{8.23}$$

Back to the proof of the theorem: Let $\upsilon_2 \in I_0^{k-\frac{d_2}{2}}(Z_2,\Lambda_{\phi_2})$ and

$$\mu = \pi_{2*}\upsilon_2.$$

Let

$$\upsilon_1 := \pi_*\upsilon_2$$

so that by (8.20) and (8.21)

$$\mu = \pi_{1*}\upsilon_1 = \pi_{1*}(\pi_*\upsilon_2)$$

and

$$\Gamma_{\pi_2*}\sigma(\upsilon_2) = \Gamma_{\pi_1*}\left(\Gamma_{\pi*}\sigma(\upsilon_2)\right).$$

So to prove that the two definitions of $\sigma_{\mathrm{new}}(\mu)$ coincide, it is enough to show that the two definitions of $\sigma(\upsilon_1)$ - the one associated with the trivial fibration of Z_1 over itself and the generating function ϕ_1, and the one associated the the fibration $\pi : Z_2 \to Z_1$ and ϕ_2 - coincide.

Write

$$\upsilon_2 = \hbar^{k-\frac{d_2}{2}} a(z_1,y,\hbar)e^{i\frac{\phi_2(z,y)}{\hbar}}\tau_{Z_2}$$

so that

$$\upsilon_1 = \hbar^{k-\frac{d_2}{2}}\left(\int a(z_1,y,\hbar)e^{i\frac{\langle Ay,y\rangle}{2\hbar}}dy\right)e^{i\frac{\phi_1}{\hbar}}\tau_{Z_1}.$$

By stationary phase, this last expression is of the form

$$\frac{\hbar^{k-\frac{d_1}{2}}}{(2\pi)^{\frac{m}{2}}}\,|\det A|^{-\frac{1}{2}}\,a(z_1,0,0)e^{i\frac{\pi}{4}\operatorname{sgn}A}e^{i\frac{\phi_1}{\hbar}}\tau_{Z_1}+O(\hbar^{k-\frac{d_1}{2}+1}).$$

Hence $\sigma_{\text{new}}(\upsilon_1)$ computed for the trivial fibration according to (8.22) is

$$\frac{\hbar^{k-\frac{d_1}{2}}}{(2\pi)^{\frac{m}{2}}}\,|\det A|^{-\frac{1}{2}}\,(\gamma_{\phi_1}^{-1})^*\left(a(z_1,0,0)e^{i\frac{\phi_1}{\hbar}}\right)e^{i\frac{\pi}{4}\operatorname{sgn}A}\gamma_{\phi_1 *}\tau_{Z_1}. \tag{8.24}$$

We now do the computation of the symbol via the pushforward by $\Gamma_{\pi *}$ of a $\frac{1}{2}$-density on Λ_{ϕ_2}. The $\frac{1}{2}$-density in question is

$$\sigma(\upsilon_2)=\hbar^{k-\frac{d_2}{2}}(\gamma_{\phi_2}^{-1})^*\left(a(z_1,y,0)e^{i\frac{\phi_2}{\hbar}}\right)\gamma_{\phi_2 *}(\tau_{Z_1}\otimes dy^{\frac{1}{2}}).$$

We apply (8.23) to the fibration $\pi: Z_2\to Z_1$ which says that we must use the preceding expression for $\sigma(\upsilon_2)$ in

$$\left(\frac{\hbar}{2\pi}\right)^{\frac{m}{2}}e^{\frac{\pi i}{4}\operatorname{sgn}}\Gamma_{\pi *}\sigma(\upsilon_2).$$

where sgn is the signature of the fibration π and the function

$$Q:\ y\mapsto\frac{1}{2}\langle Ay,y\rangle$$

on the fibers. This signature is just $\operatorname{sgn}A$. So we get for our second computation:

$$\frac{\hbar^{k-\frac{d_1}{2}}}{(2\pi)^{\frac{m}{2}}}e^{\frac{\pi i}{4}\operatorname{sgn}A}\Gamma_{\pi *}\left[(\gamma_{\phi_2}^{-1})^*\left(a(z_1,y,0)e^{i\frac{\phi_2}{\hbar}}\right)\gamma_{\phi_2 *}(\tau_{Z_1}\otimes dy^{\frac{1}{2}})\right].$$

The critical set C_{ϕ_2} for the fibration π is the set $y=0$. Identifying this set with Z_1, we see that the map

$$\lambda_{\phi_2,\pi}:C_{\phi_2}\to\Lambda_{\phi_1}$$

is just the map

$$\gamma_{\phi_1}:\ Z_1\to\Lambda_{\phi_1}$$

so our second computation becomes

$$\frac{\hbar^{k-\frac{d_1}{2}}}{(2\pi)^{\frac{m}{2}}}e^{\frac{\pi i}{4}\operatorname{sgn}A}(\gamma_{\phi_1}^{-1})^*\left(a(z_1,0,0)e^{i\frac{\phi_1}{\hbar}}\right)\Gamma_{\pi *}\left(\gamma_{\phi_2 *}(\tau_{Z_1}\otimes dy^{\frac{1}{2}})\right).$$

If we compare this with (8.24) we see that the proof of the theorem hinges on showing that

$$\Gamma_{\pi *}\left(\gamma_{\phi_2 *}(\tau_{Z_1}\otimes dy^{\frac{1}{2}})\right)=|\det A|^{-\frac{1}{2}}\gamma_{\phi_1 *}\tau_{Z_1}. \tag{8.25}$$

Now $Z_2 = Z_1 \times \mathbb{R}^m$ and the map γ_{ϕ_2} factors as

$$\gamma_{\phi_2} = \gamma_{\phi_1} \times \gamma_Q$$

where

$$\gamma_{\phi_1} : Z_1 \to \Lambda_{\phi_1}$$

and

$$\gamma_Q : \mathbb{R}^m \to \Lambda_Q. \quad \gamma_Q(y) = (y, \eta), \quad \eta = Ay.$$

Similarly, the map π factors as

$$\pi = \mathrm{id} \times \wp$$

where

$$\wp : \mathbb{R}^m \to \{0\}.$$

So the theorem amount to showing that

$$\Gamma_{\wp *}\left(\gamma_{Q*}|dy|^{\frac{1}{2}}\right) = |\det A|^{-\frac{1}{2}}.$$

For this let us go back to the exact secquence (8.13) where now $\phi = Q$ so $C_\phi = \{0\}$ is a point. Here $\frac{1}{2}$-density $\gamma_{Q*}|dy|^{\frac{1}{2}}$ assigns the value one to the basis

$$\left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_m}, A\frac{\partial}{\partial y_1}, \dots, A\frac{\partial}{\partial y_m}\right)$$

of $T_0(\Lambda_Q)$. The Lagrangian submanifold Γ_π consists of the zero section of $T^*(\mathbb{R}^m$ and the enhancement by $dy|^{\frac{1}{2}}$ of Γ_π assigns the value one to the basis

$$\left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_m}, 0, \dots, 0\right)$$

of $T_0\Gamma_\pi$.

So the tensor product (8.14) assigns the value one to the basis of $T_0(T^*\mathbb{R}^m))$ obtained by combining these two bases. But the symplectic $\frac{1}{2}$-density assigns the value $|\det A|^{\frac{1}{2}}$ to this combined basis. This proves that $\Gamma_{\wp *}\left(\gamma_{Q*}|dy|^{\frac{1}{2}}\right) = |\det A|^{-\frac{1}{2}}$. □

Whew!

The general definition of the symbol.

Let Λ be an arbitrary Lagrangian submanifold of T^*X. We can cover Λ by open sets U_i each described by a generating function ϕ_i relative to a fibration fibration $\pi_i : Z_i \to U_i$. By definition, if $\mu \in I_0^k(X, \Lambda)$, we can write μ as a finite sum

$$\mu = \sum_{i=1}^{N} \mu_i, \quad \text{with} \quad \mu_i = \pi_{i*}\upsilon_i, \quad \upsilon_i \in I_0^{k-\frac{d_i}{2}}(Z_i, \Lambda_{\phi_i}$$

where d_i is the fiber dimension of $Z_i \to X_i$. Let

$$\mathbb{L} = \mathbb{L}_\Lambda := \mathbb{L}_{\text{phase}} \otimes \mathbb{L}_{\text{Maslov}}. \tag{8.26}$$

Define

$$\sigma(\mu) \in C_0^\infty(|\Lambda|^{\frac{1}{2}} \otimes \mathbb{L}$$

by

$$\sigma(\mu) := \sum_{i=1}^{N} \sigma(\mu_i). \tag{8.27}$$

From Theorems 30 and 31 we conclude

Theorem 32 *$\sigma(\mu)$ is well defined and independent of the choices that went into (8.27). The map*

$$\sigma : \quad \mu \mapsto \sigma(\mu)$$

induces a bijection

$$I^k(X,\Lambda)/I^{k+1}(X,\Lambda) \cong C^\infty(|\Lambda|^{\frac{1}{2}} \otimes \mathbb{L}_\Lambda).$$

8.4 Symbols of semi-classical Fourier integral operators.

Let X_1 and X_2 be manifolds, and

$$\Gamma \in \text{Morph}(T^*X_1, T^*X_2)$$

be a canonical relation. Let

$$\Lambda = (\varsigma_1 \times \text{id})(\Gamma)$$

where $\varsigma(x_1,\xi_1) = (x_1, -\xi_1)$ so that Λ is a Lagrangian submanifold of $T^*(X_1 \times X_2)$. We have associated to Γ the space of compactly supported semi-classical Fourier integral operators

$$\mathcal{F}_0^k(\Gamma)$$

where $F \in \mathcal{F}_0^k(\Gamma)$ is an integral operator with kernel

$$\mu \in I_0^k(X_1 \times X_2, \Lambda).$$

We define the symbol of F to be

$$\sigma(F) = (\varsigma_1 \times \text{id})_* \sigma(\mu)$$

so that

$$\sigma(F) \in C_0^\infty(|\Gamma|^{\frac{1}{2}} \otimes \mathbb{L}_\Gamma)$$

where

$$(\mathbb{L}_\Gamma)_{(x_1,\xi_1,x_2,\xi_2)} = (\mathbb{L}_\Lambda)_{(x_1,-\xi_1,x_2,\xi_2)}.$$

By Theorem 32 we have an isomorphism

$$\mathcal{F}_0^k(\Gamma)/\mathcal{F}_0^{k+1}(\Gamma) \cong C_0^\infty(|\Gamma|^{\frac{1}{2}} \otimes \mathbb{L}_\Gamma). \tag{8.28}$$

Suppose that X_1, X_2 and X_3 are manifolds and that

$$\Gamma_1 \in \mathrm{Morph}(T^*X_1, T^*X_2) \quad \text{and } \Gamma_2 \in \mathrm{Morph}(T^*X_2, T^*X_3)$$

are transversally composible. Let $n = \dim X_2$ and

$$F_i \in \mathcal{F}_0^{k_i}(\Gamma), \quad i = 1, 2.$$

By Theorem 29 we know that

$$F_2 \circ F_1 \in \mathcal{F}_0^{m_1+m_2+\frac{n}{2}}(\Gamma_2 \circ \Gamma_1).$$

So if

$$\sigma_i := \sigma(F_i)$$

we may define

$$\sigma_2 \circ \sigma_1 := \sigma(F_2 \circ F_1).$$

By (8.28) we know that this is well defined and hence gives us a composition law

$$C_0^\infty(|\Gamma_1|^{\frac{1}{2}} \otimes \mathbb{L}_{\Gamma_1}) \times C_0^\infty(|\Gamma_2|^{\frac{1}{2}} \otimes \mathbb{L}_{\Gamma_2}) \to C_0^\infty(|\Gamma_2 \circ \Gamma_1|^{\frac{1}{2}} \otimes \mathbb{L}_{\Gamma_2 \circ \Gamma_1}).$$

This modifies our composition formula for $\frac{1}{2}$-densities in the enhanced symplectic category in that it takes the line bundle $\mathbb{L}_\Gamma$ into account.

8.5 Differential operators on oscillatory $\frac{1}{2}$-densities.

Let

$$P : \quad C^\infty(|X|^{\frac{1}{2}}) \to C^\infty(|X|^{\frac{1}{2}})$$

be an m-th order differential operator as discussed in Section 1.3.7. Let $\sigma(P)$ denote the principal symbol of P as discussed there. In particular, $\sigma(P)$ is a function on T^*X.

Let Λ be a Lagrangian submanifold of T^*X, let $\mu \in I^k(X, \Lambda)$ and let $\sigma(\mu)$ denote the symbol of μ as defined in Theorem 32.

Theorem 33 *If $\mu \in I^k(X, \Lambda)$ then*

$$P\mu \in I^{k-m}(X, \Lambda)$$

and

$$\sigma(P\mu) = \hbar^{-m}\sigma(P)_{|\Lambda}\sigma(\mu). \tag{8.29}$$

Proof. Let (x_0, ξ_0) be a point of Λ and let ϕ be a generating function for Λ near $(x_0, \xi_0$ relative to a fibration $\pi : Z \to X$. Then μ has the form (8.2) relative to (Z, π, ϕ) and the choices made in Section 8.1.1. We may differentiate under the integral sign and it is clear that applying D^α to $e^{i\frac{\phi}{\hbar}}$ will have a term $(d_X\phi)^\alpha \cdot \hbar^{-|\alpha|}$ with all other terms being of higher order in $\hbar$. This proves the first statement in the theorem. Equation (8.29) then follows from the local expression (8.10) for the symbol. □

We can be more explicit near points $(x_0, \xi_0) \in \Lambda$ where $\xi_0 \neq 0$. According to the result that we proved in Section 5.8, we can find a coordinate patch $(U, x_1, \ldots, x_n)$ about x_0 such that with the property that near (x_0, ξ_0), Λ can be described by a generating function

$$\phi(x, \xi) = x \cdot \xi - \rho(\xi), \quad \rho \in C^\infty(\mathbb{R}^n)$$

relative to the fibration

$$U \times \mathbb{R}^n \to U.$$

See equation (5.11) of Section 5.8.

So near (x_0, ξ_0)

$$\Lambda = \{(x, \xi) | x = \frac{\partial \rho}{\partial \xi}\}$$

and $\mu|U$ is of the form

$$\left(\hbar^{k-\frac{n}{2}} \int b(x, \xi, \hbar) e^{i\frac{\phi}{\hbar}} d\xi\right) dx^{\frac{1}{2}} \tag{8.30}$$

where $b \in C^\infty$ is supported on a set $|\xi| \leq N$. By Proposition 12 we may replace $b(x, \xi, \hbar)$ by $b(\frac{\partial \rho}{\partial \xi}, \xi, \hbar)$ up to adding a term in $I^{k+1}(X, \Lambda)$. So mod $I^{k+1}(X, \Lambda)$ we may write μ as

$$\mu = \left(\hbar^{k-\frac{n}{2}} \int b_0(\xi, \hbar) e^{i\frac{\phi}{\hbar}} d\xi\right) dx^{\frac{1}{2}} \tag{8.31}$$

where

$$b_0(\xi, \hbar) = b(\frac{\partial \rho}{\partial \xi}, \xi, \hbar).$$

Since we have chosen the nowhere vanishing $\frac{1}{2}$-density $dx^{\frac{1}{2}}$, we can regard P as a differential operator on functions, and hence by (8.31)

$$\begin{aligned} P\mu &= \left(\hbar^{k-\frac{n}{2}} \int P(x, D) e^{i\frac{x\cdot\xi}{\hbar}} b_0(\xi, \hbar) e^{-i\frac{\rho(\xi)}{\hbar}} d\xi\right) dx^{\frac{1}{2}} \\ &= \left(\hbar^{k-\frac{n}{2}} \int P(x, \xi) b_0(\xi, \hbar) e^{i\frac{\phi}{\hbar}} d\xi\right) dx^{\frac{1}{2}} \end{aligned}$$

where $P(x, \xi)$ is the total symbol of P as defined in Section 1.3.2. So

$$P\mu = \left(\sum_{\ell=1}^{m} \hbar^{k-\ell-\frac{n}{2}} \int p_\ell(x, \xi) b_0(\xi, \hbar) e^{i\frac{\phi}{\hbar}} d\xi\right) dx^{\frac{1}{2}}. \tag{8.32}$$

This proves that $P\mu \in I^{k-m}(X, \Lambda)$ and gives (8.29).

8.6 The transport equations redux.

Let us write H for the principal symbol of P as in Section 1.2.1 and let us assume that

$$H \equiv 0 \quad \text{on } \Lambda$$

as in Sections 1.2.10 and 1.3. Then by (8.29) and Theorem 32, we know that $P\mu \in I^{k-m+1}(X,\Lambda)$. the first main result of this section will be to compute the symbol of $P\mu$ considered as an element of $I^{k-m+1}(X,\Lambda)$. See formula (8.36) below. To prove (8.36) it is enough to prove it on an open dense subset of Λ since the symbol of $P\mu$ (as an element of $I^{k-m+1}(X,\Lambda)$) is a smooth $\frac{1}{2}$-density on Λ. We will assume in this section that Λ has the property that the set of points $(x,\xi) \in \Lambda,\ \xi \neq 0$ is dense in Λ. So it is enough to prove (8.36) at points (x,ξ) where $\xi \neq 0$ near which Λ has a generating function of the form $\phi(x,\xi) = x\cdot\xi - \rho(\xi), \quad \rho \in C^\infty(\mathbb{R}^n)$ as in the preceding section.

Since Λ is defined by the equations

$$x_i = \frac{\partial\rho}{\partial\xi_i}, \qquad i =, \ldots, n,$$

the fact that $H = p_m$ vanishes identically on Λ implies that

$$H = \sum_{i=1}^{n} q_k(x,\xi)\left(x_i - \frac{\partial\rho}{\partial\xi_i}\right). \tag{8.33}$$

Thus the highest order term in the multiple of $dx^{\frac{1}{2}}$ in (8.32) can be written as

$$\begin{aligned} \hbar^{k-\frac{n}{2}-m}\int p_m(x,\xi)b_0(\xi,\hbar)e^{i\frac{\phi}{\hbar}}d\xi &= \hbar^{k-\frac{n}{2}-m}\sum_j\int q_j(x,\xi)b_0(x,\xi)\frac{\partial\phi}{\partial\xi_j}e^{i\frac{\phi}{\hbar}}d\xi \\ &= \hbar^{k-\frac{n}{2}-m}\sum_j\int q_jb_0\frac{\hbar}{i}\frac{\partial}{\partial\xi_j}\left(e^{i\frac{\phi}{\hbar}}\right)d\xi \\ &= \hbar^{k-\frac{n}{2}-m+1}\int i\sum_j\frac{\partial}{\partial\xi_j}(q_jb_0)\,e^{i\frac{\phi}{\hbar}}d\xi. \end{aligned}$$

So, by (8.32), we may write

$$P\mu = \hbar^{k-\frac{n}{2}-m+1}\left(\int a(\xi,\hbar)e^{i\frac{\phi}{\hbar}}d\xi\right)dx^{\frac{1}{2}} \mod I^{k-m+2}(X,\Lambda)$$

where

$$a = \iota^*\left(i\sum_j\frac{\partial}{\partial\xi_j}(q_jb_0) + p_{m-1}b_0\right)$$

and ι denotes the inclusion

$$\iota : \mathbb{R}^n \to U \times \mathbb{R}^n, \qquad \xi \mapsto \left(\frac{\partial\rho}{\partial\xi}, \xi\right).$$

We decompose a into two terms

$$a = a_I + a_{II}$$

where

$$a_I := \iota^* \left(i \sum_j q_j \frac{\partial}{\partial \xi_j} b_0 \right)$$

and

$$a_{II} := \iota^* \left(\left(p_{m-1} + i \sum_j \frac{\partial}{\partial \xi_j} q_j \right) b_0 \right)$$

and will give a geometric interpretation to each of these terms.

We begin with a_I. Since H has the form (8.33),

$$\iota^* \frac{\partial H}{\partial x_j} = q_j \left(\frac{\partial \rho}{\partial \xi}, \xi \right).$$

Let π denote the diffeomorphism

$$\pi : \Lambda \to \mathbb{R}^n, \qquad (x, \xi) \mapsto \xi.$$

Since

$$v_H = \sum_i \left(\frac{\partial H}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial H}{\partial x_j} \frac{\partial}{\partial \xi_j} \right)$$

is tangent to Λ, we see that the diffeomorphism π maps the restriction of v_H to Λ to

$$\tilde{v} := - \sum_j q_j \left(\frac{\partial \rho}{\partial \xi}, \xi \right) \frac{\partial}{\partial \xi_j}$$

and so

$$a_I = \frac{1}{i} D_{v_H|\Lambda} \pi^* b_0. \tag{8.34}$$

We now turn to a_{II}. Let ν be the $\frac{1}{2}$-density on Λ given by

$$\nu := \pi^* d\xi^{\frac{1}{2}}.$$

Then

$$\begin{aligned}
D_{v_H|\Lambda}\nu &= \pi^* \left(D_{\tilde{v}} d\xi^{\frac{1}{2}} \right) \\
&= \frac{1}{2} \pi^* \left(\text{div } (\tilde{v}) d\xi^{\frac{1}{2}} \right) \\
&= \frac{1}{2} \pi^* \left(\text{div } (\tilde{v}) \right) \nu.\text{and} \\
(\text{div } (\tilde{v}) &= \sum_j \left(- \frac{\partial}{\partial \xi_j} \left(q_j (\frac{\partial \rho}{\partial \xi}, \xi \right) \right) \\
&= \iota^* \left(- \sum_j \frac{\partial q_j}{\partial \xi_j}(x, \xi) - \sum_{j,\ell} \frac{\partial q_j}{\partial x_\ell}(x, \xi) \frac{\partial^2 \rho}{\partial \xi_j \partial \xi_\ell} \right).
\end{aligned}$$

So

$$D_{v_H|\Lambda}\nu = \frac{1}{2}\iota^*\left(-\sum_j \frac{\partial q_j}{\partial \xi_j}(x,\xi) - \sum_{j,\ell}\frac{\partial q_j}{\partial x_\ell}(x,\xi)\frac{\partial^2 \rho}{\partial \xi_j \partial \xi_\ell}\right)\nu. \tag{8.35}$$

On the other hand from the formula (8.33) for $H = p_m$ we have

$$\iota^*\sum_\ell \frac{\partial}{\partial x_\ell}\frac{\partial}{\partial \xi_\ell}p_m = \iota^*\left(-\sum_{j,\ell}\frac{\partial q_j}{\partial x_\ell}\frac{\partial^2\rho}{\partial \xi_\ell \partial \xi_j} + \sum_j \frac{\partial q_j}{\partial \xi_j}\right).$$

Multiplying this by $\frac{1}{2}$ and comparing with (8.35) and recalling the formula (1.19) for the sub-principal symbol, we see that

$$a_{II} = \left(\iota^*\sigma_{sub}(P) + \frac{1}{i}\frac{D_{v_H|\lambda}\nu}{\nu}\right)b_0.$$

Hence the symbol of $P\mu$ is given as

$$\sigma(P\mu) = \hbar^{-(m-1)}\left(\frac{1}{i}D_{v_h|\Lambda} + \sigma_{sub}\right)\sigma(\mu). \tag{8.36}$$

We can now go back to the iterative procedure of Chapter 1 for the semi-classical solution of hyperbolic partial differential equations. The first step is to find a Λ on which $H \equiv 0$. The next step is to solve the transport equation

$$\left(\frac{1}{i}D_{v_h|\Lambda} + \sigma_{sub}\right)\sigma(\mu) = 0. \tag{8.37}$$

Along an integral curve $\gamma(t)$ of v_H on λ this reduces to a first order linear ordinary differential equation of the form

$$\frac{d}{dt}\sigma(\mu)(\gamma(t)) + \sigma_{sub}(\gamma(t))\sigma(mu)(\gamma(t)) = 0$$

with given initial conditions.

Assuming that the integral curves of v_H lying of Λ are well behaved in the sense that they are defined for all t and that there are no periodic or recurrent trajectories, the solution of (8.37) is reduced to the solution of a system of first order linear differential equations.

If we solved (8.37), then we know from Theorem 32 that

$$P\mu \in I^{k-m+2}(X,\Lambda).$$

Let $\sigma_{k+m-2}(\mu)$ now denote the symbol of $P\mu$ considered as an element of $I^{k-m+2}(X,\Lambda)$. We look for a $\nu \in I^{k-1}(X,\Lambda)$ such that

$$P(\mu+\nu) \in I^{k-m+3}(X,\Lambda).$$

For this to be the case $\sigma(\nu)$ must satisfy the inhomogeneous transport equation

$$\hbar^{-m-1}\left(\frac{1}{i}D_{v_h|\Lambda}+\sigma_{sub}\right)\sigma(\nu)=-\sigma_{k+m-2}(\mu). \tag{8.38}$$

This reduces to a system of first order inhomogeneous linear differential equations.

We can now proceed recursively to find $\frac{1}{2}$-densities in $I^k(X,\Lambda)$ such that

$$P\mu\in I^N(X,\Lambda)$$

for arbitrarily large N.

This completes the program outlined in Chapter 1.

8.7 Semi-classical pseudo-differential operators.

These are a special case of the semi-classical Fourier integral operators described in Section 8.2 specialized to the case

$$X_1=X_2=X$$

and

$$\Gamma=\mathrm{id}\in\mathrm{Morph}(T^*X,T^*X)$$

so

$$(\varsigma\times\mathrm{id})(\Gamma)=N^*(\Delta)$$

where

$$\Delta\subset X\times X$$

So we definitely need to generalize Theorem 29 so as to include clean composition.

is the diagonal. Clearly Γ is composable with itself so $\mathcal{F}_0(\Gamma)$ is an algebra. If $F_1\in\mathcal{F}^{k_1}(\Gamma)$ and $F_2\in\mathcal{F}^{k_2}(\Gamma)$ and either F_1 or F_2 is in $\mathcal{F}_0(\Gamma)$ then their composition is defined and

$$F_2\circ F_1\in\mathcal{F}^{k_1+k_2+\frac{n}{2}}(\Gamma)$$

where

$$n=\dim X.$$

In order to avoid the nuisance of accumulating the $\frac{n}{2}$-s we define

$$\Psi^k(X):=\mathcal{F}^{k-\frac{n}{2}}(\Gamma),\qquad \Psi_0^k(X):=\mathcal{F}_0^{k-\frac{n}{2}}(\Gamma). \tag{8.39}$$

Thus if $A_1\in\Psi^{k_1}(X)$ and $A_2\in\Psi^{k_2}(X)$ and one or the other is in $\Psi_0(X)$ then

$$A_2\circ A_1\in\Psi^{k_1+k_2}(X)$$

We call $\Psi_0(X)$ the algebra of **compactly supported semi-classical pseudo-differential operators** on X.

We will now examine the local expression for the composition law in this algebra. So we assume that X is an open convex subset of $\mathbb{R}^n$ and that we have chosen the standard $\frac{1}{2}$ density $dx^{\frac{1}{2}}$ on X. A generating function for $N^*\Delta$ is given as follows: Let

$$\pi : X \times X \times \mathbb{R}^n \to X \times X, \quad (x, y, \xi) \mapsto (x, y).$$

Then according to (5.4) (with a slightly more compact notation)

$$\phi : X \times X \times \mathbb{R}^n \to \mathbb{R}, \quad \phi(x, y, \xi) = (x - y) \cdot \xi$$

is a generating function for $N^*\Delta$.

Dropping the ubiquitous factors of $dx^{\frac{1}{2}}$ we can write $A \in \Psi_0^k(X)$ as being given by the integral kernel

$$A(x_1, x_2, \hbar) = \hbar^{k-\frac{n}{2}} \int a(x_1, x_2, \xi, \hbar) e^{i\frac{(x_1-x_2)\cdot\xi}{\hbar}} d\xi$$

where

$$a \in C_0^\infty(X \times X \times \mathbb{R}^n \times \mathbb{R}).$$

Then

$$(A_1 \circ A_2)(x_1, x_2) = \int A_1(x, y, \hbar) A_2(y, x_2) dy$$

so

$$(A_1 \circ A_2)(x_1, x_2) = \hbar^\ell \int a_1(x_1, y, \xi_1, \hbar) a_2(y, x_2, \xi_2, \hbar) e^{i\frac{\xi_1\cdot(x_1-y)+\xi_2\cdot(y-x_2)}{\hbar}} d\xi_1 d\xi_2 dy \tag{8.40}$$

where

$$\ell = k_1 + k_2 - n.$$

Our task is to disentangle this formula.

8.7.1 The right handed symbol calculus of Kohn and Nirenberg.

Make the changes of coordinates

$$\xi = \xi_1, \ \eta = \xi_1 - \xi_2, \ z = y - x_2$$

so

$$\xi_1 = \xi, \ \xi_2 = \xi - \eta, \ y = z + x_2$$

in (8.40). Thus

$$\xi_1 \cdot (x_1 - y) + \xi_2 \cdot (y - x_2) = \xi \cdot (x_1 - x_2) - \eta \cdot z$$

is the phase function in the new coordinates.

The amplitude in the new coordinates is a_R where

$$a_R(x_1, x_2, \xi, z, \eta, \hbar) := a_1(x_1, z + x_2, \xi, \hbar) a_2(z + x_2, x_2, \xi - \eta, \hbar). \tag{8.41}$$

Thus the right hand side of (8.40) is equal to

$$\int e^{i\frac{\xi\cdot(x_1-x_2)}{\hbar}} \left(\int a_R(x_1, x_2, \xi, z, \eta, \hbar) e^{-i\frac{\eta\cdot z}{\hbar}} d\eta dz \right) d\xi. \tag{8.42}$$

We are now going to apply stationary phase to the integral with respect to z and η occurring in (8.42) for fixed $x_1, x-2, \xi$. This integral is of the form

$$I(\hbar) = \int f(w) e^{i\frac{\langle Aw, w\rangle}{2\hbar}} dw.$$

In the case at hand

$$w = \begin{pmatrix} z \\ \eta \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}.$$

The general stationary phase prescription says that an integral of the above form has the asymptotic expansion

$$I(\hbar) \sim \left(\frac{\hbar}{2\pi}\right)^{\frac{d}{2}} \gamma_A \sum \exp\left(-\frac{i\hbar}{2} b(D)\right) f(0)$$

where

$$b(D) = \sum_{ij} b_{ij} D_i D_j. \qquad (b_{ij}) = B = A^{-1},$$

$$\gamma_A = |\det A|^{-\frac{1}{2}}\, e^{\frac{\pi i}{4}\operatorname{sgn} A}$$

and d is the dimension of the space over which we are integrating. In the case at hand

$$B = A$$

and

$$\operatorname{sgn} A = 0$$

so

$$\gamma_A = 1.$$

Also $d = 2n$. Let us denote the result of applying this stationary phase formula to the a_R of (8.41) by

$$a_1 \star_R a_2.$$

Then we have the formula

$$a_1 \star_R a_2 = \left(\frac{\hbar}{2\pi}\right)^n \sum_k \left(\frac{i\hbar}{2}\right)^k \frac{1}{k!} \left(D_z D_\eta\right)^k a_R \Big|_{z=\eta=0}. \tag{8.43}$$

Example. Suppose we take $a_1 = a$ and

$$a_2 = a_2(\xi) = \left(\frac{2\pi}{\hbar}\right)^n \rho(\xi)$$

where

$$\rho \in C_0^\infty(\mathbb{R}^n)$$

and $\rho \equiv 1$ on $\mathrm{supp}(a)$.

Then

$$a_R = \left(\frac{2\pi}{\hbar}\right)^n a(x_1, z + x_2, \xi, \hbar)\rho(\xi - \eta)$$

so (8.43) gives

$$a_1 \star_R a_2 = \sum_k \left(\frac{i\hbar}{2}\right)^k \frac{1}{k!} (D_z D_\eta)^k a_1(x_1, z + x_2, \xi, \hbar)\rho(\xi - \eta)\bigg|_{z=\eta=0}.$$

But since $\rho \equiv 1$ on a neighborhod of supp a, all terms except the first vanish. Hence

$$a \star_R \left(\left(\frac{2\pi}{\hbar}\right)^n \rho\right) = a.$$

The element

$$\left(\frac{2\pi}{\hbar}\right)^n \rho$$

acts as a right identity on all a whose support is contained in the set where $\rho \equiv 1$.

Remark. If at the beginning of this section we had made the change of variables

$$\xi = \xi_2,\ \eta = \xi_2 - \xi_1,\ z = y - x_1$$

we would obtain an alternative symbol calculus, the "left handed calculus". The same argument will then show that $\left(\frac{2\pi}{\hbar}\right)^n \rho$ is a left identity on all a whose support is contained in the set where $\rho \equiv 1$.

8.8 $I(X,\Lambda)$ as a module over $\Psi_0(X)$.

Let X be a manifold and Λ a Lagrangian submanifold of T^*X. Since semi-classical pseudo-differential operators are special kinds of semi-classical Fourier integral operators - ones associated with the identity morphism of T^*X - we may apply the results of Section 8.2 to conclude that $I_0(X,\Lambda)$ is a module over $\Psi_0(X)$. More precisely, if $A \in \Psi_0^k(X)$ and $\nu \in I_0^\ell(X,\Lambda)$ then it follows from Theorem 29 and our convention on the exponent in $\Psi(X)$ that $A\nu \in I_0^{k+\ell}(X,\Lambda)$. In this section we will use stationary phase once again to obtain a local description of this module structure.

We may assume that X is an open subset of $\mathbb{R}^n$, since we are interested in a local description. For simplicity, we will assume that Λ does not intersect the zero section. So we know from Section 5.8 , see equation (5.11), that locally Λ can be described by the fibration

$$\pi : X \times \mathbb{R}^n \to X, \quad (x, \xi) \mapsto x$$

and a generating function of the form

$$\phi^\sharp(x, \xi) = x \cdot \xi - \phi(\xi).$$

We will work locally where this generating function is valid. So we are assuming that $\nu \in I_0^\ell(X, \Lambda)$ is of the form $f dx^{\frac{1}{2}}$ where

$$f(x, \hbar) = \hbar^{\ell - \frac{n}{2}} \int b(x, \xi, \hbar) e^{i \frac{x \cdot \xi - \phi(\xi)}{\hbar}} d\xi.$$

with

$$b \in C_0^\infty(X \times \mathbb{R}^n \times \mathbb{R}).$$

Let $A \in \Psi^k(X)$, so $= u dx^{\frac{1}{2}} dy^{\frac{1}{2}}$ where

$$u(x, y, \hbar) = \hbar^{k-n} \int a(x, y, \xi, \hbar) e^{i \frac{(x-y)\cdot \xi}{\hbar}} d\xi$$

with $a \in C^\infty(X \times X \times \mathbb{R}^n, \mathbb{R})$ supported in a set

$$\|\xi\| \leq C.$$

By definition $Af = g(x, \hbar) dx^{\frac{1}{2}}$ where

$$g(x, \hbar) = \int u(x, y, \hbar) f(y, \hbar) dy$$

and hence is given by $\hbar^{k + \ell - \frac{3}{2} n} \times$ the integral

$$\int a(x, y, \xi, \hbar) b(y, \xi_1, \hbar) e^{i \frac{(x-y)\cdot\xi + y \cdot \xi_1 - \phi(\xi_1)}{\hbar}} d\xi d\xi_1 dy.$$

The amplitude in this integral is

$$a(x, y, \xi, \hbar) b(y, \xi_1, \hbar)$$

and the phase is

$$(x - y) \cdot \xi + y \cdot \xi_1 - \phi(\xi_1) = x \cdot \xi - \phi(\xi) + y \cdot (\xi_1 - \xi) - (\phi(\xi_1) - \phi(\xi)).$$

Let

$$\phi(\xi_1) - \phi(\xi) = \psi(\xi, \xi_1) \cdot (\xi_1 - \xi)$$

so that

$$\psi(\xi,\xi) = \frac{\partial\phi}{\partial\xi}(\xi).$$

Holding x and ξ fixed, make the change of variables

$$\eta := \xi_1 - \xi, \qquad z := y - \psi(\xi_1, \xi)$$

so that in the new coordinates the phase is

$$x \cdot \xi - \phi(\xi) + z \cdot \eta.$$

and the amplitude is

$$a^\sharp(x,\xi,\eta,z,\hbar) := a(x, z + \psi(\xi+\eta,\xi), \xi, \hbar) b(z + \psi(\xi+\eta,\xi), \xi+\eta, \hbar).$$

So we have $Af = gdx^{\frac{1}{2}}$ with

$$g = \hbar^m \int b^\sharp(x,\xi,\hbar) e^{i\frac{x\cdot\xi-\phi(\xi)}{\hbar}} d\xi, \quad m = k + \ell - \frac{3n}{2}$$

and

$$b^\sharp(x,\xi,\hbar) = \int a^\sharp(x,\xi,\eta,z,\hbar) e^{-\frac{z\cdot\eta}{\hbar}} dz d\eta.$$

Once again, stationary phase applied to this integral gives

$$b^\sharp(x,\xi,\hbar) \sim \left(\frac{\hbar}{2\pi}\right)^n \exp\left(\frac{i\hbar}{2} D_z D_\eta\right) a^\sharp(x,\xi,\eta,z,\hbar)\bigg|_{z=\eta=0}.$$

The leading term in this expansion is

$$a^\sharp(x,\xi,0,0) = a(x,\psi(,\xi,\xi),\xi,\hbar) b(\psi(\xi,\xi),\xi,\hbar) =$$

$$a\left(x, \frac{\partial\phi}{\partial\xi}(\xi), \xi, \hbar\right) b\left(\frac{\partial\phi}{\partial\xi}(\xi), \xi, \hbar\right).$$

Since Λ is the submanifold consisting of all

$$(x,\xi) = \left(\frac{\partial\phi}{\partial\xi}(\xi), \xi,\right)$$

in T^*X we see that the leading term depends only on $b_{|\Lambda}$.

8.9 The trace of a semiclassical Fourier integral operator.

Let X be an n-dimensional manifold, let $M = T^*X$ and let

$$\Gamma : T^*X \twoheadrightarrow T^*X$$

be a canonical relation. Let $\Delta_M \subseteq M \times M$ be the diagonal and let us assume that

$$\Gamma \,\overline{\pitchfork}\, \Delta_M \, .$$

Our goal in this section is to show that if $F \in \mathcal{F}_0^k(\Gamma)$ is a semi-classical Fourier integral operator "quantizing" the canonical relation Γ then one has a trace formula of the form:

$$\operatorname{tr} F = \hbar^{k+n} \sum a_p(h) e^{\frac{i\pi}{\eta_p}} e^{iT_p^*/\hbar} \tag{8.44}$$

summed over $p \in \Gamma \cap \Delta_M$. In this formula n is the dimension of X, the η_p's are Maslov factors, the T_p^* are symplectic invariants of Γ at $p \in \Gamma \circ \Delta_M$ which will be defined below, and $a_p(h) \in C^\infty(\mathbb{R})$.

Let $\varsigma : M \to M$ be the involution, $(x, \xi) \to (x, -\xi)$ and let $\Lambda = \varsigma \circ \Gamma$. We will fix a non-vanishing density, dx, on X and denote by

$$\mu = \mu(x, y, \hbar)\, dx^{\frac{1}{2}}\, dy^{\frac{1}{2}} \tag{8.45}$$

the Schwartz kernel of the operator, F. By definition

$$\mu \in I^k(X \times X, \Lambda)$$

and by (8.47) the trace of F is given by the integral

$$\operatorname{tr} F =: \int \mu(x, x)\, dx \, . \tag{8.46}$$

Here are the details:

Let $\varsigma : M \to M$ be the involution, $(x, \xi) \to (x, -\xi)$ and let $\Lambda = \varsigma \circ \Gamma$. We will fix a non-vanishing density, dx, on X and denote by

$$\mu = \mu(x, y, \hbar)\, dx^{\frac{1}{2}}\, dy^{\frac{1}{2}} \tag{8.47}$$

the Schwartz kernel of the operator, F. By definition

$$\mu \in I^k(X \times X, \Lambda)$$

and by (8.47) the trace of F is given by the integral

$$\operatorname{tr} F =: \int \mu(x, x)\, dx \, . \tag{8.48}$$

We can without loss of generality assume that Λ is defined by a generating function, i.e., that there exists a d-dimensional manifold, S, and a function $\varphi(x, y, s) \in C^{infty}(X \times X \times S)$ which generates Λ with respect to the fibration, $X \times X \times S \to X \times X$. Let C_φ be the critical set of φ and $\lambda_\varphi : C_\varphi \to \Lambda$ the diffeomorphism of this set onto Λ. Denoting by $\varphi^\sharp$ the restriction of φ to C_φ and by ψ the function, $\varphi^\sharp \circ \lambda_\varphi^{-1}$, we have by (8.19)

$$d\psi = \alpha_\Lambda \tag{8.49}$$

where α_Λ is the restriction to Λ of the canonical one form, α, on $T^*(X \times X)$.

Lets now compute the trace of F. By assumption μ can be expressed as an oscillatory integral

$$(dx)^{\frac{1}{2}}(dy)^{\frac{1}{2}}\left(h^{k-d/2}\int a(x,y,s,h)e^{\frac{i\varphi(x,y,s)}{\hbar}}\,ds\right)$$

and hence by (8.48)

$$\operatorname{tr} F = \hbar^{k-d/2}\int a(x,y,s,\hbar)e^{i\frac{\varphi(x,y,s)}{\hbar}}\,ds\,dx\,. \tag{8.50}$$

We claim that: *The function*

$$\varphi(x,y,s) : X \times S \to \mathbb{R} \tag{8.51}$$

is a Morse function, and its critical points are in one-one correspondence with the points, $p \in \Gamma \cap \Delta_M$.

Proof. Let Δ_X be the diagonal in $X \times X$ on $\Lambda_\Delta = N^*\Delta_X$ its conormal bundle in $T^*(X \times X) = M \times M$. Then $\varsigma \circ \Lambda_\Delta = \Delta_M$ and hence $\Gamma \pitchfork \Delta_M \Leftrightarrow \Lambda \pitchfork \Lambda_\Delta$. Thus the canonical relations

$$\Lambda : pt \to M \times M$$

and

$$\Lambda_\Delta^t : M \times M \to pt$$

are composable and hence the function (8.51) is a generating function for the Lagrangian manifold "*pt*" with respect to the fibration $X \times S \to pt$. In other words, in more prosaic language, the function (8.51) is a Morse function. Its critical points are the points where

$$\frac{\partial\varphi}{\partial s} = 0$$

and

$$\xi = \frac{\partial\varphi}{\partial x}(x,x,s) = -\frac{\partial\varphi}{\partial y}(x,x,s) = \eta\,;$$

in other words, points $(x,y,s) \in C_\varphi$ with the property $\gamma_\varphi(x,y,s) = (x,\xi,y,\eta)$, $p = (x,\xi) = (y,-\eta)$, hence these points are in one-one correspondence with the points $p \in \Gamma \cap \Delta_M$. $\square$

Since the function (8.51) is a Morse function we can evaluate (8.49) by stationary phase obtaining

$$\operatorname{tr} F = \sum h^{k+\eta} a_p(h) e^{i\frac{\pi}{4}\operatorname{sgn}_p} e^{i\psi(p)/\hbar} \tag{8.52}$$

where sgn_p is the signature of $\varphi(x,x,s)$ at the critical point corresponding to p and

$$\psi(p) = \varphi(x,x,s)\,,$$

the value of $\varphi(x,x,s)$ at this point. This gives us the trace formula (8.44) with $T_p^\sharp = \psi(p)$.

8.9.1 Examples.

Let's now describe how to compute these $T_p^\sharp$'s in some examples: Suppose Γ is the graph of a symplectomorphism

$$f : M \to M\,.$$

Let pr_1 and pr_2 be the projections of $T^*(X \times X) = M \times M$ onto its first and second factors, and let α_X be the canonical one form on T^*X. Then the canonical one form, α, on $T^*(X \times X)$ is

$$(pr_1)^*\alpha_X + (pr_2)^*\alpha_X\,,$$

so if we restrict this one form to Λ and then identify Λ with M via the map, $M \to \Lambda$, $p \to (p, \sigma f(p))$, we get from (8.49)

$$\alpha_X - f^*\alpha_X = d\psi \tag{8.53}$$

and $T_p^\sharp$ is the value of ψ at the point, p.

Let's now consider the Fourier integral operator

$$F^m = \overbrace{F \circ \cdots \circ F}$$

and compute its trace. This operator "quantizes" the symplectomorphism f^m, hence if

$$\text{graph}\, f^m \pitchfork \Delta_M$$

we can compute its trace by (8.44) getting the formula

$$\operatorname{tr} F^m = \hbar^\ell \sum a_{m,p}(\hbar) e^{i\frac{\pi}{4}\sigma_{m,p}} e^{iT^\sharp_{m,p}/\hbar}\,. \tag{8.54}$$

with $\ell = km + (\frac{m-1}{2})n$, the sum now being over the fixed points of f^m. As above, the oscillations, $T^\sharp_{m,p}$, are computed by evaluating at p the function, ψ_m, defined by

$$\alpha_X - (f^m)^*\alpha_X = d\psi_m\,.$$

However,

$$\begin{aligned} \alpha_X - (f^m)^*\alpha_X &= \alpha_X - f^*\alpha_X + \cdots + (f^{m-1})^*\alpha_X - (f^m)^*\alpha\,, \\ &= d(\psi + f^x\psi + \cdots + (f^{m-1})^x\psi) \end{aligned}$$

where ψ is the function (8.49). Thus at $p = f^m(p)$

$$T^\sharp_{m,p} = \sum_{i=1}^{m-1} \psi(p_i)\,, \quad p_i = f^i(p)\,. \tag{8.55}$$

In other words $T^\sharp_{m,p}$ is the sum of ψ over the periodic trajectory $(p_1, \ldots, p_{m-1})$ of the dynamical system

$$f^k\,, \quad -\infty < k < \infty\,.$$

We refer to the next subsection "The period spectrum of a symplectomorphism" for a proof that the $T^\sharp_{m,p}$'s are *intrinsic* symplectic invariants of this dynamical system, i.e., depend only on the symplectic structure of M not on the canonical one form, α_X. (We will also say more about the "geometric" meaning of these $T^\sharp_{m,p}$'s in the next lecture.)

Finally, what about the amplitudes, $a_p(h)$, in formula (8.44)? There are many ways to quantize the symplectomorphism, f, and no canonical way of choosing such a quantization; however, one condition which one can impose on F is that its symbol be of the form:

$$h^{-n}\sigma_\Gamma e^{\frac{i\psi}{\hbar}} e^{i\frac{\pi}{4}\sigma_\varphi}\,, \tag{8.56}$$

in the vicinity of $\Gamma\cap\Delta_M$, where ν_Γ is the $\frac{1}{2}$ density on Γ obtained from the symplectic $\frac{1}{2}$ density, ν_M, on M by the identification, $M\leftrightarrow\Gamma$, $p\to(p,f(p))$. We can then compute the symbol of $a_p(h)\in I^0(pt)$ by pairing the $\frac{1}{2}$ densities, ν_M and ν_Γ at $p\in\Gamma\cap\Delta_M$ as in (7.29) obtaining

$$a_p(0) = |\det(I-df_p)|^{\frac{1}{2}}\,. \tag{8.57}$$

Remark. The condition (8.56) on the symbol of F can be interpreted as a "unitarity" condition. It says that "microlocally" near the fixed points of f:

$$FF^t = I + O(h)\,.$$

8.9.2 The period spectrum of a symplectomorphism.

Let (M,ω) be a symplectic manifold. We will assume that the cohomology class of ω is zero; i.e., that ω is exact, and we will also assume that M is connected and that

$$H^1(M,\mathbb{R}) = 0\,. \tag{*}$$

Let $f: M\to M$ be a symplectomorphism and let $\omega = d\alpha$. We claim that $\alpha - f^*\alpha$ s exact. Indeed $d\alpha - f^*d\alpha = \omega - f^*\omega = 0$, and hence by (*) $\alpha - f^*\alpha$ s exact. Let

$$\alpha - f^*\alpha = d\psi$$

for $\psi\in C^\infty(M)$. This function is only unique up to an additive constant; however, there are many ways to normalize this constant. For instance if W is a connected subset of the set of fixed points of f, and $j: W\to M$ is the inclusion map, then $f\circ j = j$; so

$$j^*\,d\psi = j^*\alpha - j^*f^*\alpha = 0$$

and hence ψ is constant on W. Thus one can normalize ψ by requiring it to be zero on W.

Example. Let Ω be a smooth convex compact domain in $\mathbb{R}^n$, let X be its boundary, let U be the set of points, (x,ξ), $|\xi|<1$, in T^*X. If $B: U\to U$

is the billiard map and α the canonical one form on T^*X one can take for $\psi = \psi(x,\xi)$ the function

$$\psi(x,\xi) = |x-y| + C$$

where $(y,n) = B(x,\xi)$. B has no fixed points on U, but it extends continuously to a mapping of $\bar{U}$ on $\bar{U}$ leaving the boundary, W, of U fixed and we can normalize ψ by requiring that $\psi = 0$ on W, i.e., that $\psi(x,\xi) = |x-y|$.

Now let

$$\gamma = p_1, \dots, p_{k+1}$$

be a periodic trajectory of f, i.e.,

$$f(p_i) = p_{i+1} \qquad i = 1, \dots k$$

and $p_{k+1} = p_1$. We define the *period* of γ to be the sum

$$p(\gamma) = \sum_{i=1}^{k} \psi(p_i)\,.$$

Claim: *$P(\gamma)$ is independent of the choice of α and ψ. In other words it is a* symplectic invariant *of f.*

Proof. Suppose $\omega = d\alpha - d\alpha'$. Then $d(\alpha-\alpha') = 0$; so, by (*), $\alpha' - \alpha = dh$ for some function, $h \in C^\infty(M)$. Now suppose $\alpha - f^*\alpha = d\psi$ and $\alpha' - f^*\alpha' = d\psi'$ with $\psi = \psi'$ on the set of fixed points, W. Then

$$d\psi' - d\psi = d(f^*h - h)$$

and since $f^* = 0$ on W

$$\psi' - \psi = f^*h - h\,.$$

Thus

$$\begin{aligned} \sum_{i=1}^{k} \psi'(p_i) - \psi(p_i) &= \sum_{i=1}^{k} h(f(p_i)) - h(p_i) \\ &= \sum_{i=1}^{k} h(p_{i+1}) - h(p_i) \\ &= 0\,. \end{aligned}$$

Hence replacing ψ by ψ' doesn't change the definition of $P(\gamma)$. $\square$

Example Let $p_i = (x_i, \xi_i)$ $i = 1, \dots, k+1$ be a periodic trajectory of the billiard map. Then its period is the sum

$$\sum_{i=1}^{k} |x_{i+1} - x_i|\,,$$

i.e., is the *perimeter* of the polygon with vertices at $x_1, \dots, x_k$. (It's far from obvious that this is a symplectic invariant of B.)

8.10 The mapping torus of a symplectic mapping.

We'll give below a geometric interpretation of the oscillations, $T^{\sharp}_{m,p}$, occurring in the trace formula (8.54). First, however, we'll discuss a construction used in dynamical systems to convert "discrete time" dynamical systems to "continuous time" dynamical systems. Let M be a manifold and $f : M \to M$ a diffeomorphism. From f one gets a diffeomorphism

$$g : M \times \mathbb{R} \to M \times \mathbb{R}\,, \quad g(p,q) = (f(p), q+1)$$

and hence an action

$$\mathbb{Z} \to Diff(M \times \mathbb{R})\,, \quad k \to g^k\,, \tag{8.58}$$

of the group, $\mathbb{Z}$ on $M \times \mathbb{R}$. This action is free and properly discontinuous so the quotient

$$Y = M \times \mathbb{R}/\mathbb{Z}$$

is a smooth manifold. The manifold is called the *mapping torus* of f. Now notice that the translations

$$\tau_t : M \times \mathbb{R} \to M \times \mathbb{R}\,, \, (p,q) \to (p, q+t)\,, \tag{8.59}$$

commute with the action (8.58), and hence induce on Y a one parameter group of translations

$$\tau_t^{\sharp} : Y \to Y\,, \, -\infty < t < \infty\,. \tag{8.60}$$

Thus the mapping torus construction converts a "discrete time" dynamical system, the "discrete" one-parameter group of diffeomorphisms, $f^k : M \to M$, $-\infty < k < \infty$, into a "continuous time" one parameter group of diffeomorphisms (8.60).

To go back and reconstruct f from the one-parameter group (8.60) we note that the map

$$\iota : M = M \times \{0\} \to M \times \mathbb{R} \to (M \times \mathbb{R})/Z$$

imbeds M into Y as a global cross-section, M_0, of the flow (8.60) and for $p \in M_0$ $\gamma_t(p) \in M_0$ at $t = 1$ and via the identification $M_0 \to M$, the map, $p \to \gamma_1(p)$, is just the map, f. In other words, $f : M \to M$ is the "first return map" associated with the flow (8.60).

We'll now describe how to "symplecticize" this construction. Let $\omega \in \Omega^2(M)$ be an exact symplectic form and $f : M \to M$ a symplectomorphism. For $\alpha \in \Omega^1(M)$ with $d\alpha = \omega$ let

$$\alpha - f^*\alpha = d\varphi \tag{8.61}$$

and lets assume that φ is bounded from below by a positive constant. Let

$$g : M \times \mathbb{R} \to M \times \mathbb{R}$$

be the map

$$g(p,q) = (p, q + \varphi(x))\,. \tag{8.62}$$

As above one gets from g a free properly discontinuous action, $k \to g^k$, of $\mathbb{Z}$ on $M \times \mathbb{R}$ and hence one can form the mapping torus

$$Y = (M \times \mathbb{R})/\mathbb{Z}\,.$$

Moreover, as above, the group of translations,

$$\tau_t : M \times \mathbb{R} \to M \times \mathbb{R}\,,\; \tau_t(p,q) = (p, q+t)\,,$$

commutes with (8.62) and hence induces on Y a one-parameter group of diffeomorphisms

$$\tau_t^\sharp : Y \to Y\,,$$

just as above. We will show, however, that these are not just diffeomorphisms, they are *contacto-morphisms*. To prove this we note that the one-form,

$$\tilde{\alpha} = \alpha + dt\,,$$

on $M \times \mathbb{R}$ is a contact one-form. Moreover,

$$\begin{aligned} g^*\tilde{\alpha} &= f^*\alpha + d(\varphi + t) \\ &= \alpha + (f^*\alpha - \alpha) + d\varphi + dt \\ &= \alpha + dt = \tilde{\alpha} \end{aligned}$$

by (8.61) and

$$(\tau_a)^*\tilde{\alpha} = \alpha + d(t+a) = \alpha + dt = \tilde{\alpha}$$

so the action of $\mathbb{Z}$ on $M \times \mathbb{R}$ and the translation action of $\mathbb{R}$ on $M \times \mathbb{R}$ are both actions by groups of contacto-morphisms. Thus, $Y = (M \times \mathbb{R})/\mathbb{Z}$ inherits from $M \times \mathbb{R}$ a contact structure and the one-parameter group of diffeomorphisms, $\tau_t^\sharp$, preserves this contact structure.

Note also that the infinitesimal generator, of the group translations, τ_t, is just the vector field, $\frac{\partial}{\partial t}$, and that this vector field satisfies

$$\iota(\frac{\partial}{\partial t})\tilde{\alpha} = 1$$

and

$$\iota(\frac{\partial}{\partial t})\, d\tilde{\alpha} = 0\,.$$

Thus $\frac{\partial}{\partial t}$ is the contact vector field associated with the contract form $\tilde{\alpha}$, and hence the infinitesimal generator of the one-parameter group, $\tau_t^\sharp : Y \to Y$ is the contact vector field associated with the contract form on Y.

Comments:

1. The construction we've just outlined involves the choice of a one-form, α, on M with $d\alpha = \omega$ and a function, φ, with $\alpha = f^x\alpha = d\varphi$; however, it is easy to see that the contact manifold, Y, and one-parameter group of contacto-morphisms are uniquely determined, up to contracto-morphism, independent of these choices.

2. Just as in the standard mapping torus construction f can be shown to be "first return map" associated with the one-parameter group, $\tau_t^\sharp$.

We can now state the main result of this section, which gives a geometric description of the oscillations, $T_{m,p}^\sharp$, in the trace formula.

Theorem 34 *The periods of the periodic trajectories of the flow, $\tau_t^\sharp$, $-\infty < t < \infty$, coincide with the "length" spectrum of the symplectomorphism, $f : M \to M$.*

Proof. For $(p,a) \in M \times \mathbb{R}$,

$$g^m(p,a) = (f^m(p), q + \varphi(p) + \varphi(p_1) + \cdots + \varphi(p_{m-1})$$

with $p_i = f^i(p)$. Hence if $p = f^m(p)$

$$g^m(p,a) = \tau_{T^\sharp}(p,a)$$

with

$$T^\sharp = T_{m,p}^\sharp = \sum_{i=1}^m \varphi(p_i)\,, \quad p_i = f^i(p)\,.$$

Thus if q is the projection of (p,a) onto Y the trajectory of $\tau^\sharp$ through q is periodic of period $T_{m,p}^\sharp$. $\square$

Via the mapping torus construction one discovers an interesting connection between the trace formula in the preceding section and a trace formula which we described in Section 7.9.4.

Let β be the contact form on Y and let

$$M^\sharp = \{(y,\eta) \in T^*Y\,,\, \eta = t\beta_y\,,\, t \in \mathbb{R}_+\}\,.$$

It's easy to see that $M^\sharp$ is a symplectic submanifold of T^*Y and hence a symplectic manifold in its own right. Let

$$H : M^\sharp \to \mathbb{R}^+$$

be the function $H(y, tB_y) = t$. Then Y can be identified with the level set, $H = 1$ and the Hamiltonian vector field ν_H restricted to this level set coincides with the contact vector field, ν, on Y. Thus the flow, $\tau_t^\sharp$, is just the Hamiltonian flow, $\exp t\nu_H$, restricted to this level set. Let's now compute the "trace" of $\exp t\nu_H$ as an element in the category $\tilde{S}$ (the enhanced symplectic category).

The computation of this trace is essentially identical with the computation we make at the end of Section 7.9.4 and gives as an answer the union of the Lagrangian manifold

$$\Lambda_{T^\sharp_{m,p}} \subset T^*\mathbb{R}\,,\; m \in \mathbb{Z}\,,$$

where the $T^\sharp$'s are the elements of the period spectrum of ν_H and $\Lambda_{T^\sharp}$ is the cotangent fiber at $t = T$. Moreover, each of these $\Lambda_{T^\sharp}$'s is an element of the enhanced symplectic category, i.e. is equipped with a $\frac{1}{2}$-density $\nu_{T^\sharp_{m,p}}$ which we computed to be

$$\overline{T}^\sharp_{m,p}|I - df^m_p|^{-\frac{1}{2}}|d\tau|^{\frac{1}{2}}\,.$$

$\overline{T}^\sharp_{m,p}$ being the *primitive* period of the period trajectory of f through p (i.e., if $p_i = f^i(p)$ $i = 1,\dots,m$ and $p, p_1, \dots, p_{k-1}$ are all distinct but $p = p_k$ then $\overline{T}^\sharp_{m,p} = \overline{T}^\sharp_{k,p}$) . Thus these expressions are just the symbols of the oscillatory integrals

$$\hbar^{-1}a_{m,p}e^{I\imath\overline{T}^\sharp_{p,m}t/\hbar}$$

with $a_{m,p} = \overline{T}^\sharp_{m,p}|I - df^m_p|^{\frac{1}{2}}$.

Chapter 9

Differential calculus of forms, Weil's identity and the Moser trick.

The purpose of this chapter is to give a rapid review of the basics of the calculus of differential forms on manifolds. We will give two proofs of Weil's formula for the Lie derivative of a differential form: the first of an algebraic nature and then a more general geometric formulation with a "functorial" proof that we learned from Bott. We then apply this formula to the "Moser trick" and give several applications of this method.

9.1 Superalgebras.

A (commutative associative) **superalgebra** is a vector space

$$A = A_{even} \oplus A_{odd}$$

with a given direct sum decomposition into even and odd pieces, and a map

$$A \times A \to A$$

which is bilinear, satisfies the associative law for multiplication, and

$$\begin{aligned} A_{even} \times A_{even} &\to A_{even} \\ A_{even} \times A_{odd} &\to A_{odd} \\ A_{odd} \times A_{even} &\to A_{odd} \\ A_{odd} \times A_{odd} &\to A_{even} \\ \omega \cdot \sigma &= \sigma \cdot \omega \text{ if either } \omega \text{ or } \sigma \text{ are even,} \\ \omega \cdot \sigma &= -\sigma \cdot \omega \text{ if both } \omega \text{ and } \sigma \text{ are odd.} \end{aligned}$$

We write these last two conditions as

$$\omega \cdot \sigma = (-1)^{\deg\sigma \deg\omega} \sigma \cdot \omega.$$

Here deg $\tau = 0$ if τ is even, and deg $\tau = 1$ (mod 2) if τ is odd.

9.2 Differential forms.

A **linear differential form** on a manifold, M, is a rule which assigns to each $p \in M$ a linear function on TM_p. So a linear differential form, ω, assigns to each p an element of TM_p^*. We will, as usual, only consider linear differential forms which are smooth.

The superalgebra $\Omega(M)$ is the superalgebra generated by smooth functions on M (taken as even) and by the linear differential forms, taken as odd.

Multiplication of differential forms is usually denoted by $\wedge$. The number of differential factors is called the *degree* of the form. So functions have degree zero, linear differential forms have degree one.

In terms of local coordinates, the most general *linear* differential form has an expression as $a_1 dx_1 + \cdots + a_n dx_n$ (where the a_i are functions). Expressions of the form

$$a_{12} dx_1 \wedge dx_2 + a_{13} dx_1 \wedge dx_3 + \cdots + a_{n-1,n} dx_{n-1} \wedge dx_n$$

have degree two (and are even). Notice that the multiplication rules require

$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$

and, in particular, $dx_i \wedge dx_i = 0$. So the most general sum of products of two linear differential forms is a differential form of degree two, and can be brought to the above form, locally, after collections of coefficients. Similarly, the most general differential form of degree $k \leq n$ on an n dimensional manifold is a sum, locally, with function coefficients, of expressions of the form

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \quad i_1 < \cdots < i_k.$$

There are $\begin{pmatrix} n \\ k \end{pmatrix}$ such expressions, and they are all even, if k is even, and odd if k is odd.

9.3 The d operator.

There is a linear operator d acting on differential forms called *exterior* differentiation, which is completely determined by the following rules: It satisfies Leibniz' rule in the "super" form

$$d(\omega \cdot \sigma) = (d\omega) \cdot \sigma + (-1)^{\deg\omega}\, \omega \cdot (d\sigma).$$

On functions it is given by

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

and, finally,

$$d(dx_i) = 0.$$

Since functions and the dx_i generate, this determines d completely. For example, on linear differential forms

$$\omega = a_1 dx_1 + \cdots a_n dx_n$$

we have

$$\begin{aligned} d\omega &= da_1 \wedge dx_1 + \cdots + da_n \wedge dx_n \\ &= \left(\frac{\partial a_1}{\partial x_1} dx_1 + \cdots \frac{\partial a_1}{\partial x_n} dx_n\right) \wedge dx_1 + \cdots \\ &\qquad \left(\frac{\partial a_n}{\partial x_1} dx_1 + \cdots + \frac{\partial a_n}{\partial x_n} dx_n\right) \wedge dx_n \\ &= \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right) dx_1 \wedge dx_2 + \cdots + \left(\frac{\partial a_n}{\partial x_{n-1}} - \frac{\partial a_{n-1}}{\partial x_n}\right) dx_{n-1} \wedge dx_n. \end{aligned}$$

In particular, equality of mixed derivatives shows that $d^2 f = 0$, and hence that $d^2\omega = 0$ for any differential form. Hence the rules to remember about d are:

$$\begin{aligned} d(\omega \cdot \sigma) &= (d\omega) \cdot \sigma + (-1)^{\deg \omega}\, \omega \cdot (d\sigma) \\ d^2 &= 0 \\ df &= \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n. \end{aligned}$$

9.4 Derivations.

A linear operator $\ell : A \to A$ is called an *odd derivation* if, like d, it satisfies

$$\ell : A_{even} \to A_{odd}, \quad \ell : A_{odd} \to A_{even}$$

and

$$\ell(\omega \cdot \sigma) = (\ell\omega) \cdot \sigma + (-1)^{\deg \omega}\, \omega \cdot \ell\sigma.$$

A linear map $\ell : A \to A$,

$$\ell : A_{even} \to A_{even}, \quad \ell : A_{odd} \to A_{odd}$$

satisfying

$$\ell(\omega \cdot \sigma) = (\ell\omega) \cdot \sigma + \omega \cdot (\ell\sigma)$$

is called an *even derivation.* So the Leibniz rule for derivations, even or odd, is

$$\ell(\omega \cdot \sigma) = (\ell\omega) \cdot \sigma + (-1)^{\deg\ell \deg\omega}\ \omega \cdot \ell\sigma.$$

Knowing the action of a derivation on a set of generators of a superalgebra determines it completely. For example, the equations

$$d(x_i) = dx_i, \quad d(dx_i) = 0 \ \ \forall i$$

implies that

$$dp = \frac{\partial p}{\partial x_1} dx_1 + \cdots + \frac{\partial p}{\partial x_n} dx_n$$

for any polynomial, and hence determines the value of d on any differential form with polynomial coefficients. The local formula we gave for df where f is any differentiable function, was just the natural extension (by continuity, if you like) of the above formula for polynomials.

The sum of two even derivations is an even derivation, and the sum of two odd derivations is an odd derivation.

The composition of two derivations will not, in general, be a derivation, but an instructive computation from the definitions shows that the *commutator*

$$[\ell_1, \ell_2] := \ell_1 \circ \ell_2 - (-1)^{\deg\ell_1 \deg\ell_2}\ \ell_2 \circ \ell_1$$

is again a derivation which is even if both are even or both are odd, and odd if one is even and the other odd.

A derivation followed by a multiplication is again a derivation: specifically, let ℓ be a derivation (even or odd) and let τ be an even or odd element of A. Consider the map

$$\omega \mapsto \tau\ell\omega.$$

We have

$$\begin{aligned} \tau\ell(\omega\sigma) &= (\tau\ell\omega) \cdot \sigma + (-1)^{\deg\ell \deg\omega} \tau\omega \cdot \ell\sigma \\ &= (\tau\ell\omega) \cdot \sigma + (-1)^{(\deg\ell + \deg\tau)\deg\omega} \omega \cdot (\tau\ell\sigma) \end{aligned}$$

so $\omega \mapsto \tau\ell\omega$ is a derivation whose degree is

$$\deg\tau + \deg\ell.$$

9.5 Pullback.

Let $\phi : M \to N$ be a smooth map. Then the pullback map ϕ^* is a linear map that sends differential forms on N to differential forms on M and satisfies

$$\begin{aligned} \phi^*(\omega \wedge \sigma) &= \phi^*\omega \wedge \phi^*\sigma \\ \phi^* d\omega &= d\phi^*\omega \\ (\phi^* f) &= f \circ \phi. \end{aligned}$$

The first two equations imply that ϕ^* is completely determined by what it does on functions. The last equation says that on functions, ϕ^* is given by "substitution": In terms of local coordinates on M and on N ϕ is given by

$$\begin{aligned} \phi(x^1, \dots, x^m) &= (y^1, \dots, y^n) \\ y^i &= \phi^i(x^1, \dots, x^m) \quad i = 1, \dots, n \end{aligned}$$

where the ϕ_i are smooth functions. The local expression for the pullback of a function $f(y^1, \dots, y^n)$ is to substitute ϕ^i for the y^is as into the expression for f so as to obtain a function of the x's.

It is important to observe that the pull back on differential forms is defined for any smooth map, not merely for diffeomorphisms. This is the great advantage of the calculus of differential forms.

9.6 Chain rule.

Suppose that $\psi : N \to P$ is a smooth map so that the composition

$$\phi \circ \psi : M \to P$$

is again smooth. Then the *chain rule* says

$$(\phi \circ \psi)^* = \psi^* \circ \phi^*.$$

On functions this is essentially a tautology - it is the associativity of composition: $f \circ (\phi \circ \psi) = (f \circ \phi) \circ \psi$. But since pull-back is completely determined by what it does on functions, the chain rule applies to differential forms of any degree.

9.7 Lie derivative.

Let ϕ_t be a one parameter group of transformations of M. If ω is a differential form, we get a family of differential forms, $\phi_t^*\omega$ depending differentiably on t, and so we can take the derivative at $t = 0$:

$$\frac{d}{dt}\left(\phi_t^*\omega\right)_{|t=0} = \lim_{t=0} \frac{1}{t}\left[\phi_t^*\omega - \omega\right].$$

Since $\phi_t^*(\omega \wedge \sigma) = \phi_t^*\omega \wedge \phi_t^*\sigma$ it follows from the Leibniz argument that

$$\ell_\phi : \ \omega \mapsto \frac{d}{dt}\left(\phi_t^*\omega\right)_{|t=0}$$

is an even derivation. We want a formula for this derivation.

Notice that since $\phi_t^* d = d\phi_t^*$ for all t, it follows by differentiation that

$$\ell_\phi d = d\ell_\phi$$

and hence the formula for ℓ_ϕ is completely determined by how it acts on functions.

Let X be the vector field generating ϕ_t. Recall that the geometrical significance of this vector field is as follows: If we fix a point x, then

$$t \mapsto \phi_t(x)$$

is a curve which passes through the point x at $t = 0$. The tangent to this curve at $t = 0$ is the vector $X(x)$. In terms of local coordinates, X has coordinates $X = (X^1, \ldots, X^n)$ where $X^i(x)$ is the derivative of $\phi^i(t, x^1, \ldots, x^n)$ with respect to t at $t = 0$. The chain rule then gives, for any function f,

$$\begin{aligned}
\ell_\phi f &= \frac{d}{dt} f(\phi^1(t, x^1, \ldots, x^n), \ldots, \phi_n(t, x^1, \ldots, x^n))_{|t=0} \\
&= X^1 \frac{\partial f}{\partial x_1} + \cdots + X^n \frac{\partial f}{\partial x_n}.
\end{aligned}$$

For this reason we use the notation

$$X = X^1 \frac{\partial}{\partial x_1} + \cdots + X^n \frac{\partial}{\partial x_n}$$

so that the differential operator

$$f \mapsto Xf$$

gives the action of ℓ_ϕ on functions.

As we mentioned, this action of ℓ_ϕ on functions determines it completely. In particular, ℓ_ϕ depends only on the vector field X, so we may write

$$\ell_\phi = D_X$$

where D_X is the even derivation determined by

$$D_X f = Xf, \quad D_X d = dD_X.$$

9.8 Weil's formula.

But we want a more explicit formula for D_X. For this it is useful to introduce an odd derivation associated to X called the *interior product* and denoted by $i(X)$. It is defined as follows: First consider the case where

$$X = \frac{\partial}{\partial x_j}$$

and define its interior product by

$$i\left(\frac{\partial}{\partial x_j}\right) f = 0$$

for all functions while

$$i\left(\frac{\partial}{\partial x_j}\right)dx_k = 0, \quad k \neq j$$

and

$$i\left(\frac{\partial}{\partial x_j}\right)dx_j = 1.$$

The fact that it is a derivation then gives an easy rule for calculating $i(\partial/\partial x_j)$ when applied to any differential form: Write the differential form as

$$\omega + dx_j \wedge \sigma$$

where the expressions for ω and σ do not involve dx_j. Then

$$i\left(\frac{\partial}{\partial x_j}\right)[\omega + dx_j \wedge \sigma] = \sigma.$$

The operator

$$X^j i\left(\frac{\partial}{\partial x_j}\right)$$

which means first apply $i(\partial/\partial x_j)$ and then multiply by the function X^j is again an odd derivation, and so we can make the definition

$$i(X) := X^1 i\left(\frac{\partial}{\partial x_1}\right) + \cdots + X^n i\left(\frac{\partial}{\partial x_n}\right). \tag{9.1}$$

It is easy to check that this does not depend on the local coordinate system used.

Notice that we can write

$$Xf = i(X)df.$$

In particular we have

$$\begin{aligned} D_X dx_j &= dD_X x_j \\ &= dX_j \\ &= di(X)dx_j. \end{aligned}$$

We can combine these two formulas as follows: Since $i(X)f = 0$ for any function f we have

$$D_X f = di(X)f + i(X)df.$$

Since $ddx_j = 0$ we have

$$D_X dx_j = di(X)dx_j + i(X)ddx_j.$$

Hence

$$D_X = di(X) + i(X)d = [d, i(X)] \tag{9.2}$$

when applied to functions or to the forms dx_j. But the right hand side of the preceding equation is an even derivation, being the commutator of two odd derivations. So if the left and right hand side agree on functions and on the differential forms dx_j they agree everywhere. This equation, (9.2), known as *Weil's formula*, is a basic formula in differential calculus.

We can use the interior product to consider differential forms of degree k as $k-$multilinear functions on the tangent space at each point. To illustrate, let σ be a differential form of degree two. Then for any vector field, X, $i(X)\sigma$ is a linear differential form, and hence can be evaluated on any vector field, Y to produce a function. So we define

$$\sigma(X,Y) := [i(X)\sigma](Y).$$

We can use this to express exterior derivative in terms of ordinary derivative and Lie bracket: If θ is a linear differential form, we have

$$\begin{aligned} d\theta(X,Y) &= [i(X)d\theta](Y) \\ i(X)d\theta &= L_X\theta - d(i(X)\theta) \\ d(i(X)\theta)(Y) &= Y[\theta(X)] \\ [D_X\theta](Y) &= D_X[\theta(Y)] - \theta(D_X(Y)) \\ &= X[\theta(Y)] - \theta([X,Y]) \end{aligned}$$

where we have introduced the notation $D_XY =: [X,Y]$ which is legitimate since on functions we have

$$(D_XY)f = D_X(Yf) - YL_Xf = X(Yf) - Y(Xf)$$

so D_XY as an operator on functions is exactly the commutator of X and Y. (See below for a more detailed geometrical interpretation of D_XY.) Putting the previous pieces together gives

$$d\theta(X,Y) = X\theta(Y) - Y\theta(X) - \theta([X,Y]), \tag{9.3}$$

with similar expressions for differential forms of higher degree.

9.9 Integration.

Let

$$\omega = f dx_1 \wedge \cdots \wedge dx_n$$

be a form of degree n on $\mathbf{R}^n$. (Recall that the most general differential form of degree n is an expression of this type.) Then its integral is defined by

$$\int_M \omega := \int_M f dx_1 \cdots dx_n$$

where M is any (measurable) subset. This,of course is subject to the condition that the right hand side converges if M is unbounded. There is a lot of hidden subtlety built into this definition having to do with the notion of orientation. But for the moment this is a good working definition.

The *change of variables formula* says that if $\phi : M \to \mathbf{R}^n$ is a smooth differentiable map which is one to one whose Jacobian determinant is everywhere positive, then

$$\int_M \phi^* \omega = \int_{\phi(M)} \omega.$$

9.10 Stokes theorem.

Let U be a region in $\mathbf{R}^n$ with a chosen orientation and smooth boundary. We then orient the boundary according to the rule that an outward pointing normal vector, together with the a positive frame on the boundary give a positive frame in $\mathbf{R}^n$. If σ is an $(n-1)-$form, then

$$\int_{\partial U} \sigma = \int_U d\sigma.$$

A manifold is called *orientable* if we can choose an atlas consisting of charts such that the Jacobian of the transition maps $\phi_\alpha \circ \phi_\beta^{-1}$ is always positive. Such a choice of an atlas is called an orientation. (Not all manifolds are orientable.) If we have chosen an orientation, then relative to the charts of our orientation, the transition laws for an $n-$form (where $n = \dim M$) and for a density are the same. In other words, given an orientation, we can identify densities with $n-$forms and $n-$form with densities. Thus we may integrate $n-$forms. The change of variables formula then holds for orientation preserving diffeomorphisms as does Stokes theorem.

9.11 Lie derivatives of vector fields.

Let Y be a vector field and ϕ_t a one parameter group of transformations whose "infinitesimal generator" is some other vector field X. We can consider the "pulled back" vector field $\phi_t^* Y$ defined by

$$\phi_t^* Y(x) = d\phi_{-t}\{Y(\phi_t x)\}.$$

In words, we evaluate the vector field Y at the point $\phi_t(x)$, obtaining a tangent vector at $\phi_t(x)$, and then apply the differential of the (inverse) map ϕ_{-t} to obtain a tangent vector at x.

If we differentiate the one parameter family of vector fields $\phi_t^* Y$ with respect to t and set $t = 0$ we get a vector field which we denote by $D_X Y$:

$$D_X Y := \frac{d}{dt} \phi_t^* Y_{|t=0}.$$

If ω is a linear differential form, then we may compute $i(Y)\omega$ which is a function whose value at any point is obtained by evaluating the linear function $\omega(x)$ on the tangent vector $Y(x)$. Thus

$$i(\phi_t^* Y)\phi_t^*\omega(x) = \langle (d(\phi_t)_x)^*\omega(\phi_t x), d\phi_{-t}Y(\phi_t x)\rangle = \{i(Y)\omega\}(\phi_t x).$$

In other words,

$$\phi_t^*\{i(Y)\omega\} = i(\phi_t^* Y)\phi_t^*\omega.$$

We have verified this when ω is a differential form of degree one. It is trivially true when ω is a differential form of degree zero, i.e. a function, since then both sides are zero. But then, by the derivation property, we conclude that it is true for forms of all degrees. We may rewrite the result in shorthand form as

$$\phi_t^* \circ i(Y) = i(\phi_t^* Y) \circ \phi_t^*.$$

Since $\phi_t^* d = d\phi_t^*$ we conclude from Weil's formula that

$$\phi_t^* \circ D_Y = D_{\phi_t^* Y} \circ \phi_t^*.$$

Until now the subscript t was superfluous, the formulas being true for any fixed diffeomorphism. Now we differentiate the preceding equations with respect to t and set $t = 0$. We obtain,using Leibniz's rule,

$$D_X \circ i(Y) = i(D_X Y) + i(Y) \circ D_X$$

and

$$D_X \circ D_Y = D_{D_X Y} + D_Y \circ D_X.$$

This last equation says that Lie derivative (on forms) with respect to the vector field $D_X Y$ is just the commutator of D_X with D_Y:

$$D_{D_X Y} = [D_X, D_Y].$$

For this reason we write

$$[X, Y] := D_X Y$$

and call it the Lie bracket (or commutator) of the two vector fields X and Y. The equation for interior product can then be written as

$$i([X, Y]) = [D_X, i(Y)].$$

The Lie bracket is antisymmetric in X and Y. We may multiply Y by a function g to obtain a new vector field gY. Form the definitions we have

$$\phi_t^*(gY) = (\phi_t^* g)\phi_t^* Y.$$

Differentiating at $t = 0$ and using Leibniz's rule we get

$$[X, gY] = (Xg)Y + g[X, Y] \tag{9.4}$$

where we use the alternative notation Xg for $D_X g$. The antisymmetry then implies that for any differentiable function f we have

$$[fX, Y] = -(Yf)X + f[X, Y]. \tag{9.5}$$

From both this equation and from Weil's formula (applied to differential forms of degree greater than zero) we see that the Lie derivative with respect to X at a point x depends on more than the value of the vector field X at x.

9.12 Jacobi's identity.

From the fact that $[X, Y]$ acts as the commutator of X and Y it follows that for any three vector fields X, Y and Z we have

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.$$

This is known as **Jacobi's identity**. We can also derive it from the fact that $[Y, Z]$ is a natural operation and hence for any one parameter group ϕ_t of diffeomorphisms we have

$$\phi_t^*([Y, Z]) = [\phi_t^* Y, \phi_t^* Z].$$

If X is the infinitesimal generator of ϕ_t then differentiating the preceding equation with respect to t at $t = 0$ gives

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]].$$

In other words, X acts as a derivation of the "mutliplication" given by Lie bracket. This is just Jacobi's identity when we use the antisymmetry of the bracket. In the future we we will have occasion to take cyclic sums such as those which arise on the left of Jacobi's identity. So if F is a function of three vector fields (or of three elements of any set) with values in some vector space (for example in the space of vector fields) we will define the cyclic sum $\mathcal{C}yc\ F$ by

$$\mathcal{C}yc\ F(X, Y, Z) := F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y).$$

With this definition Jacobi's identity becomes

$$\mathcal{C}yc\ [X, [Y, Z]] = 0. \tag{9.6}$$

9.13 A general version of Weil's formula.

Let W and Z be differentiable manifolds, let I denote an interval on the real line containing the origin, and let

$$\phi : W \times I \to Z$$

be a smooth map. We let $\phi_t : W \to Z$ be defined by

$$\phi_t(w) := \phi(w, t).$$

We think of ϕ_t as a one parameter family of maps from W to Z. We let ξ_t denote the tangent vector field along ϕ_t. In more detail:

$$\xi_t : W \to TZ$$

is defined by letting $\xi_t(w)$ be the tangent vector to the curve $s \mapsto \phi(w, s)$ at $s = t$.

If σ is a differential form on Z of degree $k+1$, we let the expression $\phi_t^* i(\xi_t)\sigma$ denote the differential form on W of degree k whose value at tangent vectors $\eta_1, \dots, \eta_k$ at $w \in W$ is given by

$$\phi_t^* i(\xi_t)\sigma(\eta_1, \dots, \eta_k) := (i(\xi_t)(w))\sigma)(d(\phi_t)_w\eta_1, \dots, d(\phi_t)_w\eta_k). \tag{9.7}$$

It is only the combined expression $\phi_t^* i(\xi_t)\sigma$ which will have any sense in general: since ξ_t is not a vector field on Z, the expression $i(\xi_t)\sigma$ will not make sense as a stand alone object (in general).

Let σ_t be a smooth one-parameter family of differential forms on Z. The

$$\phi_t^*\sigma_t$$

is a smooth one parameter family of forms on W, which we can then differentiate with respect to t. The general form of Weil's formula is:

$$\frac{d}{dt}\phi_t^*\sigma_t = \phi_t^*\frac{d\sigma_t}{dt} + \phi_t^* i(\xi_t)d\sigma + d\phi_t^* i(\xi_t)\sigma. \tag{9.8}$$

Before proving the formula, let us note that it is functorial in the following sense: Suppose that that $F : X \to W$ and $G : Z \to Y$ are smooth maps, and that τ_t is a smooth family of differential forms on Y. Suppose that $\sigma_t = G^*\tau_t$ for all t. We can consider the maps

$$\psi_t : X \to Y, \quad \psi_t := G \circ \phi_t \circ F$$

and then the smooth one parameter familiy of differential forms

$$\psi_t^*\tau_t$$

on X. The tangent vector field ζ_t along ψ_t is given by

$$\zeta_t(x) = dG_{\phi_t(F(x))}\left(\xi_t(F(x))\right).$$

So

$$\psi_t^* i(\zeta_t)\tau_t = F^*\left(\phi_t^* i(\xi_t)G^*\tau_t\right).$$

Therefore, if we know that (9.8) is true for ϕ_t and σ_t, we can conclude that the analogous formula is true for ψ_t and τ_t.

Consider the special case of (9.8) where we take the one parameter family of maps

$$f_t : W \times I \to W \times I, \qquad f_t(w, s) = (w, s+t).$$

Let

$$G : W \times I \to Z$$

be the map ϕ, and let

$$F : W \to W \times I$$

be the map

$$F(w) = (w, 0).$$

Then

$$(G \circ f_t \circ F)(w) = \phi_t(w).$$

Thus the functoriality of the formula (9.8) shows that we only have to prove it for the special case $\phi_t = f_t : W \times I \to W \times I$ as given above!

In this case, it is clear that the vector field ξ_t along ψ_t is just the constant vector field $\frac{\partial}{\partial s}$ evaluated at $(x, s+t)$. The most general differential (t-dependent) on $W \times I$ can be written as

$$ds \wedge a + b$$

where a and b are differential forms on W. (In terms of local coordinates $s, x^1, \ldots, x^n$ these forms a and b are sums of terms that have the expression

$$cdx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

where c is a function of s, t and x.) To show the full dependence on the variables we will write

$$\sigma_t = ds \wedge a(x,s,t)dx + b(x,s,t)dx.$$

With this notation it is clear that

$$\phi_t^* \sigma_t = ds \wedge a(x, s+t, t)dx + b(x, s+t, t)dx$$

and therefore

$$\frac{d\phi_t^*\sigma_t}{dt} = ds \wedge \frac{\partial a}{\partial s}(x, s+t, t)dx + \frac{\partial b}{\partial s}(x, s+t, t)dx$$
$$+ds \wedge \frac{\partial a}{\partial t}(x, s+t, t)dx + \frac{\partial b}{\partial t}(x, s+t, t)dx.$$

So

$$\frac{d\phi_t^*\sigma_t}{dt} - \phi_t^* \frac{d\sigma_t}{dt} = ds \wedge \frac{\partial a}{\partial s}(x, s+t, t)dx + \frac{\partial b}{\partial s}(x, s+t, t)dx.$$

Now

$$i\left(\frac{\partial}{\partial s}\right)\sigma_t = adx$$

so

$$\phi_t^* i(\xi_t)\sigma_t = a(x, s+t, t)dx.$$

Therefore

$$d\phi_t^* i(\xi_t)\sigma_t = ds \wedge \frac{\partial a}{\partial s}(x, s+t, t)dx + d_W(a(x, s+t, t)dx).$$

Also

$$d\sigma_t = -ds \wedge d_W(adx) + \frac{\partial b}{\partial s} ds \wedge dx + d_W bdx$$

so

$$i\left(\frac{\partial}{\partial s}\right)d\sigma_t = -d_W(adx) + \frac{\partial b}{\partial s}dx$$

and therefore

$$\phi_t^* i(\xi_t) d\sigma_t = -d_W a(x, s+t, t)dx + \frac{\partial b}{\partial s}(x, s+t, t)dx.$$

So

$$\begin{aligned} d\phi_t^* i(\xi_t)\sigma_t + \phi_t^* i(\xi_t) d\sigma_t &= ds \wedge \frac{\partial a}{\partial s}(x, s+t, t)dx + \frac{\partial b}{\partial s}(x, s+t, t)dx \\ &= \frac{d\phi_t^*\sigma_t}{dt} - \phi_t^* \frac{d\sigma_t}{dt} \end{aligned}$$

proving (9.8).

A special case of (9.8) is the following. Suppose that $W = Z = M$ and ϕ_t is a family of diffeomporphisms $f_t : M \to M$. Then ξ_t is given by

$$\xi_t(p) = v_t(f_t(p))$$

where v_t is the vector field

$$v_t(f(p)) = \frac{d}{dt} f_t(p).$$

In this case $i(v_t)\sigma_t$ makes sense, and so we can write (9.8) as

$$\frac{d\phi_t^*\sigma_t}{dt} = \phi_t^* \frac{d\sigma_t}{dt} + \phi_t^* D_{v_t}\sigma_t. \tag{9.9}$$

9.14 The Moser trick.

Let M be a differentiable manifold and let ω_0 and ω_1 be smooth k-forms on M. Let us examine the following question: does there exist a diffeomorphism $f : M \to M$ such that $f^*\omega_1 = \omega_0$?

Moser answers this kind of question by making it harder! Let ω_t, $0 \le t \le 1$ be a family of k-forms with $\omega_t = \omega_0$ at $t = 0$ and $\omega_t = \omega_1$ at $t = 1$. We look for a one parameter family of diffeomorphisms

$$f_t : M \to M, \quad 0 \le t \le 1$$

such that

$$f_t^*\omega_t = \omega_0 \tag{9.10}$$

and

$$f_0 = \mathrm{id}\,.$$

Let us differentiate (9.10) with respect to t and apply (9.9). We obtain

$$f_t^*\dot\omega_t + f_t^* D_{v_t}\omega_t = 0$$

where we have written $\dot\omega_t$ for $\frac{d\omega_t}{dt}$. Since f_t is required to be a diffeomorphism, this becomes the requirement that

$$D_{v_t}\omega_t = -\dot\omega_t. \tag{9.11}$$

Moser's method is to use "geometry" to solve this equation for v_t if possible. Once we have found v_t, solve the equations

$$\frac{d}{dt}f_t(p) = v_t(f_t(p), \qquad f_0(p) = p \tag{9.12}$$

for f_t. Notice that for p fixed and $\gamma(t) = f_t(p)$ this is a system of ordinary differential equations

$$\frac{d}{dt}\gamma(t) = v_t(\gamma(t)), \quad \gamma(0) = p.$$

The standard existence theorems for ordinary differential equations guarantees the existence of of a solution depending smoothly on p at least for $|t| < \epsilon$. One then must make some additional hypotheses that guarantee existence for all time (or at least up to $t = 1$). Two such additional hypotheses might be

- M is compact, or
- C is a closed subset of M on which $v_t \equiv 0$. Then for $p \in C$ the solution for all time is $f_t(p) = p$. Hence for p close to C solutions will exist for a long time. Under this condition there will exist a neighborhood U of C and a family of diffeomorphisms

  $$f_t : U \to M$$

  defined for $0 \leq t \leq 1$ such

  $$f_0 = \text{id}, \qquad f_{t|C} = \text{id}\,\forall t$$

  and (9.10) is satisfied.

We now give some illustrations of the Moser trick.

9.14.1 Volume forms.

Let M be a compact oriented connected n-dimensional manifold. Let ω_0 and ω_1 be nowhere vanishing n-forms with the same volume:

$$\int_M \omega_0 = \int_M \omega_1.$$

Moser's theorem asserts that under these conditions there exists a diffeomorphism $f : M \to M$ such that

$$f^*\omega_1 = \omega_0.$$

Moser invented his method for the proof of this theorem.

The first step is to choose the ω_t. Let

$$\omega_t := (1-t)\omega_0 + t\omega_1.$$

Since both ω_0 and ω_1 are nowhere vanishing, and since they yield the same integral (and since M is connected), we know that at every point they are either both positive or both negative relative to the orientation. So ω_t is nowhere vanishing. Clearly $\omega_t = \omega_0$ at $t = 0$ and $\omega_t = \omega_1$ at $t = 1$. Since $d\omega_t = 0$ as ω_t is an n-from on an n-dimensional manifold,

$$D_{v_t} = di(v_t)\omega_t$$

by Weil's formula. Also

$$\dot{\omega}_t = \omega_1 - \omega_0.$$

Since $\int_M \omega_0 = \int_M \omega_1$ we know that

$$\omega_0 - \omega_1 = d\nu$$

for some $(n-1)$-form ν. Thus (9.11) becomes

$$di(v_t)\omega_t = d\nu.$$

We will certainly have solved this equation if we solve the harder equation

$$i(v_t)\omega_t = \nu.$$

But this equation has a unique solution since ω_t is no-where vanishing. QED

9.14.2 Variants of the Darboux theorem.

We present these in Chapter 2.

9.14.3 The classical Morse lemma.

Let $M = \mathbb{R}^n$ and $\phi_i \in C^\infty(\mathbb{R}^n),\ i = 0.1$. Suppose that 0 is a non-degenerate critical point for both ϕ_0 and ϕ_1, suppose that $\phi_0(0) = \phi_1(0) = 0$ and that they have the same Hessian at 0, i.e. suppose that

$$\left(d^2\phi_0\right)(0) = \left(d^2\phi_1\right)(0).$$

The Morse lemma asserts that there exist neighborhoods U_0 and U_1 of 0 in $\mathbb{R}^n$ and a diffeomorphism

$$f : U_0 \to U_1, \quad f(0) = 0$$

such that

$$f^*\phi_1 = \phi_0.$$

Proof. Set

$$\phi_t := (1-t)\phi_0 + t\phi_1.$$

The Moser trick tells us to look for a vector field v_t with

$$v_t(0) = 0, \quad \forall\ t$$

and

$$D_{v_t}\phi_t = -\dot{\phi}_t = \phi_0 - \phi_1.$$

The function ϕ_t has a non-degenerate critical point at zero with the same Hessian as ϕ_0 and ϕ_1 and vanishes at 0. Thus for each fixed t, the functions

$$\frac{\partial \phi_t}{\partial x^i}$$

form a system of coordinates about the origin.

If we expand v_t in terms of the standard coordinates

$$v_t = \sum_j v_j(x,t)\frac{\partial}{\partial x^j}$$

then the condition $v_j(0,t) = 0$ implies that we must be able to write

$$v_j(x,t) = \sum_i v_{ij}(x,t)\frac{\partial \phi_t}{\partial x^i}.$$

for some smooth functions v_{ij}. Thus

$$D_{v_t}\phi_t = \sum_{ij} v_{ij}(x,t)\frac{\partial \phi_t}{\partial x^i}\frac{\partial \phi_t}{\partial x^j}.$$

Similarly, since $-\dot{\phi}_t$ vanishes at the origin together with its first derivatives, we can write

$$-\dot{\phi}_t = \sum_{ij} h_{ij}\frac{\partial \phi_t}{\partial x^i}\frac{\partial \phi_t}{\partial x^j}$$

where the h_{ij} are smooth functions. So the Moser equation $D_{v_t}\phi_t = -\dot{\phi}_t$ is satisfied if we set

$$v_{ij}(x,t) = h_{ij}(x,t).$$

Notice that our method of proof shows that if the ϕ_i depend smoothly on some paramters lying in a compact manifold S then the diffeomorphism f can be chosen so as to depend smoothly on $s \in S$.

In Section 5.11 we give a more refined version of this argument to prove the Hörmander-Morse lemma for genrating functions.

In differential topology books the classical Morse lemma is usually stated as follows:

Theorem 35 *Let M be a manifold and $\phi : M \to \mathbb{R}$ be a smooth function. Suppose that $p \in M$ is a non-degenerate critical point of ϕ and that the signature of $d^2\phi_p$ is $(k, n-k)$. Then there exists a system of coordinates $(U, x_1, \ldots, x_n)$ centered at p such that in this coordinate system*

$$\phi = c + \sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{n} x_i^2.$$

Proof. Choose any coordinate system $(W, y_1, \ldots y_n)$ centered about p and apply the previous result to

$$\phi_1 = \phi - c$$

and

$$\phi_0 = \sum h_{ij} y_i y_j$$

where

$$h_{ij} = \frac{\partial^2 \phi}{\partial y_i \partial y_i}(0).$$

This gives a change of coordinates in terms of which $\phi - c$ has become a non-degenerate quadratic form. Now apply Sylvester's theorem in linear algebra which says that a linear change of variables can bring such a non-degenerate quadratic form to the desired diagonal form.