Coherent Interferometry Algorithms for Photoacoustic Imaging

Habib Ammari†  Elie Bretin‡  Josselin Garnier§  Vincent Jugnon¶

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Abstract

The aim of this paper is to develop new Coherent Interferometry (CINT) algorithms to correct the effect of an unknown cluttered sound speed (random fluctuations around a known constant) on photoacoustic images. By back-propagating the correlations between the pre-processed pressure measurements, we show that we are able to provide statistically stable photo-acoustic images. The pre-processing is exactly in the same way as when we use the circular or the line Radon inversion to obtain photo-acoustic images. Moreover, we provide a detailed stability and resolution analysis of the new CINT-Radon algorithms. We also present numerical results to illustrate their performance and to compare them with Kirchhoff-Radon migration functions.

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1 Introduction

In photoacoustic imaging, optical energy absorption causes thermo-elastic expansion of the tissue, which leads to the propagation of a pressure wave. This signal is measured by transducers distributed on the boundary of the object, which in turn is used for imaging optical properties of the object.

In pure optical imaging, optical scattering in soft tissues degrades spatial resolution significantly with depth. Pure optical imaging is very sensitive to optical absorption but can only provide a spatial resolution of the order of 1 cm at cm depths. Pure conventional ultrasound imaging is based on the detection of the mechanical properties (acoustic impedance) in biological soft tissues. It can provide good spatial resolution because of its millimetric wavelength and weak scattering at MHz frequencies. The major contribution of photo-acoustic
imaging is to provide images of optical contrasts (based on the optical absorption) with the resolution of ultrasound [30]. Potential applications include breast cancer and vascular disease.

In photo-acoustic imaging, the absorbed energy density is related to the optical absorption coefficient distribution through a model for light propagation such as the diffusion approximation or the radiative transfer equation. Although the problem of reconstructing the absorption coefficient from the absorbed energy is nonlinear, efficient techniques can be designed, specially in the context of small absorbers [2]. The absorbed optical energy density is the initial condition in the acoustic wave equation governing the pressure. If the medium is acoustically homogeneous and has the same acoustic properties as the free space, then the boundary of the object plays no role and the absorbed energy density can be reconstructed from measurements of the pressure wave by inverting a spherical or a circular Radon transform [22, 23, 19, 20].

Recently, we have been interested in reconstructing initial conditions for the wave equation with constant sound speed in a bounded domain. In [1, 3], we developed a variety of inversion approaches which can be extended to the case of variable but known sound speed and can correct for the effect of attenuation on image reconstructions. However, the situation of interest for medical applications is the case where the sound speed is perturbed by an unknown clutter noise. This means that the speed of sound of the medium is randomly fluctuating around a known value. In this situation, waves undergo partial coherence loss [18] and the designed algorithms assuming a constant sound speed may fail.

Interferometric methods for imaging have been considered in [13, 26, 27]. Coherent interferometry (CINT) was introduced and analyzed in [7, 8]. While classical methods back-propagate the recorded signals directly, CINT is an array imaging method that first computes cross-correlations of the recorded signals over appropriately chosen space-frequency windows and then back-propagates the local cross-correlations. As shown in [7, 8, 9, 10], CINT deals well with partial loss of coherence in cluttered environments.

In the present paper, combining the CINT method for imaging in clutter together with a reconstruction approach for extended targets by Radon inversions, we propose CINT-Radon algorithms for photoacoustic imaging in the presence of random fluctuations of the sound speed. We show that these new algorithms provide statistically stable photoacoustic images. We provide a detailed analysis for their stability and resolution from a simple clutter noise model and numerically illustrate their performance.

The paper is organized as follows. In Section 2 we formulate the inverse problem of photoacoustics and describe the clutter noise considered for the sound speed. In Section 3 we recall the reconstruction using the circular Radon transform when the sound speed is constant and describe the original CINT algorithm. We then propose a new CINT approach which consists in pre-processing the data (in the same way as for the circular Radon inversion) before back-propagating their correlations. Section 4 is devoted to the stability analysis of this new algorithm. Section 5 adapts the results presented in Sections 3 and 4 to the case of a bounded domain. We make a parallel between the filtered back-projection of the circular Radon inversion in free space and of the line Radon inversion when we have boundary conditions. Both algorithms end with a back-propagation step. We propose to back-propagate the correlations between the (pre-processed) data in the same way as in Section 3. The paper ends with a short discussion.
2 Problem Formulation

In photoacoustics, laser pulses are focused into biological tissues. The delivered energy is absorbed and converted into heat, which generates thermo-elastic expansion, which in turn gives rise to the propagation of an ultrasonic pressure wave \(p(x, t)\) from a source term \(p_0(x)\):

\[
\begin{cases}
\frac{\partial^2 p}{\partial t^2}(x, t) - c(x)^2 \Delta p(x, t) = 0, \\
p(x, 0) = p_0(x), \\
\frac{\partial p}{\partial t}(x, 0) = 0.
\end{cases}
\]  
(2.1)

The pressure field is measured at the surface of a domain \(\Omega\) that contains the support of \(p_0\). The imaging problem is to reconstruct the initial value of the pressure \(p_0\) in a search domain from measurements of \(p\) on the whole boundary \(\partial\Omega\) of a domain \(\Omega\). Most of the reconstruction algorithms assume constant (or known) sound speed. However, in real applications, the sound speed is not perfectly known. It seems more relevant to consider that it fluctuates randomly around a known distribution. For simplicity, we consider the model with random fluctuations around a constant that we normalize to one:

\[
\frac{1}{c(x)^2} = 1 + \sigma_c \mu(x),
\]  
(2.2)

where \(\mu\) is a zero-mean stationary random process, \(x_c\) is the correlation length of the fluctuations of \(c(x)\) and \(\sigma_c\) is their standard deviation.

Throughout this paper, we restrict ourselves to the two-dimensional case and assume that \(\Omega\) is the unit disk with center at \(0\) and radius \(X_0 = 1\). The two-dimensional case is of particular interest in photoacoustic imaging with integrating line detectors [11].

3 Imaging Algorithms

We define the Fourier transform with respect to the second variable (time-variable) by

\[
\hat{f}(y, \omega) = \int_\mathbb{R} f(y, t) e^{i\omega t} \, dt, \quad f(y, t) = \frac{1}{2\pi} \int_\mathbb{R} \hat{f}(y, \omega) e^{-i\omega t} \, d\omega.
\]  
(3.1)

In free space, it is possible to link the measurements of the pressure waves \(p(y, t)\) on the boundary \(\partial\Omega\) to the circular Radon transform of the initial condition \(p_0(x)\) as follows (see, for instance, [14, p.682] and [15]):

\[
\mathcal{R}_\Omega[p_0](y, r) = \mathcal{W}[p](y, r), \quad y \in \partial\Omega, \quad r \in \mathbb{R}^+,
\]  
(3.2)

where the circular Radon transform is defined by

\[
\mathcal{R}_\Omega[p_0](y, r) := \int_{\mathbb{S}^1} r p_0(y + r \theta) \, d\sigma(\theta), \quad y \in \partial\Omega, \quad r \in \mathbb{R}^+,
\]

and

\[
\mathcal{W}[p](y, r) := 4r \int_0^r \frac{p(y, t)}{\sqrt{r^2 - t^2}} \, dt, \quad y \in \partial\Omega, \quad r \in \mathbb{R}^+.
\]

Here \(\mathbb{S}^1\) denotes the unit circle and \(d\sigma(\theta)\) is the surface measure on \(\mathbb{S}^1\).
Since $\Omega$ is assumed to be the unit disk with center at $0$ and radius $X_0 = 1$, it follows
that, in order to find $p_0$, we can use the following exact inversion formula [16]:

$$p_0(x) = \frac{1}{4\pi^2} R^*_\Omega BW[p](x), \quad x \in \Omega, \tag{3.3}$$

where $R^*_\Omega$ (the formal adjoint of the circular Radon transform) is a back-projection operator
given by

$$R^*_\Omega[f](x) = \int_{\partial \Omega} f(y, |x-y|) d\sigma(y) = \frac{1}{2\pi} \int_{\partial \Omega} \int_{\mathbb{R}} \hat{f}(y, \omega) e^{-i\omega|x-y|} d\omega d\sigma(y), \quad x \in \Omega,$$

with $d\sigma(y)$ being the surface measure on $\partial \Omega$, and $B$ is a filter defined by

$$B[g](y, t) = \int_{\mathbb{R}^2} g(y, r) \ln(|r^2 - t^2|) dr, \quad y \in \partial \Omega.$$

Note that (3.2) and (3.3) only hold for constant speed sound (in this case speed 1).

Note also that (3.3) reads in the Fourier domain as

$$p_0(x) = \frac{1}{2\pi} \int_{\partial \Omega} \int_{\mathbb{R}} \hat{BW}[p](y, \omega) e^{-i\omega|x-y|} d\omega d\sigma(y), \quad x \in \Omega,$$

where $p_0(x) = \frac{1}{4\pi^2} BW[p]$ (3.5) is the pre-processed data.

A second imaging function is to simply back-propagate the raw data (without any pre-
processing) [5]:

$$I_{KM}(x) = R^*_\Omega[p](x) = \frac{1}{2\pi} \int_{\partial \Omega} d\sigma(y) \int_{\mathbb{R}} d\omega \hat{p}(y, \omega)e^{-i\omega|x-y|}, \quad x \in \Omega. \tag{3.6}$$

Here, the subscript KM stands for Kirchhoff Migration. As will be seen later, this simplified
function is sufficient for localizing point sources in homogeneous media, but may fail for
imaging extended targets and/or in the presence of clutter noise.

When the sound speed varies as in (2.2), the phases of the measured waves $\hat{p}(\omega, y)$ are
shifted with respect to the deterministic, unperturbed phase because of the unknown clutter.
When the data are numerically back-propagated in the homogeneous medium with speed
of propagation equal to one, the phase terms do not compensate each other in (3.6) when
$x$ is a source location, which results in instability and loss of resolution. To correct this
effect, the idea of the original CINT algorithm is to back-propagate the space and frequency
correlations between the data [8]:

$$I_{C1}(x) = \frac{1}{(2\pi)^2} \int_{\partial \Omega \times \partial \Omega} \int d\sigma(y_1) d\sigma(y_2) \int_{\mathbb{R} \times \mathbb{R}} d\omega_1 d\omega_2 \int_{\mathbb{R} \times \mathbb{R}} d\omega_1 d\omega_2 \times \hat{p}(y_1, \omega_1)e^{-i\omega_1|x-y_1|}\hat{p}(y_2, \omega_2)e^{i\omega_2|x-y_2|}. \tag{3.7}$$
The CINT function is close to the KM function. In fact, if $\Omega_d \to \infty$ and $X_d \to \infty$, then $I_{C1}$ is the square of the Kirchhoff migration function:

$$I_{C1}(x) = |I_{KM}(x)|^2.$$ 

However the cut-off parameters $\Omega_d$ and $X_d$ play a crucial role. When one writes the CINT function in the time domain:

$$I_{C1}(x) = \int \int_{\partial \Omega \times \partial \Omega, |y_2 - y_1| \leq X_d} d\sigma(y_1) d\sigma(y_2) \times \frac{\Omega_d}{\pi} \int_R dt \text{sinc}(\Omega_d t) p(y_1, |y_1 - x| + t) p(y_2, |y_2 - x| - t),$$

it is clear that it forms the image by computing the local correlation of the recorded data in a time interval scaled by $1/\Omega_d$ and by superposing the back-propagated local correlations over pairs of receivers that are not further apart than $X_d$. The idea that motivates the form (3.7) of the CINT function is that, at nearby frequencies $\omega_1, \omega_2$ and nearby locations $y_1, y_2$, the random phase shifts of the data $\hat{p}(y_1, \omega_1), \hat{p}(y_2, \omega_2)$ are similar (say, correlated) so they cancel each other in the product $\hat{p}(y_1, \omega_1) \hat{p}(y_2, \omega_2)$. We then say that the data $\hat{p}(y_1, \omega_1)$ and $\hat{p}(y_2, \omega_2)$ are coherent. In such a case, the back-propagation of this product in the homogeneous medium should be stable. The purpose of the CINT imaging function is to keep in (3.7) the pairs $(y_1, \omega_1)$ and $(y_2, \omega_2)$ for which the data $\hat{p}(y_1, \omega_1)$ and $\hat{p}(y_2, \omega_2)$ are coherent and to remove the pairs that do not bring information. It then appears intuitive that the cut-off parameters $X_d$, resp. $\Omega_d$, should be of the order of the spatial (resp. frequency) correlation radius of the recorded data, and we will confirm this intuition in the following.

As will be shown later, $I_{C1}$ is quite efficient in localizing point sources in cluttered media but not in finding the true value of $p_0$ (up to a square). Moreover, when the support of the initial pressure $p_0$ is extended, $I_{C1}$ may fail in recovering a good photoacoustic image. We propose two things. First, in order to avoid numerical oscillatory effects, we replace the sharp cut-offs in the integral by Gaussian convolutions. Then instead of taking the correlations between the back-propagated raw data, we pre-process them like we do for the Radon inversion. We thus get the following CINT-Radon imaging function:

$$I_{CIR}(x) = \frac{1}{(2\pi)^2} \int_{\partial \Omega \times \partial \Omega} d\sigma(y_1) d\sigma(y_2) \int_{R \times R} d\omega_1 d\omega_2 e^{-\frac{\omega_2 - \omega_1}{2\pi}} e^{-\frac{|y_2 - y_1|^2}{2\pi^2}}$$

$$\times \hat{q}(y_1, \omega_1) e^{-i\omega_1 |x - y_1|} \hat{q}(y_2, \omega_2) e^{i\omega_2 |x - y_2|},$$

(3.8)

where $q$ is given by (3.5). Note again that, when $\Omega_d \to \infty$ and $X_d \to \infty$, $I_{CIR}$ is the square of the Kirchhoff-Radon migration function:

$$I_{CIR}(x) = |I_{KRM}(x)|^2.$$ 

The purpose of the CINT-Radon imaging function $I_{CIR}$ is to keep in (3.8) the pairs $(y_1, \omega_1)$ and $(y_2, \omega_2)$ for which the pre-processed data $\hat{q}(y_1, \omega_1)$ and $\hat{q}(y_2, \omega_2)$ are coherent and to remove the pairs that do not bring information.

Using the exact inversion formulas in [17] for the spherical Radon transform, the CINT-Radon algorithm presented in this paper can be generalized to the three-dimensional case provided that the measurements are taken on a sphere $\partial \Omega$. 


Note that because of the pre-processing step (3.5), which is in the same way as when one inverts the Radon transform in (3.3), we called, by abuse of language, the imaging functions $\mathcal{I}_{\text{KRM}}$ and $\mathcal{I}_{\text{CIR}}$ Kirchhoff-Radon migration and CINT-Radon, respectively.

4 Stability and Resolution Analysis

In this section we perform a resolution and stability analysis that clarifies the role of the cut-off parameters $\Omega_d$ and $X_d$ in the CINT-Radon imaging function. This analysis is based on arguments quite similar to those used in [6, 7, 8, 9, 10] to study the original CINT function. As far as we know, both the introduction of the CINT-Radon imaging functions and their resolution and stability analysis are new.

4.1 Random Travel Time Model

The random travel time model is a simple model used to describe wave propagation in a medium whose index of refraction has small fluctuations. The model is valid in the high-frequency regime when the fluctuations of the random medium have small amplitude and large correlation length (compared to the typical wavelength $\lambda_0$) [28, 25]. More exactly, the random travel time model is valid when the correlation length $\sigma_c$ and the standard deviation $\sigma_c$ of the fluctuations of the wave speed $c(x)$ satisfy conditions [6]:

$$x_c \ll X_0, \quad \sigma_c^2 \ll \frac{x_c^3}{X_0^3}, \quad \sigma_c^2 \frac{X_0^3}{x_c^3} \ll \frac{\lambda_0^2}{\sigma_c^2 x_c X_0} \lesssim 1. \quad (4.1)$$

Here $X_0$ is the radius of $\Omega$ (taken equal to 1 for simplicity). In these conditions the geometric optics approximation is valid, the perturbation of the amplitude of the wave is negligible, and the perturbation of the phase of the wave is of order one or larger (see [28, Chapter 6], [25, Chapter 1], and [6]).

We assume that conditions (4.1) for the random travel time model are satisfied. Therefore there is an error $\nu(x, y)$ between the theoretical travel time $\tau_0(x, y) = |y - x|$ with the background velocity equal to one and the real travel time $\tau(x, y)$ where $y$ is a point of the surface of the observation disk $\partial \Omega$ and $x$ is a point of the search domain (where we plot the image):

$$\tau(x, y) = \tau_0(x, y) + \nu(x, y).$$

The random process $\nu(x, y)$ is given by

$$\nu(x, y) = \frac{\sigma_c |y - x|}{2} \int_0^1 \mu \left( \frac{x + (y - x)s}{x_c} \right) ds. \quad (4.2)$$

This is the integral of the fluctuations of $1/c$ along the unperturbed, straight ray from $y$ to $x$. Assuming that the search domain is relatively small we can assume that $\nu$ depends only on the sensor position $y$ and we can neglect the variations of $\nu$ with respect to $x$. This is perfectly correct if we analyze the expectations and the variances of the imaging functions for a fixed test point $x$ (then the search domain is just one point). This is still correct if we analyze the covariance of the imaging function for a pair of test points $x$ and $x'$ that are close to each other (closer than the correlation radius $x_c$ of the clutter noise).
Note finally that the random travel time model can be used to model another type of noisy data which arises when the medium is homogeneous but the positions of the sensors are poorly characterized.

We assume that the random process \( \mu \) is a random process with Gaussian statistics, mean zero, and covariance function:

\[
\mathbb{E} [\sigma_c \mu \left( \frac{x}{x_c} \right) \sigma_c \mu \left( \frac{x'}{x_c} \right)] = \sigma_c^2 \exp \left( - \frac{|x - x'|^2}{2x_c^2} \right).
\]

Here \( \mathbb{E} \) stands for the expectation (mean value). Gaussian statistics means that, for any finite collection of points \((x_j)_{j=1, \ldots, n}\) in \( \mathbb{R}^2 \), the collection of random variables \((\mu(x_j))_{j=1, \ldots, n}\) has a multivariate normal distribution.

We will need the following lemma.

**Lemma 4.1** If \( x_c \ll X_0 \), then \( \nu \) is a random process with Gaussian statistics, mean zero, and covariance function:

\[
\mathbb{E}[\nu(y)\nu(y')] = \tau_c^2 \psi \left( \frac{|y - y'|}{x_c} \right), \quad \psi(r) = \frac{1}{r} \int_0^r \exp \left( - \frac{s^2}{2} \right) ds,
\]

where

\[
\tau_c^2 = \sqrt{2\pi \sigma_c^2/4}
\]

is the variance of the fluctuations of the travel times.

**Proof.** It follows by direct calculation from (4.2) that the random process \( \nu(y) \), where its variations with respect to \( x \) are neglected, has mean zero and covariance function

\[
\mathbb{E}[\nu(y)\nu(y')] = \frac{\sigma_c^2 |y - x| |y' - x|}{4} \int_0^1 ds \int_0^1 ds' \exp \left[ - \frac{|s(y - x) - s'(y' - x)|^2}{2x_c^2} \right]
\]

\[
= \frac{\sigma_c^2 |y - x| |y' - x|}{4} \int_0^1 ds \int_{-s}^{1-s} d\tilde{s} \exp \left[ - \frac{|s(y - y') + \tilde{s}(x - y')|^2}{2x_c^2} \right].
\]

Moreover, when \( X_0 \gg x_c \), we can extend the \( \tilde{s} \) integral in (4.5) to the entire real line. Integrating in \( \tilde{s} \) then gives

\[
\mathbb{E}[\nu(y)\nu(y')] = \frac{\sqrt{2\pi} \sigma_c^2 |y - x|}{4} \int_0^1 ds \exp \left[ - \frac{s^2}{2x_c^2} \left( |y - y'|^2 - \left( \frac{x - y'}{|x - y'} \cdot (y - y') \right)^2 \right) \right].
\]

If \( x \) is close to the center of the disk \( \Omega \), then \( y - y' \) is approximately orthogonal to \( x - y' \) and we obtain (4.3). \( \square \)

Using Gaussian statistics it is straightforward to compute the moments

\[
\mathbb{E}[e^{\omega \nu(y)}] = \exp \left( - \frac{\omega^2 \tau_c^2}{2} \right),
\]

\[
\mathbb{E}[e^{i\omega \nu(y) - i\omega' \nu(y')} ] = \exp \left( - \frac{(\omega - \omega')^2 \tau_c^2}{2} - \omega \omega' \tau_c^2 \left( 1 - \psi \left( \frac{|y - y'|}{x_c} \right) \right) \right).
\]

If, additionally, we assume that \( \omega, \omega' \simeq \omega_0 \), with

\[
\omega_0 \tau_c \gg 1,
\]

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then, using a second order Taylor expansion of $\psi$ and the assumption $|y - y'| \ll x_c$, we arrive at

$$E[e^{\omega \nu (y) - i \omega' \nu(y')}] \approx \exp \left( -\frac{(\omega - \omega')^2 \tau_c^2}{2} - \frac{|y - y'|^2}{2X_c^2} \right),$$

(4.8)

with

$$X_c^2 = \frac{3 \sigma_c^2}{\omega_0^2 \tau_c^2}.$$  

(4.9)

Note that the typical wavelength $\lambda_0 = 2\pi/\omega_0$.

From (4.8) we can see that $\tau_c^{-1}$ is the frequency coherence radius and $X_c$ is the spatial coherence radius of the recorded signals.

In the following sections it will be convenient to introduce a typical frequency $\omega_0$. Since we deal with a wave equation with initial conditions (2.1) and the speed of propagation is one, the typical wavelength for this problem is of the order of the diameter of the support of the function $p_0$ when it is smooth, or of the order of the scale of variations of the function $p_0$ when it is oscillating. Since the speed of propagation is one, the typical frequency $\omega_0$ is of the order of the reciprocal of the typical wavelength. Moreover, the bandwidth (i.e., the spectral width of the data) is also of the order of the typical frequency.

### 4.2 Kirchhoff-Radon Migration

Recall that the Kirchhoff-Radon migration function is

$$I_{\text{KRM}}(x) = \frac{1}{2\pi} \int_{\partial \Omega} d\sigma(y) \int_{\mathbb{R}} d\omega \hat{q}(y, \omega) e^{-i\omega|y - x|} = \int_{\partial \Omega} d\sigma(y) q(y, |y - x|),$$

(4.10)

where $q = \mathcal{B}\mathcal{W}[p]/(4\pi^2)$. The function applied to the perfect pre-processed data

$$q^{(0)} = \frac{1}{4\pi^2} \mathcal{B}\mathcal{W}[p^{(0)}]$$

(4.11)

is

$$I_{\text{KRM}}^{(0)}(x) = \frac{1}{2\pi} \int_{\partial \Omega} d\sigma(y) \int_{\mathbb{R}} d\omega \hat{q}^{(0)}(y, \omega) e^{-i\omega|y - x|} = \int_{\partial \Omega} d\sigma(y) q^{(0)}(y, |y - x|),$$

(4.12)

and it is equal to the initial condition $p_0(x)$. Here and below we denote by $p^{(0)}$ and $q^{(0)}$ the unperturbed data obtained when the medium is homogeneous ($\mu \equiv 0$).

We consider the random travel time model to describe the recorded data set:

$$q(y, t) = q^{(0)}(y, t - \nu(y)).$$

(4.13)

This is a rough approximate model that accounts only for random time delays, but calculations can be carried out in a rather simple manner that illustrate the dual role of the cut-off parameters $\Omega_d$ and $X_d$. Even though $q$ arises from $p$ via some time integration, the model (4.13) can be considered as a valid approximation since $p(y, t) \simeq p^{(0)}(y, t - \nu(y))$ with $\nu$ being independent of the time variable $t$, for any $t > 0$.

We first consider the expectation of the function. Using (4.6) we find that the following holds.
The variance that characterizes the amplitude of the fluctuations is
generation in random media and is sometimes called extinction \cite{25}. The approxima
tion (4.14) gives the order of magnitude depending on the typical frequency

\[ \omega \]

\[ \text{distance}. \] Applying Proposition A.1, we find that

\[ \text{Theorem 4.2} \]

\[ \text{Let } I_{\text{KRM}} \text{ be defined by (4.10). Consider a travel time model for the effec}
\]

\[ \text{t of clutter noise and assume that conditions (4.1) and (4.7) are satisfied. We have}
\]

\[ \mathbb{E}[I_{\text{KRM}}(x)] = \frac{1}{2\pi} \int_{\partial \Omega} d\sigma(y) \int_{\mathbb{R}} d\omega \bar{q}^{(0)}(y, \omega) \exp \left( -\frac{\omega^2 \tau^2}{2} \right) e^{-i\omega|y-x|}
\]

\[ \simeq \exp \left( -\frac{\omega_0^2 \tau^2}{2} \right) \|I_{\text{KRM}}(x)\|, \quad (4.14)
\]

with \( A \simeq B \) iff \( A = B(1 + o(1)) \). Here, \( \tau_c \), given by (4.4), is the variance of the fluctuations of the travel times.

Theorem 4.2 shows that the mean function undergoes a strong damping compared to the

\[ \text{unperturbed function } \hat{I}_{\text{KRM}}(x). \] This phenomenon is standard when studying wave propagation in random media and is sometimes called extinction \cite{25}. The approximation (4.14) gives the order of magnitude depending on the typical frequency \( \omega_0 \) (or equivalently, the
typical wavelength \( \lambda_0 = 2\pi/\omega_0 \))

The statistics of the fluctuations can be characterized by the covariance. Using (4.8) we have

\[ \mathbb{E}[I_{\text{KRM}}(x)I_{\text{KRM}}(x')] = \frac{1}{(2\pi)^2} \int_{\partial \Omega \times \partial \Omega} d\sigma(y_1) d\sigma(y_2) \int_{\mathbb{R} \times \mathbb{R}} d\omega_1 d\omega_2 \bar{q}^{(0)}(y_1, \omega_1) \bar{q}^{(0)}(y_2, \omega_2)
\]

\[ \times \exp \left( -\frac{(\omega_1 - \omega_2)^2 \tau^2}{2} - \frac{|y_1 - y_2|^2}{2X^2_c} \right) e^{-i\omega_1|y_1-x|} e^{i\omega_2|y_2-x'|}.
\]

The variance that characterizes the amplitude of the fluctuations is

\[ \text{Var}(I_{\text{KRM}}(x)) := \mathbb{E}[|I_{\text{KRM}}(x)|^2] - |\mathbb{E}[I_{\text{KRM}}(x)]|^2.
\]

We assume that \( X_c \ll X_0 \), that is, the correlation radius is smaller than the propagation
distance. Applying Proposition A.1, we find that

\[ \mathbb{E}[|I_{\text{KRM}}(x)|^2] = \frac{X_c}{(2\pi)^3 r_c} \int_{\partial \Omega} d\sigma(y_a) \int_{\mathbb{R}} d\omega_a \int_{Y_a^\perp} d\sigma(\kappa_a) \int_{\mathbb{R}} d\tau_a
\]

\[ \times \mathcal{W}_q(Y_a, \omega_a; \kappa_a, \tau_a) \exp \left( \frac{|\kappa_a - \omega_a|}{2X^2_c} \right) - \left( \tau_a - |Y_a - x| \right)^2),
\]

where \( \mathcal{W}_q \) is the Wigner transform of \( q^{(0)} \):

\[ \mathcal{W}_q(Y_a, \omega_a; \kappa_a, \tau_a) = \int d\lambda_a \int_{Y_a^\perp} d(y_a) q^{(0)}(Y_a + \frac{\lambda_a}{2}, \omega_a + \frac{\hbar a}{2}) \gamma^{(0)}(Y_a - \frac{\lambda_a}{2}, \omega_a - \frac{\hbar a}{2})
\]

\[ \times e^{i\kappa_a \cdot y_a - i\hbar_a \tau_a}, \quad (4.15)
\]

and \( Y_a^\perp \) is defined by

\[ Y_a^\perp = \{ y_a \in \mathbb{R}^2, \ Y_a - \frac{y_a}{2} \in \partial \Omega \text{ and } Y_a + \frac{y_a}{2} \in \partial \Omega \}. \quad (4.16)
\]

If we consider the regime in which \( \tau_c^{-1} \ll \omega_0 \) and \( X_c \ll X_0 \), then we get

\[ \mathbb{E}[|I_{\text{KRM}}(x)|^2] \simeq \frac{X_c}{(2\pi)^3 r_c} \int_{\partial \Omega} d\sigma(y_a) \int_{\mathbb{R}} d\omega_a \int_{Y_a^\perp} d\sigma(\kappa_a) \int_{\mathbb{R}} d\tau_a \mathcal{W}_q(Y_a, \omega_a; \kappa_a, \tau_a)
\]

\[ \simeq \frac{X_c}{2\pi \tau_c} \int_{\partial \Omega} d\sigma(y_a) \int_{\mathbb{R}} d\omega_a |q^{(0)}(Y_a, \omega_a)|^2,
\]

\[ 9 \]
Combining (4.17) together with (4.12) yields
\[ E[|I_{KRM}(x)|^2] \approx X_c \tau_c^{-1} \int_{\partial \Omega} d\sigma(y) \int_{\mathbb{R}} dtq(y, t)^2. \] (4.17)

Combining (4.17) together with (4.12) yields
\[ E[|I_{KRM}(x)|^2] \approx \left| I_{KRM}^{(0)}(x) \right|^2 \left( \frac{1}{\omega_0 \tau_c} \right) \left( \frac{X_c}{X_0} \right). \] (4.18)

Define the signal-to-noise ratio (SNR) by
\[ \text{SNR}_{KRM} = \frac{|E[I_{KRM}(x)]|}{\text{Var}(I_{KRM}(x))^{1/2}}. \] (4.19)

Using (4.18), it follows from Theorem 4.2 that the following result holds.

**Theorem 4.3** Under the same assumptions as those in Theorem 4.2, it follows that if \( \omega_0 \tau_c \gg 1 \) and \( X_c \ll X_0 \), then SNR_{KRM} is very small:
\[ \text{SNR}_{KRM} \ll 1. \]

Here, \( A \ll B \) iff \( A/B = o(1) \).

### 4.3 CINT-Radon

We consider the random travel time model (4.13). We first note that in (3.8) the coherence is maintained as long as \( \exp(i \omega_2 \nu(y_2) - i \omega_1 \nu(y_1)) \) is close to one. From (4.8) this requires that \( |\omega_1 - \omega_2| < \tau_c^{-1} \) and \( |y_1 - y_2| < X_c \). We can therefore anticipate that the cut-off parameters \( X_d \) and \( \Omega_d \) should be related to the coherence parameters \( X_c \) and \( \tau_c^{-1} \). In the following we study the role of the cut-off parameters \( X_d \) and \( \Omega_d \) for resolution and stability of the CINT-Radon function. For doing so, we compute the expectation and variance of the imaging function \( I_{CIR} \).

Using (4.8), we have
\[ E[I_{CIR}(x)] = \frac{1}{(2\pi)^2} \int_{\mathbb{R} \times \mathbb{R}} d\omega_1 d\omega_2 \int_{\partial \Omega \times \partial \Omega} d\sigma(y_1) d\sigma(y_2) \tilde{q}(y_1, \omega_1) \tilde{q}(y_2, \omega_2) \]
\[ \times e^{-i \omega_1 |x - y_1|} e^{i \omega_2 |x - y_2|} \exp \left( -\frac{|\omega_1 - \omega_2|^2}{2 \tau_c^2} \left( \frac{1}{\Omega_d^2} \right) - \frac{|y_1 - y_2|^2}{2} \left( \frac{1}{X_d^2} + \frac{1}{X_c^2} \right) \right), \] (4.20)

where \( q^{(0)} \) is the perfect pre-processed data defined by (4.11). Applying Proposition A.1, we obtain the following result.

**Theorem 4.4** Let \( I_{CIR} \) be defined by (3.8). Consider a travel time model for the effect of clutter noise and assume that conditions (4.1) and (4.7) are satisfied. Let \( X_c \) be defined by (4.9) with \( x_c \) being the correlation length of the clutter noise, \( \tau_c \) the variance of the fluctuations of the travel times, and \( \omega_0 \) the typical frequency. If \( X_c \ll X_0 \), where \( X_0 \) is the radius of \( \Omega \), then we have
\[ E[I_{CIR}(x)] = \frac{1}{(2\pi)^3} \left( \frac{1}{X_d^2} + \frac{1}{X_c^2} \right)^2 (\tau_c^2 + \frac{1}{\Omega_d^2}) \int_{\partial \Omega} d\sigma(Y_a) \int_{\mathbb{R}} d\omega_a \int_{\mathbb{R}^2} d\sigma(\kappa_a) \int d\tau_a \]
\[ \times W_q(Y_a, \omega_a; \kappa_a, \tau_a) \exp \left( -\frac{|\kappa_a - \omega_a \kappa|}{2(\frac{1}{X_d^2} + \frac{1}{X_c^2})} - \frac{(\tau_a - |Y_a - x|^2)^2}{2(\tau_c^2 + \frac{1}{\Omega_d^2})} \right), \] (4.21)
where $W_q$ is the Wigner transform of $q^{(0)}$ defined by (4.15).

Formula (4.21) shows that the coherent part (i.e., the expectation) of the CINT-Radon function is a smoothed version of the Wigner transform of the pre-processed data. It selects a band of directions ($\kappa_d$) and time delays ($\tau_a$) that are centered around the direction and the time delay between the search point $x$ and the point $Y_a$ of the sensor array.

We observe that:

- if $\Omega_{d} > \tau_{c}^{-1}$ and $X_d > X_c$ then the cut-off parameters $\Omega_{d}$ and $X_d$ have no influence on the coherent part of the function which does not depend on $(\Omega_{d}, X_d)$.

- if $\Omega_{d} < \tau_{c}^{-1}$ and $X_d < X_c$ then the cut-off parameters $\Omega_{d}$ and $X_d$ have an influence and reduce the resolution of the coherent part of the function (they enhance the smoothing of the Wigner transform).

We next compute the covariance of the CINT-Radon function in the regime $\Omega_{d} < \tau_{c}^{-1}$ and $X_d < X_c$. Let us consider four points $(y_{j})_{j=1,..,4}$ and four frequencies $(\omega_{j})_{j=1,..,4}$ in the bandwidth. Using the Gaussian property of the process $\nu(y)$ and the fact that $\psi(0) = 1$, we have

$$E[e^{i\omega_1 \nu(y_1) - i\omega_2 \nu(y_2)}e^{i\omega_3 \nu(y_3) - i\omega_4 \nu(y_4)}] = \exp\left\{ -\frac{\tau_{c}^{2}}{2} \left[ \sum_{j=1}^{4} \omega_{j}^{2} + 2\omega_{1}\omega_{2}\psi\left(\frac{|y_1 - y_2|}{x_c}\right) \right. \right.$$

$$+ 2\omega_{2}\omega_{3}\psi\left(\frac{|y_2 - y_3|}{x_c}\right) - 2\omega_{1}\omega_{2}\psi\left(\frac{|y_1 - y_2|}{x_c}\right) - 2\omega_{1}\omega_{3}\psi\left(\frac{|y_1 - y_3|}{x_c}\right)$$

$$\left. \left. - 2\omega_{2}\omega_{3}\psi\left(\frac{|y_2 - y_3|}{x_c}\right) - 2\omega_{3}\omega_{4}\psi\left(\frac{|y_3 - y_4|}{x_c}\right) \right] \right\},$$

If

$$\omega_1 = \omega_a + \frac{h_a}{2}, \quad \omega_2 = \omega_a - \frac{h_a}{2}, \quad \omega_3 = \omega_b + \frac{h_b}{2}, \quad \omega_4 = \omega_b - \frac{h_b}{2},$$

$$y_1 = Y_a + \frac{y_a}{2}, \quad y_2 = Y_a - \frac{y_a}{2}, \quad y_3 = Y_b + \frac{y_b}{2}, \quad y_4 = Y_b - \frac{y_b}{2},$$

with $h_a, h_b = O(\Omega_d), \omega_a, \omega_b = O(\omega_0), Y_a, Y_b = O(X_0)$, and $y_a, y_b = O(X_d)$, then

$$E[e^{i\omega_1 \nu(y_1) - i\omega_2 \nu(y_2)}e^{i\omega_3 \nu(y_3) - i\omega_4 \nu(y_4)}] \approx \exp\left\{ -\frac{\tau_{c}^{2}}{2} \left[ h_a^2 + h_b^2 - 2h_a h_b \psi\left(\frac{|Y_a - Y_b|}{x_c}\right) \right] \right\},$$

(note that $X_d \ll x_c$ since $X_d \leq X_c$ and $X_c \ll x_c$) and therefore,

$$E[I_{\text{CHR}}(x)I_{\text{CHR}}(x')] = \frac{1}{(2\pi)^2} \int_{\Theta_{d}} d\sigma(Y_a) \int_{\mathbb{R}} dh_a \int_{\Theta_{d}} d\sigma(Y_b) \int_{\mathbb{R}} dh_b \int_{\mathbb{R}} d\sigma(y_a)$$

$$\times Q(Y_a, h_a, y_a; x)Q(Y_b, h_b, y_b; x')$$

$$\times \exp\left\{ -\frac{(h_a^2 + h_b^2)}{2} \left( \frac{1}{\Omega_d^2} + \tau_{c}^{2} \right) - \frac{|y_a|^2 + |y_b|^2}{2X_d^2} \right\}$$

$$\times \exp\left[ \tau_{c}^{2} h_a h_b \psi\left(\frac{|Y_a - Y_b|}{x_c}\right) \right].$$
Using (4.20) and the fact $\Omega_d < \tau_c^{-1}$ and $X_d < X_c$, we arrive at

$$E[I_{\text{CIR}}(x)] = \frac{1}{(2\pi)^2} \int_{\partial \Omega} d\sigma(Y_a) \int_{R} dh_a \int_{Y_a} d\sigma(y_a) \hat{Q}(Y_a, h_a, y_a; x) \exp \left(-\frac{h_a^2}{2\Omega_d^2} - \frac{|y_a|^2}{2X_d^2}\right),$$

so that

$$E[I_{\text{CIR}}(x)I_{\text{CIR}}(x')] \simeq E[I_{\text{CIR}}(x)]E[I_{\text{CIR}}(x')] = 1.$$

Therefore, the following theorem, where the SNR is defined analogously to (4.19), holds.

**Theorem 4.5** With the same notation and assumptions as those in Theorem 4.4, it follows that if $\Omega_d < \tau_c^{-1} \ll \omega_0$ and $X_d < X_c \ll X_0$, then we have

$$\text{SNR}_{\text{CIR}} \gg 1.$$

To conclude, we notice that the values of the parameters $X_d \simeq X_c$ and $\Omega_d \simeq \tau_c^{-1}$ achieve a good trade-off between resolution and stability. When taking smaller values $\Omega_d < \tau_c^{-1}$ and $X_d < X_c$ one increases the signal-to-noise ratio but one also reduces the resolution. In practice, these parameters are difficult to estimate directly from the data, so it is better to determine them adaptively, by optimizing over $\Omega_d$ and $X_d$ the quality of the resulting image. This is exactly what is done in adaptive CINT [9].

### 4.4 Two Particular Cases

We now discuss the following two particular cases:

(i) If we take $X_d \to 0$ then the CINT function has the form

$$I_{\text{CIR}}(x) = \frac{1}{(2\pi)^2} \int_{R \times R} d\omega_1 d\omega_2 \int_{\partial \Omega} d\sigma(y) e^{-\frac{\omega_1^2 - \omega_2^2}{2\Omega_d^2}} \hat{q}(y, \omega_1) e^{-i\omega_1|x-y|} \hat{q}(y, \omega_2) e^{i\omega_2|y-x|}.$$  \hspace{1cm} (4.22)

This case could correspond to the situation in which $X_c$ is very small, which means that the signals recorded by different sensors are so noisy that they are independent from each other.

It is easy to see that, for any $\Omega_d$, (4.22) is equal to

$$I_{\text{CIR}}(x) = \frac{\Omega_d}{\sqrt{2\pi}} \int_{\partial \Omega} d\sigma(Y_a) \int_{R} dt |q(Y_a, |Y_a - x| + t)|^2 \exp \left(-\frac{\Omega_d^2 t^2}{2}\right).$$

Moreover, if $\Omega_d$ is large, then (4.22) is (approximately) equivalent to the incoherent matched field function

$$I_{\text{CIR}}(x) \simeq \int_{\partial \Omega} d\sigma(Y_a) |q(Y_a, |Y_a - x|)|^2.$$ \hspace{1cm} (4.23)

This function is called "incoherent" because we take out the phase of the data by looking at the square modulus only.

(ii) If we take $X_d \to \infty$ then the CINT function has the form

$$I_{\text{CIR}}(x) = \frac{1}{(2\pi)^2} \int_{R \times R} d\omega_1 d\omega_2 \int_{\partial \Omega \times \partial \Omega} d\sigma(y_1) d\sigma(y_2) e^{-\frac{\omega_1^2 - \omega_2^2}{2\Omega_d^2}} \hat{q}(y_1, \omega_1) e^{-i\omega_1|y_1 - x|} \hat{q}(y_2, \omega_2) e^{i\omega_2|y_2 - x|}.$$ \hspace{1cm} (4.24)
This case could correspond to the situation in which \( X_c \) is very large, which means that the signals recorded by different sensors are strongly correlated with one another. This is a typical weak clutter noise case.

For any \( \Omega_d \), the function (4.24) is equal to

\[
I_{CIR}(x) = \frac{\Omega_d}{\sqrt{2\pi}} \int d\omega \left| \int_{\partial\Omega} d\sigma(y) q(y, |y - x| + t) \exp \left( -\frac{\Omega_d^2 t^2}{2} \right) \right|^2.
\]

If \( \Omega_d \) is large, then (4.22) is equivalent to the coherent matched field function (or square KRM function):

\[
I_{CIR}(x) \approx \left| \int_{\partial\Omega} d\sigma(y) q(y, |y - x|) \right|^2.
\]

In contrast to (4.23), this function is called "coherent" since we take into account the phase of the data.

5 CINT-Radon Algorithm in a Bounded Domain

When considering photoacoustics in a bounded domain, we developed in [3] an approach involving the line Radon transform of the initial condition. We will consider homogeneous Dirichlet conditions:

\[
\begin{align*}
\frac{\partial^2 p}{\partial t^2}(x, t) - c(x)^2 \Delta p(x, t) &= 0, \quad x \in \Omega, \\
p(x, 0) &= p_0(x), \quad \frac{\partial p}{\partial n}(x, 0) = 0, \quad x \in \Omega, \\
p(y, t) &= 0, \quad y \in \partial \Omega.
\end{align*}
\]

Here, \( \Omega \) is not necessarily a disk. Let \( n \) denote the outward normal to \( \partial \Omega \). When \( c(x) = 1 \), we can express the line Radon transform of the initial condition \( p_0(x) \) in terms of the Neumann measurements \( \partial_n p(y, t) = n(y) \cdot \nabla p(y, t) \) on \( \partial \Omega \times [0, T] \):

\[
\mathcal{R}[p_0](\theta, s) = \mathcal{W}[\partial_n p](\theta, s),
\]

where the line Radon transform is defined by

\[
\mathcal{R}[p_0](\theta, r) := \int_R p_0(r\theta + s\theta^\perp) ds, \quad \theta \in \mathbb{S}^1, \quad r \in \mathbb{R},
\]

and

\[
\mathcal{W}[g](\theta, s) := \int_0^T dt \int_{\partial\Omega} d\sigma(x) g(x, t) H(x \cdot \theta + t - s), \quad \theta \in \mathbb{S}^1, \quad s \in \mathbb{R}.
\]

Here \( H \) denotes the Heaviside function. We then invert the Radon transform using the filtered back-projection algorithm

\[
p_0 = \mathcal{R}^* \mathcal{W}[\partial_n p],
\]

where \( \mathcal{R}^* \) is the formal adjoint Radon transform:

\[
\mathcal{R}^*[f](x) = \frac{1}{2\pi} \int_{\mathbb{S}^1} d\sigma(\theta) f(\theta, x \cdot \theta) = \frac{1}{(2\pi)^2} \int_{\mathbb{S}^1} d\sigma(\theta) \int_R d\omega \hat{f}(\theta, \omega) e^{-i\omega x \cdot \theta}, \quad x \in \Omega,
\]
and $B$ is a ramp filter
\[ B[g](\theta, s) = \frac{1}{4\pi} \int_{\mathbb{R}} d\omega |\omega| g(\theta, \omega) e^{-i\omega s}, \quad \theta \in S^1. \]

Here, the hat stands for the Fourier transform (3.1) in the second (shift) variable. In the Fourier domain, the inversion reads
\[ p_0(x) = \frac{1}{(2\pi)^2} \int_{S^1} d\sigma(\theta) \int_{\mathbb{R}} d\omega \hat{B}[\hat{\partial_n}p](\theta, \omega) e^{-i\omega x \cdot \theta}, \quad x \in \Omega. \]

Therefore, a natural idea to extend the CINT imaging to bounded media is to consider the imaging function:
\[
I_{\text{CIR}}(x) := \frac{1}{(2\pi)^4} \int_{S^1 \times S^1} d\sigma(\theta_1) d\sigma(\theta_2) \int_{\mathbb{R} \times \mathbb{R}} d\omega_1 d\omega_2 e^{\frac{-(\omega_2 - \omega_1)^2}{2\Theta^2}} e^{-\frac{|\theta_2 - \theta_1|^2}{2\Theta^2}}
\times \hat{B}[\hat{\partial_n}p](\theta_1, \omega_1)e^{-i\omega_1 x \cdot \theta_1} \hat{B}[\hat{\partial_n}p](\theta_2, \omega_2)e^{i\omega_2 x \cdot \theta_2},
\]

where $\Theta$ is an angular cut-off parameter and plays the same role as $X_d$ in the previous sections. Using the arguments developed before, it would be possible to perform a stability and resolution analysis for $I_{\text{CIR}}$ given by (5.1).

### 6 Numerical Illustrations

In this section we present numerical experiments to illustrate the performance of the CINT-Radon algorithms and to compare them with the Kirchhoff-Radon imaging function. We also compare $I_{\text{CIR}}$ and $I_{\text{KM}}$ in the case where the clutter noise has high frequency components.

The wave equation (direct problem) can be rewritten as a first order partial differential equation:
\[ \partial_t P = AP + BP, \]

where
\[ P = \left( \begin{array}{c} p \\ \partial_t p \end{array} \right), \quad A = \left( \begin{array}{cc} 0 & 1 \\ \Delta & 0 \end{array} \right), \quad \text{and} \quad B = \left( \begin{array}{cc} 0 & 0 \\ (c^2 - 1)\Delta & 0 \end{array} \right). \]

This equation is solved on the box $Q = (-2, 2)^2$ containing $\Omega$. We use a splitting spectral Fourier approach [12] coupled with a perfectly matched layer (PML) technique [4] to simulate a free outgoing interface on $\partial Q$. The operator $A$ is computed exactly in the Fourier space while the operator $B$ is treated explicitly with a finite difference method and a PML formulation to simulate a free outgoing interface on $\partial Q$ in the case of free space. The inverse circular Radon formula is discretized as in [16]. Realizations of sound speed fluctuations are computed by the Fourier method (i.e., by filtering a discrete white noise). We first generate on a grid a white Gaussian noise. Then we filter the Gaussian noise in the Fourier domain by applying a low-pass filter. The parameters of the filter are linked to the correlation length of the clutter noise [21].

We consider two situations: point targets and extended targets. Since the case of interest in our analysis is when the correlation length of the clutter noise is comparable to or larger than the typical wavelength, the simulated clutter noise in the case of point targets can have high frequencies while in the case of extended targets we should use a clutter noise with only
low frequencies. For point targets, only $I_{CI}$ and $I_{KM}$ are used. The imaging functions $I_{CIR}$ and $I_{KMR}$ do not improve image reconstruction.

In Figure 1, we consider $p_0$ to be the sum of 6 Dirac masses. In order to solve the wave equation with initial data $p = p_0$ and $\partial p / \partial t = 0$ at $t = 0$, we compute the solution to the wave equation with zero initial data but with source term given by $df/dt \times$ the sum of Dirac masses, with

$$f(t) = \cos(2\pi \omega_0 t) t \delta_\omega \exp(-\pi t^2 \delta_\omega^2), \quad \text{with} \quad \delta_\omega = 10, \quad \omega_0 = 3 \delta_\omega.$$

The 6 Dirac masses emit therefore pulses of the form $f$, with $f$ being an approximation of the derivative of a Dirac mass in time at $t = 0$.

We use the random velocity $c_1$, visualized in Figure 2. It is a realization of a Gaussian process with Gaussian covariance function. Figures 3 and 4 present the pressure $p(y, t)$ computed without and with clutter noise, and the reconstruction of the source locations obtained by the Kirchhoff migration function $I_{KM}$. Figures 3 and 4 illustrate that in the presence of clutter noise, $I_{KM}$ becomes very instable and fails to really localize the targets. The images obtained by $I_{CI}$ are plotted in Figure 5 and compared to those obtained by $I_{KM}$. Note that $I_{CI}$ presents better stability properties when $X_d$ and $\Omega_d$ become small as predicted by the theory.

![Figure 1: Positions of the sensors.](image)

We now consider the case of extended targets and test the imaging function $I_{CIR}$. We use the random velocity $c_2$ shown in Figure 2. Reconstructions of the initial pressure obtained by $I_{KRM}$ are plotted in Figures 6, 7, 9, and 10 for two different configurations: the Shepp-Logan phantom and a line. These figures clearly highlight the fact that the clutter noise significantly affects the reconstruction. For example, in Figure 10 the whole line is not found.

On the other hand, plots of $I_{CIR}$ presented in Figures 8 and 11 provide more stable reconstructions of $p_0(x)$. However, note that choosing small values of the parameters $X_d$ and $\Omega_d$ can affect the reconstruction in the sense that $I_{CIR}$ becomes very different from the expected value, $p_0^0$, when $X_d$ and $\Omega_d$ tend to zero. This is a manifestation of the trade-off between resolution and stability discussed in Section 4.

In the case of a bounded domain, we consider the low frequency cluttered speed of Figure 2, on a square medium, with homogeneous Dirichlet conditions. We illustrate the performance of $I_{CIR}$ on the Shepp-Logan phantom. Figure 12 shows the reconstruction.
Figure 2: Left: random velocity $c_1$ with high frequencies; right: random velocity $c_2$ with low frequencies.

Figure 3: Test1 (a): measured data $p(y, t)$ with (right) and without (left) clutter noise. The coordinates are the pixel indices. The abscissa is time and the ordinate is arclength on $\partial \Omega$.

Figure 4: Test1 (b): source localization using Kirchhoff migration $I_{KM}$ with (right) and without (left) clutter noise.
Figure 5: Test1 (c): source localization using the standard CINT function $I_{\text{CIT}}$, with parameters $X_d$ and $\Omega_d$ given by $X_d = 0.25, 0.5, 1$ (from left to right) and $\Omega_d = 25, 50, 100$ (from top to bottom).
using the inverse Radon transform algorithm in the case with boundary conditions. We notice that the outer interface appears twice. In fact, $I_{CIR}$ can correct this effect. Figure 13 shows results with a colormap saturated at 80% of their maximum values for different values of $\Theta_d$ and $\Omega_d$. Here, $\Theta_d$ and $\Omega_d$ are the cut-off parameters in (5.1) and (3.8), respectively. The imaging function $I_{CIR}$ can get the outer interface correctly. Moreover, if the colormap is saturated appropriately (here at 80%), $I_{CIR}$ reconstructs as well the inside of the target with the good contrast.

![Figure 6: Test2 (a): measured data $p(y,t)$ with (right) and without clutter noise for the Shepp-Logan phantom. The abscissa is time and the ordinate is arclength on $\partial \Omega$.](image)

![Figure 7: Test2 (b): reconstruction of $p_0(x)$ using $I_{KRM}$ with (right) and without (left) clutter noise.](image)

7 Conclusion

In this paper we have introduced new CINT-Radon type imaging functions in order to correct the effect on photoacoustic images of random fluctuations of the background sound speed around a known constant value. We have provided a stability and resolution analysis of the
Figure 8: Test2 (c): source localization using $\mathcal{I}_\text{CIR}$, with $X_d$ and $\Omega_d$ given by $X_d = 0.5, 1, 2$ (from left to right) and $\Omega_d = 50, 100, 200$ (from top to bottom).

Figure 9: Test3 (a): measured data $p(y, t)$ with (right) and without (left) clutter noise for a line target. The abscissa is time and the ordinate is arclength on $\partial\Omega$. 
Figure 10: Test3 (b): reconstruction of $p_0$ using $\mathcal{I}_{\text{KRM}}$ with (right) and without (left) clutter noise.

Figure 11: Test3 (c): source localization using $\mathcal{I}_{\text{CIR}}$, with $X_d$ and $\Omega_d$ given by $X_d = 0.5, 1, 2$ (from left to right) and $\Omega_d = 50, 100, 200$ (from top to bottom).
proposed algorithms and found the values of the cut-off parameters which achieve a good trade-off between resolution and stability. We have presented numerical reconstructions of both small and extended targets and compared our algorithms with Kirchhoff-Radon migration functions. The CINT-Radon imaging functions give better reconstruction than Kirchhoff-Radon migration, specially for extended targets in the presence of a low-frequency clutter noise.

A Calculation of the second-order moments

**Proposition A.1** Let $X_d, \Omega_d > 0$. Let us consider

$$I(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R} \times \mathbb{R}} d\omega_1 d\omega_2 \int_{\partial \Omega \times \partial \Omega} d\sigma(y_1)d\sigma(y_2)q(y_1, y_2; y_1^0, y_2^0)$$

$$\times e^{-i\omega_1|y_1-x|} e^{i\omega_2|y_2-x|} \exp \left( -\frac{(\omega_1 - \omega_2)^2}{2\Omega_d^2} - \frac{|y_1 - y_2|^2}{2X_d^2} \right).$$

(A.1)

If $X_d \ll X_0$, then we have

$$I(x) \simeq \frac{X_d \Omega_d}{(2\pi)^3} \int_{\partial \Omega} d\sigma(Y_a) \int_{\mathbb{R}} d\omega \int_{Y_a^+} d\sigma(\kappa_a) \int_{\mathbb{R}} d\tau \omega_a \kappa_a$$

$$\times W_q(Y_a, \omega_a; \kappa_a, \tau_a) \exp \left( -\frac{X_d^2 |\kappa_a - \omega_a \frac{x - Y_a}{|x - Y_a|}|^2}{2\Omega_d^2} - \frac{\Omega_d^2 (\tau_a - |Y_a - x|)^2}{2} \right).$$

(A.2)

where $W_q$ is the Wigner transform of $q(y)$ defined by (4.15) and $Y_{a,\perp}$ is defined by (4.16).

**Proof.** Let $Y_{a,\perp}$ be defined by (4.16). Using the change of variables

$$\omega_1 = \omega_a + \frac{h_a}{2}, \quad \omega_2 = \omega_a - \frac{h_a}{2}, \quad y_1 = Y_a + \frac{y_a}{2}, \quad y_2 = Y_a - \frac{y_a}{2},$$

$$\omega_1 = \omega_a + \frac{h_a}{2}, \quad \omega_2 = \omega_a - \frac{h_a}{2}, \quad y_1 = Y_a + \frac{y_a}{2}, \quad y_2 = Y_a - \frac{y_a}{2},$$

Figure 12: Reconstruction of an extended target using line Radon transform in the case of imposed boundary conditions.
Figure 13: Extended target reconstruction with boundary conditions using $I_{CIR}$, with $\Theta_d$ and $\Omega_d$ given by $\Theta_d = 1, 3, 6$ (from top to bottom) and $\Omega_d = 50, 100, 200$ (from left to right). Colormaps are saturated at 80% of the maximum values of the images.
the function $I(x)$ can be written as

$$I(x) = \frac{1}{(2\pi)^2} \int_{\partial \Omega} d\sigma(Y_a) \int_{\mathbb{R}} dh_a \int_{Y_a^\perp} d\sigma(y_a) \hat{Q}(Y_a, h_a, y_a; x)$$

$$\times \exp \left( -\frac{h_a^2}{2\Omega^2_a} - \frac{|y_a|^2}{2X_a^2} \right),$$

with

$$\hat{Q}(Y_a, h_a, y_a; x) = \int_{\mathbb{R}} d\omega_a \hat{q}(0)(Y_a + \frac{y_a}{2}, \omega_a + \frac{h_a}{2}) \hat{q}^*(0)(Y_a - \frac{y_a}{2}, \omega_a - \frac{h_a}{2})$$

$$\times e^{-i(\omega_a + \frac{h_a}{2} - \frac{y_a}{2} - x) \cdot Y_a} + e^{i(\omega_a - \frac{h_a}{2} - \frac{y_a}{2} + x) \cdot Y_a}.$$

We have assumed that $X_d$ is much smaller than $X_0 = 1$, so that $y_1 - y_2$ is approximately orthogonal to $(y_1 + y_2)/2$ when $y_1, y_2 \in \partial \Omega$ and $|y_1 - y_2| \leq X_d$. Then

$$Y_a^\perp \simeq \{ y_a \in \mathbb{R}^2, Y_a \cdot y_a = 0 \},$$

and

$$\hat{Q}(Y_a, h_a, y_a; x) \simeq \int_{\mathbb{R}} d\omega_a \int_{\mathbb{R}} d\tau_a \int_{Y_a^\perp} d\sigma(\kappa_a) \mathcal{W}_q(Y_a, \omega_a; \kappa_a, \tau_a)$$

$$\times e^{i(\omega_a \frac{x - Y_a}{|x - Y_a|} - \kappa_a) \cdot y_a + i(\tau_a - |x - Y_a|)h_a},$$

where $\mathcal{W}_q$ is the Wigner transform of $q(0)$ defined by (4.15). Therefore, we get

$$I(x) \simeq \frac{X_d\Omega_d}{(2\pi)^3} \int_{\partial \Omega} d\sigma(Y_a) \int_{\mathbb{R}} d\omega_a \int_{Y_a^\perp} d\sigma(\kappa_a) \int_{\mathbb{R}} d\tau_a$$

$$\times \mathcal{W}_q(Y_a, \omega_a; \kappa_a, \tau_a)$$

$$\times \exp \left( -\frac{X_d^2|\kappa_a - \omega_a(\frac{x - Y_a}{|x - Y_a|} - \frac{Y_a}{|Y_a|})|^2}{2} - \frac{\Omega^2_a(\tau_a - |x - Y_a|)^2}{2} \right).$$

Since, for any $s$,

$$\mathcal{W}_q(Y_a, \omega_a; \kappa_a + sY_a, \tau_a) = \mathcal{W}_q(Y_a, \omega_a; \kappa_a, \tau_a),$$

we obtain the desired result. □

References


