Quantitative invertibility of random matrices: a combinatorial perspective

Vishesh Jain

Massachusetts Institute of Technology

UIC Computer Science Theory Seminar

October 9, 2019
Motivation I: The strong circular law

Figure 1: Eigenvalue plots for randomly generated $5000 \times 5000$ matrices using Rademacher random variables (left) and Gaussian random variables (right). Figure by P.M. Wood
Motivation I: The strong circular law

- Fix a complex random variable $\xi$ with $\mathbb{E}[\xi] = 0$ and $\mathbb{E}[|\xi|^2] = 1$.
- For each $n$, let $N_n$ be an $n \times n$ random matrix each of whose entries is an i.i.d. copy of $\xi$. 

Theorem (Strong circular law. Tao and Vu, 2008)

Almost surely, as $n \to \infty$, the rescaled ESD

$$\mu_1 = \frac{1}{n} \sum_{j=1}^n \delta_{\lambda_j(1/\sqrt{n} N_n)}$$

converges weakly to the uniform measure on the complex unit disc.


Vishesh Jain (MIT)
Motivation I: The strong circular law

- Fix a complex random variable $\xi$ with $\mathbb{E}[\xi] = 0$ and $\mathbb{E}[|\xi|^2] = 1$.
- For each $n$, let $N_n$ be an $n \times n$ random matrix each of whose entries is an i.i.d. copy of $\xi$.

**Theorem (Strong circular law. Tao and Vu, 2008)**

*Almost surely, as $n \to \infty$, the rescaled ESD*

\[
\mu \frac{1}{\sqrt{n}} N_n := \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j(\frac{1}{\sqrt{n}} N_n)}
\]

*converges weakly to the uniform measure on the complex unit disc.*


Motivation II: Numerical linear algebra

- Solving $A_n x = b$, where $A_n$ is an $n \times n$ matrix is one of the most fundamental tasks in computational linear algebra. In principle, this can be solved efficiently and exactly using Gaussian elimination.
Motivation II: Numerical linear algebra

- Solving $A_n x = b$, where $A_n$ is an $n \times n$ matrix is one of the most fundamental tasks in computational linear algebra. In principle, this can be solved efficiently and exactly using Gaussian elimination.

- However, since computers can only store numbers with finite bit precision, errors produced due to rounding can lead to very inaccurate solutions.

Vishesh Jain (MIT)
Solving $A_n x = b$, where $A_n$ is an $n \times n$ matrix is one of the most fundamental tasks in computational linear algebra. In principle, this can be solved efficiently and exactly using Gaussian elimination.

However, since computers can only store numbers with finite bit precision, errors produced due to rounding can lead to very inaccurate solutions.

In the worst-case, theoretical guarantees show that we may need $O(n)$ bits of precision. However, in practice, we typically obtain accurate solutions using much less precision.
Solving $A_n x = b$, where $A_n$ is an $n \times n$ matrix is one of the most fundamental tasks in computational linear algebra. In principle, this can be solved efficiently and exactly using Gaussian elimination.

However, since computers can only store numbers with finite bit precision, errors produced due to rounding can lead to very inaccurate solutions.

In the worst-case, theoretical guarantees show that we may need $O(n)$ bits of precision. However, in practice, we typically obtain accurate solutions using much less precision.

**Question**

Why do numerical linear algebra algorithms work so well in practice?
A connecting theme

The least singular value of an $n \times n$ matrix $M_n$ is defined by

$$s_n(M_n) := \inf_{v \in \mathbb{S}^{n-1}} \| M_n v \|_2.$$
A connecting theme

The least singular value of an $n \times n$ matrix $M_n$ is defined by

$$s_n(M_n) := \inf_{v \in S^{n-1}} \| M_n v \|_2.$$

Quantitative invertibility problem

What is the probability that $s_n(M_n)$ is smaller than $\eta$?
A connecting theme

The least singular value of an $n \times n$ matrix $M_n$ is defined by

$$s_n(M_n) := \inf_{v \in S^{n-1}} \| M_n v \|_2.$$ 

Quantitative invertibility problem

What is the probability that $s_n(M_n)$ is smaller than $\eta$?

- For the strong circular law, well-known reductions show that we need to consider $M_n = z \sqrt{n} \cdot \text{Id}_n - N_n$ for fixed $z \in \mathbb{C}$ (Girko, 1984; Bai, 1997).
A connecting theme

The least singular value of an $n \times n$ matrix $M_n$ is defined by

$$s_n(M_n) := \inf_{v \in S^{n-1}} \| M_n v \|_2.$$ 

Quantitative invertibility problem

What is the probability that $s_n(M_n)$ is smaller than $\eta$?

- For the strong circular law, well-known reductions show that we need to consider $M_n = z \sqrt{n} \cdot \text{Id}_n - N_n$ for fixed $z \in \mathbb{C}$ (Girko, 1984; Bai, 1997).

- For numerical linear algebra, we need to consider $M_n = A_n + N_n$, where $N_n$ represents the random noise in the system (Spielman and Teng, 2001).
The quantitative invertibility problem

- Asked in various forms by many researchers in numerical analysis (von Neumann et al., Birkhoff et al., Smale, Demmel...)

- Early work e.g. Edelman (1988), Sankar, Spielman, and Teng (2006) focused on the case when the randomness is i.i.d. Gaussian; proofs relied on explicit computations/symmetry properties and do not generalize.

For numerical analysis, the case of discrete noise is especially important. For universality phenomena in random matrix theory, such as the strong circular law, we want to work with minimal possible assumptions on the distribution of the entries.
The quantitative invertibility problem

- Asked in various forms by many researchers in numerical analysis (von Neumann et al., Birkhoff et al., Smale, Demmel...)

- Early work e.g. Edelman (1988), Sankar, Spielman, and Teng (2006) focused on the case when the randomness is i.i.d. Gaussian; proofs relied on explicit computations/symmetry properties and do not generalize.

- For numerical analysis, the case of discrete noise is especially important.
The quantitative invertibility problem

- Asked in various forms by many researchers in numerical analysis (von Neumann et al., Birkhoff et al., Smale, Demmel...)

- Early work e.g. Edelman (1988), Sankar, Spielman, and Teng (2006) focused on the case when the randomness is i.i.d. Gaussian; proofs relied on explicit computations/symmetry properties and do not generalize.

- For numerical analysis, the case of discrete noise is especially important.

- For universality phenomena in random matrix theory, such as the strong circular law, we want to work with minimal possible assumptions on the distribution of the entries.
A motivating conjecture

**Conjecture (Spielman and Teng, ICM 2002)**

*If the entries of $M_n$ are independent Rademacher random variables, then there exists some $c \in (0, 1)$ such that for all $\eta \geq 0$,*

$$\Pr \left( s_n(M_n) \leq \eta \right) \leq \sqrt{n\eta} + c^n.$$
A motivating conjecture

**Conjecture (Spielman and Teng, ICM 2002)**

*If the entries of $M_n$ are independent Rademacher random variables, then there exists some $c \in (0, 1)$ such that for all $\eta \geq 0$,*

$$\Pr (s_n(M_n) \leq \eta) \leq \sqrt{n\eta} + c^n.$$  

- Resolved (up to overall constant) in the celebrated work of Rudelson and Vershynin (2007) for $M_n$ with i.i.d. subgaussian entries with mean 0.
A motivating conjecture

Conjecture (Spielman and Teng, ICM 2002)

*If the entries of $M_n$ are independent Rademacher random variables, then there exists some $c \in (0, 1)$ such that for all $\eta \geq 0$,*

\[
\Pr \left( s_n(M_n) \leq \eta \right) \leq \sqrt{m} \eta + c^n.
\]

- Resolved (up to overall constant) in the celebrated work of Rudelson and Vershynin (2007) for $M_n$ with i.i.d. subgaussian entries with mean 0. Extended by Rebrova and Tikhomirov (2015) for $M_n$ with i.i.d. entries with mean 0 and finite variance.
A motivating conjecture

**Conjecture (Spielman and Teng, ICM 2002)**

*If the entries of $M_n$ are independent Rademacher random variables, then there exists some $c \in (0, 1)$ such that for all $\eta \geq 0$,*

$$
\Pr \left( s_n(M_n) \leq \eta \right) \leq \sqrt{n} \eta + c^n.
$$

- Resolved (up to overall constant) in the celebrated work of Rudelson and Vershynin (2007) for $M_n$ with i.i.d. subgaussian entries with mean 0. Extended by Rebrova and Tikhomirov (2015) for $M_n$ with i.i.d. entries with mean 0 and finite variance.

**Question**

What about other models of random matrices such as non-centered entries, symmetric matrices, non-i.i.d. entries, adjacency matrices of random graphs...?
Aside: Qualitative invertibility is already difficult

- Even the qualitative invertibility of random matrices i.e. the probability that $M_n$ is non-invertible is a highly non-trivial question and a very active area of research.
Aside: Qualitative invertibility is already difficult

- Even the qualitative invertibility of random matrices i.e. the probability that $M_n$ is non-invertible is a highly non-trivial question and a very active area of research.

- As an illustration, consider random Rademacher matrices. These are singular with probability at least $2^{-n}$ (which is the probability that the first two rows are equal).
Aside: Qualitative invertibility is already difficult

- Even the qualitative invertibility of random matrices i.e. the probability that $M_n$ is non-invertible is a highly non-trivial question and a very active area of research.

- As an illustration, consider random Rademacher matrices. These are singular with probability at least $2^{-n}$ (which is the probability that the first two rows are equal).

- It was a long-standing conjecture that this probability is actually $(1/2 + o_n(1))^n$. This was only recently resolved by Tikhomirov (2018), improving on intermediate results of Bourgain, Vu, and Wood (2010), Tao and Vu (2007), Kahn, Komlós, and Szemerédi (1995), Komlós (1967).
Aside: Qualitative invertibility is already difficult

- Despite this progress, qualitative invertibility questions for most discrete random matrix models are wide open!
Aside: Qualitative invertibility is already difficult

- Despite this progress, qualitative invertibility questions for most discrete random matrix models are wide open!

- For instance, in the next simplest model of **symmetric** random Rademacher matrices, the best known upper bound is only of the form $2^{-\sqrt{n}}$ Campos, Mattos, Morris, and Morrison (2019), improving intermediate bounds of Ferber and J. (2018) and Vershynin (2014).
Aside: Qualitative invertibility is already difficult

- Despite this progress, qualitative invertibility questions for most discrete random matrix models are wide open!

- For instance, in the next simplest model of symmetric random Rademacher matrices, the best known upper bound is only of the form $2^{-\sqrt{n}}$ Campos, Mattos, Morris, and Morrison (2019), improving intermediate bounds of Ferber and J. (2018) and Vershynin (2014).

- For combinatorial models, such as adjacency matrices of regular (di)graphs, the situation is much more dire – bounds of the form $n^{-1}$ are not known except in special cases and even $o_n(1)$ bounds were only very recently obtained e.g. Huang (2018), Mészáros (2018), Nguyen and Wood (2018), Cook (2017), Landon, Sosoe, and Yau (2016)...
Existing techniques for quantitative invertibility

- **High-dimensional geometric methods**, pioneered by Rudelson and Vershynin (2007, 2008)
Existing techniques for quantitative invertibility

- **High-dimensional geometric methods**, pioneered by Rudelson and Vershynin (2007, 2008)
  - Give nearly optimal bounds in many cases.
  - Unclear how to extend to general complex matrices (important for circular law), non-centered matrices (important for numerical linear algebra)...

  - Work for complex matrices and non-centered matrices.
  - Give much weaker bounds e.g. cannot get an upper bound better than \(2^{-\log_2(n)}\) on the singularity probability of random Rademacher matrices.
  - Use heavy machinery from additive combinatorics e.g. Frieman's inverse theorem.

For both of these methods, there are significant obstacles when dealing with dependent entries.
Existing techniques for quantitative invertibility

- **High-dimensional geometric methods**, pioneered by Rudelson and Vershynin (2007, 2008)
  - Give nearly optimal bounds in many cases.
  - Unclear how to extend to general complex matrices (important for circular law), non-centered matrices (important for numerical linear algebra)...


  Use heavy machinery from additive combinatorics e.g. Frieman's inverse theorem.

For both of these methods, there are significant obstacles when dealing with dependent entries.
Existing techniques for quantitative invertibility

- **High-dimensional geometric methods**, pioneered by Rudelson and Vershynin (2007, 2008)
  - Give nearly optimal bounds in many cases.
  - Unclear how to extend to general complex matrices (important for circular law), non-centered matrices (important for numerical linear algebra)...

  - Work for complex matrices and non-centered matrices.
  - Give much weaker bounds e.g. cannot get an upper bound better than $2^{-\log^2(n)}$ on the singularity probability of random Rademacher matrices.
  - Use heavy machinery from additive combinatorics e.g. Frieman’s inverse theorem.
Existing techniques for quantitative invertibility

- **High-dimensional geometric methods**, pioneered by Rudelson and Vershynin (2007, 2008)
  - Give nearly optimal bounds in many cases.
  - Unclear how to extend to general complex matrices (important for circular law), non-centered matrices (important for numerical linear algebra)...

  - Work for complex matrices and non-centered matrices.
  - Give much weaker bounds e.g. cannot get an upper bound better than $2^{-\log^2(n)}$ on the singularity probability of random Rademacher matrices.
  - Use heavy machinery from additive combinatorics e.g. Frieman’s inverse theorem.

For both of these methods, there are significant obstacles when dealing with dependent entries.
Summary

We introduce new tools and a new approach to the quantitative invertibility problem.
Our results (and goals)

Summary

We introduce new tools and a new approach to the quantitative invertibility problem.

- Our goal is to prove bounds of the form
  \[ \Pr (s_n(M_n) \leq \eta) \lesssim n^C \eta^\delta + \exp(-n^c) \]
  in a simple and unified way for very general random matrix models;
Our results (and goals)

Summary

We introduce new tools and a new approach to the quantitative invertibility problem.

- Our goal is to prove bounds of the form

\[
\Pr (s_n(M_n) \leq \eta) \lesssim n^C \eta^\delta + \exp(-n^c)
\]

in a simple and unified way for very general random matrix models; even in the case of symmetric random Rademacher matrices, bounds of this form are best known to date (Vershynin, 2011).
Our results (and goals)

Summary

We introduce new tools and a new approach to the quantitative invertibility problem.

- Our goal is to prove bounds of the form
  \[ \Pr (s_n(M_n) \leq \eta) \lesssim n^C \eta^\delta + \exp(-n^c) \]
  in a **simple and unified** way for very general random matrix models; even in the case of symmetric random Rademacher matrices, bounds of this form are best known to date (Vershynin, 2011).

- Our method works for complex matrices, non-centered matrices, symmetric matrices, combinatorial matrices...
Our results (and goals)

Summary
We introduce new tools and a new approach to the quantitative invertibility problem.

- Our goal is to prove bounds of the form
  \[ \text{Pr} (s_n(M_n) \leq \eta) \lesssim n^C \eta^\delta + \exp(-n^c) \]
  in a **simple and unified** way for very general random matrix models; even in the case of symmetric random Rademacher matrices, bounds of this form are best known to date (Vershynin, 2011).

- Our method works for complex matrices, non-centered matrices, symmetric matrices, combinatorial matrices...

- For many of these models, our methods provide the first bounds of this form.
Recall that the operator norm of an $n \times n$ matrix $M_n$ is defined by

$$\sup_{v \in S^{n-1}} \| M_n v \|_2.$$
Our Results - Non centered complex matrices

Recall that the operator norm of an $n \times n$ matrix $M_n$ is defined by

$$\sup_{\mathbf{v} \in \mathbb{S}^{n-1}} \| M_n \mathbf{v} \|_2.$$ 

For the two motivating applications discussed at the start of this talk, the previous best-known result is:


Let $F_n$ be a fixed $n \times n$ complex matrix with operator norm $O(n^B)$. If the entries of $N_n$ are i.i.d. complex random variables with mean 0 and variance 1, then for all $\eta \geq 0$,

$$\Pr \left( s_n (F_n + N_n) \leq \eta \right) \lesssim n^C \eta^{\delta \eta} + n^{-\omega(1)}.$$ 

Vishesh Jain (MIT)  
Quantitative invertibility of random matrices  
October 9, 2019 12 / 34
Our results - Non centered complex matrices


Let $F_n$ be a fixed $n \times n$ complex matrix with operator norm $O(n^B)$. If the entries of $N_n$ are i.i.d. complex random variables with mean 0 and variance 1, then for all $\eta \geq 0$,

$$\Pr(s_n(F_n + N_n) \leq \eta) \lesssim n^C \eta^{\delta \eta} + n^{-\omega(1)}.$$
Our results - Non centered complex matrices


Let $F_n$ be a fixed $n \times n$ complex matrix with operator norm $O(n^B)$. If the entries of $N_n$ are i.i.d. complex random variables with mean 0 and variance 1, then for all $\eta \geq 0$,

$$\Pr(s_n(F_n + N_n) \leq \eta) \lesssim n^C \eta^{\delta \eta} + n^{-\omega(1)}.$$

Theorem (J., 2019+)

Under the same assumptions,

$$\Pr(s_n(F_n + N_n) \leq \eta) \lesssim n^C \eta^{\delta \eta} + \exp(-n^C).$$
Our results - combinatorially dependent entries

As a ‘proof of concept’, we consider the simplest such setting.

**Theorem (J., 2019+)**

Let $M_n$ be an $n \times n$ random matrix with independent rows in $\{0, 1\}^n$, each of which sums to $n/2$. Then, for any $\eta \geq 0$,

$$\Pr (s_n(M_n) \leq \eta) \preceq n^2 \eta + \exp(-n^c).$$
Our results - combinatorially dependent entries

As a ‘proof of concept’, we consider the simplest such setting.

**Theorem (J., 2019+)**

Let $M_n$ be an $n \times n$ random matrix with independent rows in $\{0, 1\}^n$, each of which sums to $n/2$. Then, for any $\eta \geq 0$,

$$\Pr (s_n(M_n) \leq \eta) \lesssim n^2 \eta + \exp(-n^c).$$

- First result of this form in a genuinely combinatorial setting. $\eta = 0$ case first proved by Ferber, J., Luh, and Samotij (2018+).
- Previously (Nguyen and Vu, 2012), not even an upper bound on the singularity probability of the form $\exp(-\log^2 n)$ was known.
Our results - symmetric random matrices

Theorem (Ferber and J., 2018)

Let $M_n$ be an $n \times n$ symmetric matrix whose above diagonal entries are independent Rademacher random variables. Then,

$$\Pr (s_n(M_n) = 0) \lesssim \exp(-n^{1/4}).$$


Extension to quantitative invertibility of complex heavy tailed non-centered symmetric matrices is work in progress.
Our results - symmetric random matrices

Theorem (Ferber and J., 2018)

Let $M_n$ be an $n \times n$ symmetric matrix whose above diagonal entries are independent Rademacher random variables. Then,

$$\Pr (s_n(M_n) = 0) \lesssim \exp(-n^{1/4}).$$


- Extension to quantitative invertibility of complex heavy tailed non-centered symmetric matrices is work in progress.
A proof template

- In the remainder of this talk, I will sketch some of the ideas and techniques that go into the proofs of our results.

- In order to motivate them, I will first present a high-level ‘proof template’.
In the remainder of this talk, I will sketch some of the ideas and techniques that go into the proofs of our results.

In order to motivate them, I will first present a high-level ‘proof template’.

To keep technicalities to a minimum, I will consider the case of the qualitative invertibility of i.i.d. matrices i.e. our goal in the next few slides will be to discuss how to obtain upper bounds on the probability that an i.i.d matrix is singular in the light-tailed setting.
Definition (Small ball probability)

The *r-ball probability* of a vector \( \mathbf{v} := (v_1, \ldots, v_n) \in \mathbb{R}^n \) with respect to a random variable \( \xi \) is defined by

\[
\rho_{r,\xi}(\mathbf{v}) := \sup_{x \in \mathbb{R}} \Pr \left( |v_1\xi_1 + \cdots + v_n\xi_n - x| \leq r \right),
\]

where \( \xi_1, \ldots, \xi_n \) are independent copies of \( \xi \).
Definition (Small ball probability)

The \( r \)-ball probability of a vector \( \mathbf{v} := (v_1, \ldots, v_n) \in \mathbb{R}^n \) with respect to a random variable \( \xi \) is defined by

\[
\rho_{r, \xi}(\mathbf{v}) := \sup_{x \in \mathbb{R}} \Pr \left( |v_1 \xi_1 + \cdots + v_n \xi_n - x| \leq r \right),
\]

where \( \xi_1, \ldots, \xi_n \) are independent copies of \( \xi \).

Examples (when \( \xi \) is Bernoulli):

- If \( \mathbf{v} = (10, 100, 1000, \ldots, 10^n) \), then \( \rho_{1/4, \xi}(\mathbf{v}) = 2^{-n} \).
A proof template: the anti-concentration phenomenon

**Definition (Small ball probability)**

The *r-ball probability* of a vector $\mathbf{v} := (v_1, \ldots, v_n) \in \mathbb{R}^n$ with respect to a random variable $\xi$ is defined by

$$
\rho_{r, \xi}(\mathbf{v}) := \sup_{x \in \mathbb{R}} \Pr \left( |v_1\xi_1 + \cdots + v_n\xi_n - x| \leq r \right),
$$

where $\xi_1, \ldots, \xi_n$ are independent copies of $\xi$.

**Examples (when $\xi$ is Bernoulli):**

- If $\mathbf{v} = (10, 100, 1000, \ldots, 10^n)$, then $\rho_{1/4, \xi}(\mathbf{v}) = 2^{-n}$.
- If $\mathbf{v} = (1, \ldots, 1)$, then $\rho_{1/4, \xi}(\mathbf{v}) = 2^{-n} \left( \left\lfloor \frac{n}{2} \right\rfloor \right) = \Theta \left( \frac{1}{\sqrt{n}} \right)$. 

A proof template: reduction to anti-concentrating normals

- Let $X_1, \ldots, X_n$ denote the rows of $M_n$. 

$S_n$ denotes the event that $M_n$ is singular.
A proof template: reduction to anti-concentrating normals

- Let $X_1, \ldots, X_n$ denote the rows of $M_n$.
- Let $S$ denote the event that $M_n$ is singular.
- Let $S_i$ denote the event that $X_i$ lies in the span of the other rows.
A proof template: reduction to anti-concentrating normals

- Let $X_1, \ldots, X_n$ denote the rows of $M_n$.
- Let $S$ denote the event that $M_n$ is singular.
- Let $S_i$ denote the event that $X_i$ lies in the span of the other rows.
- Since $1_S \leq 1_{S_1} + \cdots + 1_{S_n}$, 

$$ \Pr(S) \leq n \Pr(S_n). $$
Suppose we could prove the following:

Random normals anti-concentrate

Let $G_\rho$ denote the event that any $\mathbf{v} \in S^{n-1}$ which is orthogonal to $X_1, \ldots, X_{n-1}$ satisfies $\rho_{0,\xi}(\mathbf{v}) \leq \rho$. Then,

$$\Pr(G_\rho) \geq 1 - \exp(-n).$$
Suppose we could prove the following:

**Random normals anti-concentrate**

Let $G_\rho$ denote the event that any $v \in S^{n-1}$ which is orthogonal to $X_1, \ldots, X_{n-1}$ satisfies $\rho_0,\xi(v) \leq \rho$. Then,

$$\Pr(G_\rho) \geq 1 - \exp(-n).$$

Then, $\Pr(S_n) \leq \rho + \exp(-n)$, and we would be done.
Let $G_{\rho}$ denote the event that any $v \in S^{n-1}$ which is orthogonal to $X_1, \ldots, X_{n-1}$ satisfies $\rho_0,\xi(v) \leq \rho$. Then, $\Pr(G_{\rho}) \geq 1 - \exp(-n)$. 

Proof:

Let $v \in S^{n-1}$ have $\rho_0,\xi(v) \in [2^{-k-1}, 2^{-k}]$. By the independence of $X_1, \ldots, X_{n-1}$,

$\Pr \left( X_1 \cdot v = 0 \land \cdots \land X_{n-1} \cdot v = 0 \right) \leq (2^{-k})^{n-1}$.

Suppose we could show that the 'number' of $v \in S^{n-1}$ with $\rho_0,\xi(v) \in [2^{-k-1}, 2^{-k}]$ is at most $\left( \frac{2^k}{n^\gamma} \right)^n$. Then, by a union bound,

$\Pr(G_{\rho}) \leq \log(1/\rho) \sum_{k=0}^{\infty} \left( \frac{2^k}{n^\gamma} \right)^n \cdot (2^{-k})^{n-1} \leq \log(1/\rho) \sum_{k=0}^{\infty} 2^{-k} \leq \log(1/\rho) \rho_n^{-\gamma} \ll \exp(-n)$.
Let $\mathcal{G}_\rho$ denote the event that any $v \in S^{n-1}$ which is orthogonal to $X_1, \ldots, X_{n-1}$ satisfies $\rho_{0,\xi}(v) \leq \rho$. Then, $Pr(\mathcal{G}_\rho) \geq 1 - \exp(-n)$.

**Proof:** Union bound!
Proof template: Random normals anti-concentrate

Random normals anti-concentrate

Let $G_\rho$ denote the event that any $\mathbf{v} \in S^{n-1}$ which is orthogonal to $X_1, \ldots, X_{n-1}$ satisfies $\rho_{0,\xi}(\mathbf{v}) \leq \rho$. Then, $\Pr(G_\rho) \geq 1 - \exp(-n)$.

Proof: Union bound!

- Let $\mathbf{v} \in S^{n-1}$ have $\rho_{0,\xi}(\mathbf{v}) \in [2^{-k-1}, 2^{-k}]$. By the independence of $X_1, \ldots, X_{n-1}$,

$$\Pr(X_1 \cdot \mathbf{v} = 0 \land \cdots \land X_{n-1} \cdot \mathbf{v} = 0) \leq (2^{-k})^{n-1}.$$
Proof template: Random normals anti-concentrate

**Random normals anti-concentrate**

Let $\mathcal{G}_\rho$ denote the event that any $v \in \mathbb{S}^{n-1}$ which is orthogonal to $X_1, \ldots, X_{n-1}$ satisfies $\rho_{0,\xi}(v) \leq \rho$. Then, $\Pr(\mathcal{G}_\rho) \geq 1 - \exp(-n)$.

**Proof:** Union bound!

- Let $v \in \mathbb{S}^{n-1}$ have $\rho_{0,\xi}(v) \in (2^{-k-1}, 2^{-k}]$. By the independence of $X_1, \ldots, X_{n-1}$,

$$\Pr(X_1 \cdot v = 0 \land \cdots \land X_{n-1} \cdot v = 0) \leq (2^{-k})^{n-1}.$$ 

- **Suppose we could show that the ‘number’ of $v \in \mathbb{S}^{n-1}$ with $\rho_{0,\xi}(v) \in (2^{-k-1}, 2^{-k}]$ is at most $(2^k/n^\gamma)^n$.**
Proof template: Random normals anti-concentrate

Let $G_\rho$ denote the event that any $v \in S^{n-1}$ which is orthogonal to $X_1, \ldots, X_{n-1}$ satisfies $\rho_{0,\xi}(v) \leq \rho$. Then, $\Pr(G_\rho) \geq 1 - \exp(-n)$.

**Proof:** Union bound!

- Let $v \in S^{n-1}$ have $\rho_{0,\xi}(v) \in (2^{-k-1}, 2^{-k}]$. By the independence of $X_1, \ldots, X_{n-1}$,

  $$\Pr(X_1 \cdot v = 0 \land \cdots \land X_{n-1} \cdot v = 0) \leq (2^{-k})^{n-1}.$$

- Suppose we could show that the ‘number’ of $v \in S^{n-1}$ with $\rho_{0,\xi}(v) \in (2^{-k-1}, 2^{-k}]$ is at most $(2^k/n^{\gamma})^n$.

- Then, by a union bound,

  $$\Pr(\overline{G_\rho}) \leq \sum_{k=0}^{\log(1/\rho)} \left( \frac{2^k}{n^{\gamma}} \right)^n (2^{-k})^{n-1} \leq \sum_{k=0}^{\log(1/\rho)} 2^k n^{-\gamma n} \lesssim \log(1/\rho) \rho n^{-\gamma n} \ll \exp(-n).$$
Can we count?

Not directly, since $\mathbb{S}^{n-1}$ has uncountably many points! Overcoming this obstacle is the heart of the matter.
Can we count?

Not directly, since $\mathbb{S}^{n-1}$ has uncountably many points! Overcoming this obstacle is the heart of the matter.

- **High-dimensional geometric framework**
  - Discretize the unit sphere using a (Euclidean) net based on a very exact relation between Diophantine approximation and anti-concentration.

- Extension to genuinely complex setting is unclear.
- Requires strong control on the operator norm, which is often not available when dealing with non-centered entries.

- Additive combinatorial framework
  - Uses deep structure vs. randomness ideas from additive combinatorics to build an $\ell^\infty$-norm net on the sphere.
  - Only effective for $\rho \geq n - C$.
- Extensions to dependent models require much more work e.g. quadratic inverse Littlewood–Offord theory of Nguyen (2011).
Can we count?

Not directly, since $\mathbb{S}^{n-1}$ has uncountably many points! Overcoming this obstacle is the heart of the matter.

- **High-dimensional geometric framework**
  - Discretize the unit sphere using a (Euclidean) net based on a very exact relation between Diophantine approximation and anti-concentration.
  - Extension to genuinely complex setting is unclear.
  - Requires strong control on the operator norm, which is often not available when dealing with non-centered entries.
Can we count?

Not directly, since $S^{n-1}$ has uncountably many points! Overcoming this obstacle is the heart of the matter.

- **High-dimensional geometric framework**
  - Discretize the unit sphere using a (Euclidean) net based on a very exact relation between Diophantine approximation and anti-concentration.
  - Extension to genuinely complex setting is unclear.
  - Requires strong control on the operator norm, which is often not available when dealing with non-centered entries.

- **Additive combinatorial framework**
  - Uses deep structure vs. randomness ideas from additive combinatorics to build an $\infty$-norm net on the sphere.
Can we count?

Not directly, since $\mathbb{S}^{n-1}$ has uncountably many points! Overcoming this obstacle is the heart of the matter.

- **High-dimensional geometric framework**
  - Discretize the unit sphere using a (Euclidean) net based on a very exact relation between Diophantine approximation and anti-concentration.
  - Extension to genuinely complex setting is unclear.
  - Requires strong control on the operator norm, which is often not available when dealing with non-centered entries.

- **Additive combinatorial framework**
  - Uses deep structure vs. randomness ideas from additive combinatorics to build an $\infty$-norm net on the sphere.
  - Only effective for $\rho \geq n^{-c}$.
  - Extensions to dependent models require much more work e.g. quadratic inverse Littlewood–Offord theory of **Nguyen (2011)**.
We can count something!

Theorem (Ferber, J., Luh, and Samotij, 2018+; J., 2019+)

For all $\rho \geq \exp(-n^{c_1})$, the number of vectors $v \in \mathbb{Z}^n$ with

$$\|v\|_\infty \leq \exp(n^{c_2}); \quad \rho_{1,\xi}(v) \geq \rho$$

is at most

$$\left(\rho^{-1}/n^{0.5-\epsilon}\right)^n.$$  

The crucial parts in the above theorem are:

- The ability to take $\rho$ as small as $\exp(-n^{c_1})$ and $\|v\|_\infty$ as large as $\exp(n^{c_2})$.
- The presence of the $n^{-\gamma n}$ term in the upper bound.
We can count something!

**Theorem (Ferber, J., Luh, and Samotij, 2018+; J., 2019+)**

*For all $\rho \geq \exp(-n^{c_1})$, the number of vectors $v \in \mathbb{Z}^n$ with*

$$\|v\|_\infty \leq \exp(n^{c_2}); \quad \rho_{1,\xi}(v) \geq \rho$$

*is at most*

$$(\rho^{-1}/n^{0.5-\epsilon})^n.$$ 

- The crucial parts in the above theorem are:
  - The ability to take $\rho$ as small as $\exp(-n^{c_1})$ and $\|v\|_\infty$ as large as $\exp(n^{c_2})$.
  - The presence of the $n^{-\gamma n}$ term in the upper bound.
- Proved by a (perhaps surprisingly!) short and elementary double counting argument!
What can we do with the counting theorem?

Recall that we wanted to prove:

**Random normals anti-concentrate**

Except with probability \(\exp(-n)\), any \(\mathbf{v} \in \mathbb{S}^{n-1}\) which is orthogonal to \(X_1, \ldots, X_{n-1}\) i.e. for which \(\sum_{i=1}^{n-1} |X_i \cdot \mathbf{v}|^2 = 0\) satisfies \(\rho_{0,\xi}(\mathbf{v}) \leq \rho\).
What can we do with the counting theorem?

Recall that we wanted to prove:

**Random normals anti-concentrate**

Except with probability $\exp(-n)$, any $\mathbf{v} \in \mathbb{S}^{n-1}$ which is orthogonal to $X_1, \ldots, X_{n-1}$ i.e. for which $\sum_{i=1}^{n-1} |X_i \cdot \mathbf{v}|^2 = 0$ satisfies $\rho_{0,\xi}(\mathbf{v}) \leq \rho$.

Let us show how to use the counting theorem to prove:

**Random integer ‘approximate normals’ anti-concentrate**

Except with probability $\exp(-n)$, any non-zero $\mathbf{z} \in \mathbb{Z}^n$, $\|\mathbf{z}\|_\infty \leq \exp(n^c)$ for which

$$\sqrt{\sum_{i=1}^{n-1} |X_i \cdot \mathbf{z}|^2} \leq n^{1-2\epsilon}$$

satisfies $\rho_{1,\xi}(\mathbf{z}) \leq \rho$. 
What can we do with the counting theorem?

Random integer ‘approximate normals’ anti-concentrate

Except with probability \( \exp(-n) \), any non-zero \( z \in \mathbb{Z}^n \), \( \|z\|_\infty \leq \exp(nc) \) for which

\[
\sqrt{\sum_{i=1}^{n-1} |X_i \cdot z|^2} \leq n^{1-2\epsilon}
\]

satisfies \( \rho_{1,\xi}(z) \leq \rho \).

**Proof:** By a similar union bound to what we have seen.
What can we do with the counting theorem?

Random integer ‘approximate normals’ anti-concentrate

Except with probability \( \exp(-n) \), any non-zero \( z \in \mathbb{Z}^n \), \( \|z\|_{\infty} \leq \exp(n^c) \) for which

\[
\sqrt{\sum_{i=1}^{n-1} |X_i \cdot z|^2} \leq n^{1-2\epsilon}
\]

satisfies \( \rho_{1,\xi}(z) \leq \rho \).

**Proof:** By a similar union bound to what we have seen.

- Let \( z \) be an integer vector with \( \|z\|_{\infty} \leq \exp(n^c) \) and \( \rho_{1,\xi}(z) \in (2^{-k-1}, 2^{-k}] \).
- By independence, the probability that the vector \( (X_1 \cdot z, \ldots X_{n-1} \cdot z) \) lies in any fixed hypercube with side length 1 is at most \( (2^{-k})^{n-1} \).
What can we do with the counting theorem?

- **Key point:** Since the volume of the $n^{1-2\epsilon}$-ball in $\mathbb{R}^{n-1}$ is at most $(n^{1-2\epsilon}/\sqrt{n})^{(n-1)}$, 

Vishesh Jain (MIT)  
Quantitative invertibility of random matrices  
October 9, 2019  
25 / 34
What can we do with the counting theorem?

- **Key point:** Since the volume of the $n^{1-2\epsilon}$-ball in $\mathbb{R}^{n-1}$ is at most $(n^{1-2\epsilon}/\sqrt{n})^{(n-1)}$, the probability that $\sqrt{\sum_{i=1}^{n-1} |X_i \cdot z|^2} \leq n^{1-2\epsilon}$ is at most

  $$(2^{-k})^{n-1} \cdot n^{(0.5-2\epsilon)n}.$$
What can we do with the counting theorem?

- **Key point:** Since the volume of the $n^{1-2\epsilon}$-ball in $\mathbb{R}^{n-1}$ is at most $(n^{1-2\epsilon}/\sqrt{n})^{n-1}$, the probability that $\sqrt{\sum_{i=1}^{n-1} |X_i \cdot z|^2} \leq n^{1-2\epsilon}$ is at most

  $$(2^{-k})^{n-1} \cdot n^{(0.5-2\epsilon)n}. $$

- On the other hand, by the counting theorem, the number of such $z$ is at most

  $$\left(2^{k+1}\right)^n \cdot n^{(-0.5+\epsilon)n}. $$
What can we do with the counting theorem?

- **Key point:** Since the volume of the $n^{1-2\epsilon}$-ball in $\mathbb{R}^{n-1}$ is at most $(n^{1-2\epsilon}/\sqrt{n})^{(n-1)}$, the probability that $\sqrt{\sum_{i=1}^{n-1} |X_i \cdot z|^2} \leq n^{1-2\epsilon}$ is at most
  \[
  (2^{-k})^{n-1} \cdot n^{(0.5-2\epsilon)n}.
  \]

- On the other hand, by the counting theorem, the number of such $z$ is at most
  \[
  \left(2^{k+1}\right)^n \cdot n^{(-0.5+\epsilon)n}.
  \]

- Therefore, the contribution of such $z$ to the union bound is at most
  \[
  \left(2^{k+1}\right)^n n^{(-0.5+\epsilon)n} \cdot (2^{-k})^{n-1} n^{(0.5-2\epsilon)n} = 2^n \cdot 2^k \cdot n^{-\epsilon n}.
  \]
Getting around by rounding?

Recall that we wanted to prove:

**Random normals anti-concentrate**

Except with probability $\exp(-n)$, any $\mathbf{v} \in S^{n-1}$ which is orthogonal to $X_1, \ldots, X_{n-1}$ satisfies $\rho_{0,\xi}(\mathbf{v}) \leq \rho$. 

Can we reduce it to what we can prove?

**Random integer 'approximate normals' anti-concentrate**

Except with probability $\exp(-n)$, any non-zero $\mathbf{z} \in \mathbb{Z}^n$, $\|\mathbf{z}\|_\infty \leq \exp(n^c)$ for which

$$\sqrt{n-1} \sum_{i=1}^{n} |X_i \cdot \mathbf{z}|^2 \leq n^{1-2\epsilon}$$

satisfies $\rho_{1,\xi}(\mathbf{z}) \leq \rho$. 

Vishesh Jain (MIT)
Quantitative invertibility of random matrices
October 9, 2019 26 / 34
Getting around by rounding?

Recall that we wanted to prove:

**Random normals anti-concentrate**

Except with probability \( \exp(-n) \), any \( \mathbf{v} \in \mathbb{S}^{n-1} \) which is orthogonal to \( X_1, \ldots, X_{n-1} \) satisfies \( \rho_{0,\xi}(\mathbf{v}) \leq \rho \).

Can we reduce it to what we can prove?

**Random integer ‘approximate normals’ anti-concentrate**

Except with probability \( \exp(-n) \), any non-zero \( \mathbf{z} \in \mathbb{Z}^n \), \( \|\mathbf{z}\|_{\infty} \leq \exp(n^{c}) \) for which

\[
\sqrt{\sum_{i=1}^{n-1} |X_i \cdot \mathbf{z}|^2 \leq n^{1-2\epsilon}}
\]

satisfies \( \rho_{1,\xi}(\mathbf{z}) \leq \rho \).
Failed attempt: naïve rounding

- By rounding \( \mathbf{v} \in \mathbb{S}^{n-1} \) to the nearest integer multiple of \( 1/\sqrt{n} \), we obtain some \( \mathbf{z} \in \mathbb{Z}^n \) such that \( \| \mathbf{v} - (\mathbf{z}/\sqrt{n}) \|_2 \leq 1/2 \) i.e.

\[
\| \sqrt{n} \mathbf{v} - \mathbf{z} \|_2 \leq \sqrt{n}/2.
\]
Failed attempt: naïve rounding

- By rounding \( \mathbf{v} \in S^{n-1} \) to the nearest integer multiple of \( 1/\sqrt{n} \), we obtain some \( \mathbf{z} \in \mathbb{Z}^n \) such that \( \| \mathbf{v} - (\mathbf{z}/\sqrt{n}) \|_2 \leq 1/2 \) i.e.

  \[
  \| \sqrt{n} \mathbf{v} - \mathbf{z} \|_2 \leq \sqrt{n}/2.
  \]

- Let \( \tilde{\mathbf{M}}_{n-1} \) denote the matrix consisting of the first \( n - 1 \) rows of \( \mathbf{M}_n \).

- So, if \( \tilde{\mathbf{M}}_{n-1} \mathbf{v} = 0 \) and \( \| \tilde{\mathbf{M}}_{n-1} \| \leq \sqrt{n} \), we get
Failed attempt: naïve rounding

- By rounding $v \in S^{n-1}$ to the nearest integer multiple of $1/\sqrt{n}$, we obtain some $z \in \mathbb{Z}^n$ such that $\|v - (z/\sqrt{n})\|_2 \leq 1/2$ i.e.

$$\|\sqrt{n}v - z\|_2 \leq \sqrt{n}/2.$$ 

- Let $\tilde{M}_{n-1}$ denote the matrix consisting of the first $n-1$ rows of $M_n$.

- So, if $\tilde{M}_{n-1}v = 0$ and $\|\tilde{M}_{n-1}\| \leq \sqrt{n}$, we get

$$\sqrt{\sum_{i=1}^{n-1} |X_i \cdot z|^2} = \|\tilde{M}_{n-1}z\|_2 \leq \|\tilde{M}_{n-1}\| \cdot \|\sqrt{n}v - z\|_2 \leq \sqrt{n} \cdot \frac{1}{2} \sqrt{n} = \frac{n}{2}.$$
Failed attempt: naïve rounding

- By rounding $v \in S^{n-1}$ to the nearest integer multiple of $1/\sqrt{n}$, we obtain some $z \in \mathbb{Z}^n$ such that $\|v - (z/\sqrt{n})\|_2 \leq 1/2$ i.e.

$$\|\sqrt{n}v - z\|_2 \leq \sqrt{n}/2.$$  

- Let $\tilde{M}_{n-1}$ denote the matrix consisting of the first $n - 1$ rows of $M_n$.

- So, if $\tilde{M}_{n-1}v = 0$ and $\|\tilde{M}_{n-1}\| \leq \sqrt{n}$, we get

$$\sqrt{\sum_{i=1}^{n-1} |X_i \cdot z|^2} = \|\tilde{M}_{n-1}z\|_2 \leq \|\tilde{M}_{n-1}\| \cdot \|\sqrt{n}v - z\|_2$$

$$\leq \sqrt{n} \cdot \frac{1}{2} \sqrt{n} = \frac{n}{2}.$$  

- But we wanted something of the form $n^{1-2\epsilon}$ on the right hand side...
Successful attempt: non-trivial rounding available!

- We saw that naïve rounding ‘just’ fails. However, we are not trying to round any \( \mathbf{v} \in \mathbb{S}^{n-1} \) but only those with \( \rho_{0,\xi}(\mathbf{v}) \geq \rho \).
Successful attempt: non-trivial rounding available!

- We saw that naïve rounding ‘just’ fails. However, we are not trying to round any \( \mathbf{v} \in \mathbb{S}^{n-1} \) but only those with \( \rho_{0,\xi}(\mathbf{v}) \geq \rho \).

- But such vectors are already special and have a better-than-trivial integer approximation!

Proposition (Diophantine approximation vs. small-ball probability)

For \( \mathbf{v} \in \mathbb{S}^{n-1} \) with \( \rho_{0,\xi}(\mathbf{v}) \geq \rho \geq \exp(-n^{c}) \), there exists some \( \lambda \in [1, \exp(n^{c})] \) and non-zero \( \mathbf{z} \in \mathbb{Z}^{n} \) such that

\[
\|\lambda \mathbf{v} - \mathbf{z}\|_2 \leq n^{\delta}.
\]
Putting everything together

**Proposition (Diophantine approximation vs. small-ball probability)**

For $\mathbf{v} \in S^{n-1}$ with $\rho_{0,\xi}(\mathbf{v}) \geq \rho \geq \exp(-n^c)$, there exists some $\lambda \in [1, \exp(n^c)]$ and non-zero $\mathbf{z} \in \mathbb{Z}^n$ such that $\|\lambda \mathbf{v} - \mathbf{z}\|_2 \leq n^\delta$.

- Suppose $\tilde{M}_{n-1} \mathbf{v} = 0$ for $\mathbf{v} \in S^{n-1}$ satisfying $\rho_{0,\xi}(\mathbf{v}) \geq \rho$. 

Vishesh Jain (MIT)  
Quantitative invertibility of random matrices  
October 9, 2019 29 / 34
Proposition (Diophantine approximation vs. small-ball probability)

For $v \in S^{n-1}$ with $\rho_{0,\xi}(v) \geq \rho \geq \exp(-n^c)$, there exists some $\lambda \in [1, \exp(n^c)]$ and non-zero $z \in \mathbb{Z}^n$ such that $\|\lambda v - z\|_2 \leq n^\delta$.

- Suppose $\tilde{M}_{n-1}v = 0$ for $v \in S^{n-1}$ satisfying $\rho_{0,\xi}(v) \geq \rho$.
- Then, by the theorem, we can find a non-zero $z \in \mathbb{Z}^n$ with $\|\lambda v - z\|_2 \leq n^\delta$ and $\|z\|_\infty \leq \exp(n^c)$. 

Therefore, if $\|\tilde{M}_{n-1}\| \leq \sqrt{n}$, we get

$$\sum_{i=1}^{\sqrt{n}} |X_i \cdot z|^2 = \|\tilde{M}_{n-1}z\|^2 \leq \|\tilde{M}_{n-1}\| \cdot \|\lambda v - z\|^2 \leq n \cdot n^\delta \ll n^{1-2\epsilon}.$$ 

This is exactly what we have ruled out with high probability!
Proposition (Diophantine approximation vs. small-ball probability)

For $\mathbf{v} \in S^{n-1}$ with $\rho_{0,\xi}(\mathbf{v}) \geq \rho \geq \exp(-n^c)$, there exists some $\lambda \in [1, \exp(n^c)]$ and non-zero $\mathbf{z} \in \mathbb{Z}^n$ such that $\|\lambda \mathbf{v} - \mathbf{z}\|_2 \leq n^{\delta}$.

- Suppose $\tilde{M}_{n-1} \mathbf{v} = 0$ for $\mathbf{v} \in S^{n-1}$ satisfying $\rho_{0,\xi}(\mathbf{v}) \geq \rho$.
- Then, by the theorem, we can find a non-zero $\mathbf{z} \in \mathbb{Z}^n$ with $\|\lambda \mathbf{v} - \mathbf{z}\|_2 \leq n^{\delta}$ and $\|\mathbf{z}\|_{\infty} \leq \exp(n^c)$.
- Therefore, if $\|\tilde{M}_{n-1}\| \leq \sqrt{n}$, we get

$$\sqrt{\sum_{i=1}^{n-1} |X_i \cdot \mathbf{z}|^2} = \|\tilde{M}_{n-1} \mathbf{z}\|_2 \leq \|\tilde{M}_{n-1}\| \cdot \|\lambda \mathbf{v} - \mathbf{z}\|_2$$

$$\leq \sqrt{n} \cdot n^{\delta} \ll n^{1-2\epsilon},$$

so that $\mathbf{z}$ is an integer ‘approximate normal’.
Proposition (Diophantine approximation vs. small-ball probability)

For $\mathbf{v} \in S^{n-1}$ with $\rho_{0,\xi}(\mathbf{v}) \geq \rho \geq \exp(-n^c)$, there exists some $\lambda \in [1, \exp(n^c)]$ and non-zero $\mathbf{z} \in \mathbb{Z}^n$ such that $\|\lambda \mathbf{v} - \mathbf{z}\|_2 \leq n^\delta$.

- Suppose $\tilde{M}_{n-1} \mathbf{v} = 0$ for $\mathbf{v} \in S^{n-1}$ satisfying $\rho_{0,\xi}(\mathbf{v}) \geq \rho$.
- Then, by the theorem, we can find a non-zero $\mathbf{z} \in \mathbb{Z}^n$ with $\|\lambda \mathbf{v} - \mathbf{z}\|_2 \leq n^\delta$ and $\|\mathbf{z}\|_{\infty} \leq \exp(n^c)$.
- Therefore, if $\|\tilde{M}_{n-1}\| \leq \sqrt{n}$, we get

$$\sqrt{n-1} \sum_{i=1}^{n-1} |X_i \cdot \mathbf{z}|^2 = \|\tilde{M}_{n-1} \mathbf{z}\|_2 \leq \|\tilde{M}_{n-1}\| \cdot \|\lambda \mathbf{v} - \mathbf{z}\|_2$$

$$\leq \sqrt{n} \cdot n^\delta \ll n^{1-2\epsilon},$$

so that $\mathbf{z}$ is an integer ‘approximate normal’.
- This is exactly what we have ruled out with high probability!
Diophantine approximation vs. small-ball probability

**Proposition (Diophantine approximation vs. small-ball probability)**

For $\mathbf{v} \in \mathbb{S}^{n-1}$ with $\rho_{0,\xi}(\mathbf{v}) \geq \rho \geq \exp(-n^c)$, there exists some $\lambda \in [1, \exp(n^c)]$ and non-zero $\mathbf{z} \in \mathbb{Z}^n$ such that

$$\|\lambda \mathbf{v} - \mathbf{z}\|_2 \leq n^\delta.$$ 

- Straightforward consequence of the following bound resulting from a short Fourier analytic computation:

$$\rho_{1,\xi}(\mathbf{v}) \lesssim \int_{\mathbb{R}} \exp \left( -\text{dist} (\lambda \mathbf{v}, \mathbb{Z}^n)^2 - \lambda^2 \right) d\lambda,$$

which essentially appears in a classical work of Halász (1977).
Some technical issues

- The above ideas essentially suffice to prove our results for random matrices whose entries have ‘light tails’.

For combinatorial settings, we exploit the spectral gap in random graphs to round despite a large operator norm. For random matrices with heavier-tailed entries, we introduce a new rounding procedure based on controlling the $\infty$-to-2 norm of a matrix after eliminating outlier rows and columns. For perturbations of large matrices, we need to choose the ‘correct’ scale at which to round the vector.
Some technical issues

- The above ideas essentially suffice to prove our results for random matrices whose entries have ‘light tails’.

- For **combinatorial settings**, we exploit the spectral gap in random graphs to round despite a large operator norm.
Some technical issues

- The above ideas essentially suffice to prove our results for random matrices whose entries have ‘light tails’.

- For **combinatorial settings**, we exploit the spectral gap in random graphs to round despite a large operator norm.

- For random matrices with **heavier-tailed entries**, we introduce a new rounding procedure based on controlling the $\infty$-to-2 norm of a matrix after eliminating outlier rows and columns.
Some technical issues

- The above ideas essentially suffice to prove our results for random matrices whose entries have ‘light tails’.

- For **combinatorial settings**, we exploit the spectral gap in random graphs to round despite a large operator norm.

- For random matrices with **heavier-tailed entries**, we introduce a new rounding procedure based on controlling the $\infty$-to-2 norm of a matrix after eliminating outlier rows and columns.

- For **perturbations of large matrices**, we need to choose the ‘correct’ scale at which to round the vector.
References


For an introduction to the geometric framework of Rudelson-Vershynin, see their survey *Non-asymptotic theory of random matrices: extreme singular values* from the 2010 ICM Proceedings.

For an introduction to inverse Littlewood-Offord theory and its applications to random matrix theory, see *Small Ball Probability, Inverse Theorems, and Applications*, Nguyen-Vu. Erdős Centennial pp 409-463.
THANK YOU!

References available at math.mit.edu/~visheshj

For any questions or comments: visheshj@mit.edu