Let $X_1, \ldots, X_n$ be non-negative independent random variables, each of which has mean 1. Let $X = X_1 + \cdots + X_n$. We will show (following Nazarov https://mathoverflow.net/questions/187938/lower-bound-for-prx-geq-ex/188087#188087) that there exists an absolute constant $c > 0$ such that

$$\Pr(X \leq n + 1) \geq c.$$  

We will find it more convenient to work instead with $Y_i = 1 - X_i$. Note that $Y_i \leq 1$ and $\mathbb{E}[Y_i] = 0$ for all $i$. The above statement is equivalent to showing that

$$P := \Pr(Y \geq -1) \geq c.$$  

**Proof:** The proof will proceed by case analysis, depending on the value of $\mathbb{E}e^Y$.

**Case I:** $\mathbb{E}e^Y \leq 2$. In this case, we have

$$1 \leq e^{\mathbb{E}[Y]/2} \leq e^{\mathbb{E}[Y^2]} = \mathbb{E}[e^{Y^2}(Y < -1)] + \mathbb{E}[e^{Y^2}(Y \geq -1)] \leq e^{-1/2}(1 - P) + \sqrt{2P};$$

the second inequality is Jensen’s inequality, and the rightmost inequality is by Cauchy-Schwarz. Hence, in this case, we get that

$$1 \leq e^{-1/2}(1 - P) + \sqrt{2P}$$

which translates to $P \geq \approx 0.104$.

**Case II:** $\mathbb{E}e^Y > 2$. In this case, there exists some $t \in (0, 1)$ such that $\mathbb{E}e^{tY} = 2$. We claim that there exists some absolute constant $K > 1$ for which $\mathbb{E}e^{2tY} \leq 2^K$. Before proving this claim, let’s see how this finishes the proof.

Setting $q := 2^{-K-1}$, we have

$$\mathbb{E}[e^{tY} - qe^{2tY} - 1] \geq 2 - 2^{-1} - 1 \geq \frac{1}{2}.$$  

Moreover, the function $x - qx^2 - 1$ is bounded above by $1/4q = 2^{K-1}$, and is negative whenever $x < 0$. Therefore, we have

$$\frac{1}{2} \leq \mathbb{E}[e^{tY} - qe^{2tY} - 1] \leq 2^{K-1} \Pr(Y \geq 0) \leq 2^{K-1} P.$$  

It remains to prove the claim. It suffices to show that there exists some absolute constant $K > 1$ such that if $Z \leq 1$ is a mean zero random variable, then

$$\mathbb{E}e^{2Z} \leq (\mathbb{E}e^Z)^K.$$  

From this, the claim follows since

$$\mathbb{E}e^{2tY} = \prod_{i=1}^n \mathbb{E}e^{2tY_i}$$

$$\leq \left( \prod_{i=1}^n \mathbb{E}e^{Y_i} \right)^K$$

$$= (\mathbb{E}e^Y)^K = 2^K.$$
Finally, the inequality for $Z$ follows by noting that there is some absolute constant $K > 1$ for which the following numerical inequality is true: $e^{2z} - 1 - 2z \leq K (e^z - 1 - z)$ for all $z \leq 1$, and the following chain of inequalities:

\[
(\mathbb{E} e^Z)^K = (1 + \mathbb{E}[e^Z - 1 - Z])^K \\
\geq 1 + K \mathbb{E}[e^Z - 1 - Z] \\
\geq 1 + \mathbb{E}[e^{2Z} - 1 - 2Z] \\
= \mathbb{E}[e^{2Z}].
\]

**Application:** A fractional matching in a $k$-graph $H = (V, E)$ is a function $w : E \to [0, 1]$ such that for every $v \in V$, $\sum_{e \ni v} w(e) \leq 1$ (observe that if $w : E \to \{0, 1\}$, then the same condition gives a matching). The size of a fractional matching is defined to be $\sum_{e \in E} w(e)$. We say that $w$ is a perfect fractional matching if its size is $|V|/k$ (or equivalently, if $\sum_{e \ni v} w(e) = 1$ for all $v \in V$).

For an integer $0 \leq d \leq k - 1$ and a real number $0 \leq s \leq n/k$, we let $f^*_d(k, n)$ denote the smallest integer $m$ such that every $n$-vertex $k$-graph $H$ with $\delta_d(H) \geq m$ has a fractional matching of size $s$. We denote $f^{n/k}_d(k, n)$ simply by $f_d(k, n)$. Also, let

\[
f_d(k) := \limsup_{n \to \infty} \frac{f_d(k, n)}{(n - d)\binom{k}{n-d}}.
\]

It was proved by Alon, Frankl, Huang, Rödl, Ruciński, and Sudakov that for all $k \geq 3$ and $1 \leq d \leq k - 1$,

\[
f_d(k) \leq f^d(k - d),
\]

where

\[
f^d(\ell) := \limsup_{m \to \infty} \frac{f^{m+d/\ell+d}_0(\ell, m)}{\binom{m}{\ell}}.
\]

Recently, together with Asaf Ferber, we observed that

\[
f^d(\ell) = \Theta^d(\ell).
\]

Here

\[
\Theta^d(\ell) := \sup \Pr[X_1 + \cdots + X_\ell \geq \ell + d],
\]

where the supremum is taken over all collections of non-negative i.i.d. random variables $X_1, \ldots, X_\ell$ with mean 1.