PSET 9 Solutions

18.314

November 30, 2013

P1. a) True. Run Kruskal's algorithm with that edge being the unique minimal-weight edge.

b) True. Run Kruskal's algorithm with the two edges (and only these two edges) having the smallest weights.

c) False. If the three edges form a triangle.

d) True. Run Kruskal's algorithm with the edges in F (and only these edges) having the smallest weights.

P2. a) The eigenvalues of the adjacency matrix are ± 1, so the those of the Laplacian are 0 and 2. The number of spanning trees is then $\frac{1}{2}2 = 1$, by Matrix-Tree.

b) The (list of) eigenvalues of the adjacency matrix are $-2, 0, 0, 2$, so those of the Laplacian are $0, 2, 2, 4$. By Matrix-Tree, $\frac{1}{4} \cdot 2 \cdot 2 \cdot 4 = 4$ is the number of spannig trees.

c) The (list of) eigenvalues of the adjacency matrix are $-3, -1, -1, -1, 1, 1, 1, 3$, so those of the Laplacian are $0, 2, 2, 2, 4, 4, 4, 6$. By Matrix-Tree, $\frac{1}{8} \cdot 2^3 \cdot 4^3 \cdot 6 = 4^3 \cdot 6 = 384$ is the number of spanning trees.

d) As numbers, the eigenvalues of the adajcency matrix are $-n, -n+2, -n+4, \ldots, n-4, n-2, n$, where the multiplicity of $n-2k$ is $\binom{n}{k}$. Hence, those of the Laplacian are $0, 2, 4, 6, \ldots, 2n$, where the multiplicity of $2k$ is $\binom{n}{k}$. Hence, the number of spanning trees is $\dfrac{\prod_{k=1}^{n} (2k)^{\binom{n}{k}}}{2^n}$.

A1. a) By inspection we check that $\mathcal{L} - rI$ has s identical rows, so it has rank at most $r+1$. But then, the multiplicity of the 0 eigenvalue is at least $s-1$, so the multiplicity of the eigenvalue r in $\mathcal{L}$ is at least $s-1$.

b) Analogous to a), the multiplicity of eigenvalue s in $\mathcal{L}$ is at least $r-1$.

c) $\mathcal{L}$ has two more eigenvalues, one of which is 0. The remaining one can be quickly found by computing the trace, which is $2rs$, and subtracting the known eigenvalues to obtain $r+s$.

d) Form the previous points this is $\dfrac{r^{s-1}s^{r-1}(r+s)}{r+s} = r^{s-1}s^{r-1}$.

A2. a) The number of vertices is the number of given eigenvalues, so 9. The degree of each vertex is equal to the largest eigenvalue, so 4. Hence, the number of edges is $\frac{1}{2} \cdot 9 \cdot 4 = 18$.

b) The eigenvalues of the Laplacian are $6, 6, 6, 6, 3, 3, 3, 3, 0$, so by Matrix-Tree, the number of spanning trees is $\frac{1}{9} \cdot 6^4 \cdot 3^4 = 11664$.