PSET 8 Solutions.

18.314

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P1. The connected components of a forest on \([n]\) induce a partition of \([n]\), elements of the same block are vertices of the same connected components.

P2. It suffices to prove that in any connected graph on at least two vertices, there exists two vertices such that removing either one of them still leaves a connected graph. Let \(G\) be a counterexample to this claim with minimal number of vertices, and consider a cut vertex \(v\) in \(G\). Removing \(v\) from \(G\) produces at least two disjoint connected components \(G_1\) and \(G_2\) of smaller size. Each of these is either a single vertex or a connected graph with at least two non-cut vertices. If \(G_1\) is a single vertex \(u_1\), select \(u_1\), and if \(G_1\) is larger, then select a non-cut vertex \(u_1\) in \(G_1\) such that \(u_1\) is not the only neighbor of \(v\) in \(G\) that lies in \(G_1\). Similarly, choose \(u_2\) in \(G_2\). Then, both \(u_1\) and \(u_2\) are non-cut.

Leaves of spanning trees also work.

P3. Take two disjoint maximal-length paths, say with length \(m\), one path \(P_1\) from \(u\) to \(v\) and one path \(P_2\) from \(w\) to \(y\). There exist two vertices \(a\) in \(P_1\) and \(b\) in \(P_2\) such that the path \(Q\) from \(a\) to \(b\) does not share edges with either \(P_1\) or \(P_2\), so \(Q\) has length \(k > 0\). Either the path from \(u\) to \(a\) or the path from \(v\) to \(a\) has length at least \([m/2]\), and similarly for \(x\), \(y\) and \(b\). But then, using these longest pieces from \(P_1\) and \(P_2\), and \(Q\), we obtain a path of length \(2[m/2] + k > m\), a contradiction.

P5. Several examples:

P6. This can be easily calculated with the Matrix-Tree Theorem as a determinant of the modified Laplacian of the graph. For a different proof, let \(n \geq 2\). The answer to the problem is clearly equal to \(n^{n-2} - r\), where \(r\) is the number of trees on \([n]\) that contain edge 12. Now, for each tree on \([n]\) that contains 12, we can put a new vertex \(\ell\) in the interior of edge 12 to form a new graph \(G\). Then, select a number \(k\) from \([n]\), and draw the edge \(\ell k\) in \(G\). This will form a cycle that contains either 1 or 2. If the cycle contains 1, delete edge 1\(\ell\), collapse edge 2\(\ell\), and call the new vertex 2. Analogously, if the cycle contains 2, delete edge 2\(\ell\), collapse edge 1\(\ell\), and call the new vertex 1. This map is a clearly a bijective map from the set of pairs \((T, k)\) where \(T\) is a tree on \([n]\) that contains
edge 12 and $k \in [n]$, to the set of trees on $[n]$. Furthermore, the restriction of the map to $k \in \{1, 2\}$ maps trees that contain 12 to themselves, so the fraction of all trees on $[n]$ that contain edge 12 is $\frac{2}{n}$, so $r = 2n^{n-3}$.

A1. We have $3n + (n - m) = 2(n - 1)$, where $n$ is the number of vertices, so the tree has $n - 1$ edges. Solving this we find $n = 2m + 2$.

A2. There is clearly nothing better than the tree obtained from the greedy algorithm, i.e. Kruskal’s algorithm for finding a minima-weight tree.

A3. (Due to C-C. Lien) All vertices have degree 3 and there are no 3-cycles or 4-cycles. Hence, in a 7-cycle, there would be no chords. But then, the third edge of every vertex in the cycle must be connected to one of three vertices outside the cycle, giving a total of 7 edges sprouting out from the cycle. By pigeonhole, one of the vertices outside the 7-cycle is connected to exactly three vertices in the cycle, and this will form either a 3-cycle or 4-cycle again, yielding a contradiction.