PSET 8 Solutions.

18.314

November 28, 2013

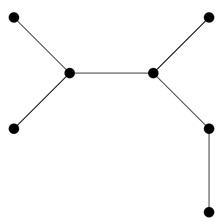
P1. The connected components of a forest on $[n]$ induce a partition of $[n]$, elements of the same block are vertices of the same connected components.

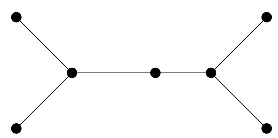
P2. It suffices to prove that in any connected graph on at least two vertices, there exists two vertices such that removing either one of them still leaves a connected graph. Let G be a counterexample to this claim with minimal number of vertices, and consider a cut vertex v in G. Removing v from G produces at least two disjoint connected components G_1 and G_2 of smaller size. Each of these is either a single vertex or a connected graph with at least two non-cut vertices. If G_1 is a single vertex u_1, select u_1, and if G_1 is larger, then select a non-cut vertex u_1 in G_1 such that u_1 is not the only neighbor of v in G that lies in G_1. Similarly, choose u_2 in G_2. Then, both u_1 and u_2 are non-cut.

Leaves of spanning trees also work.

P3. Take two disjoint maximal-length paths, say with length m, one path P_1 from u to v and one path P_2 from w to y. There exist two vertices a in P_1 and b in P_2 such that the path Q from a to b does not share edges with either P_1 or P_2, so Q has length $k > 0$. Either the path from u to a or the path from v to a has length at least $\lceil m/2 \rceil$, and similarly for x, y and b. But then, using these longest pieces from P_1 and P_2, and Q, we obtain a path of length $2\lceil m/2 \rceil + k > m$, a contradiction.

P5. Several examples:





P6. This can be easily calculated with the Matrix-Tree Theorem as a determinant of the *modified* Laplacian of the graph. For a different proof, let $n \geq 2$. The answer to the problem is clearly equal to $n^{n-2} - r$, where r is the number of trees on $[n]$ that contain edge 12. Now, for each tree on $[n]$ that contains 12, we can put a new vertex ℓ in the interior of edge 12 to form a new graph G. Then, select a number k from $[n]$, and draw the edge ℓk in G. This will form a cycle that contains either 1 or 2. If the cycle contains 1, delete edge 1ℓ, collapse edge 2ℓ, and call the new vertex 2. Analogously, if the cycle contains 2, delete edge 2ℓ, collapse edge 1ℓ, and call the new vertex 1. This map is a clearly a bijective map from the set of pairs (T, k) where T is a tree on $[n]$ that contains edge 12 and $k \in [n]$, to the set of trees on $[n]$. Furthermore, the restriction of the map to $k \in \{1, 2\}$ maps trees that contain 12 to themselves, so the fraction of all trees on $[n]$ that contain edge 12 is $\frac{2}{n}$, so $r = 2n^{n-3}$.

A1. We have $3m + (n - m) = 2(n - 1)$, where n is the number of vertices, so the tree has $n - 1$ edges. Solving this we find $n = 2m + 2$.

A2. There is clearly nothing better than the tree obtained from the greedy algorithm, i.e. Kruskal's algorithm for finding a minima-weight tree.

A3. (Due to C-C. Lien) All vertices have degree 3 and there are no 3-cycles or 4-cycles. Hence, in a 7-cycle, there would be no chords. But then, the third edge of every vertex in the cycle must be connected to one of three vertices outside the cycle, giving a total of 7 edges sprouting out from the cycle. By pigeonhole, one of the vertices outside the 7-cycle is connected to exactly three vertices in the cycle, and this will form either a 3-cycle or 4-cycle again, yielding a contradiction.