PSET 7 Solutions

18.314

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P1. A transitive tournament on n vertices is the same as a ranking of the vertices. Hence, there are $n!$ of them.

P2. There are n vertices, and each can have at most degree $n-1$, so each can have degree $1, 2, \ldots, n-2$ or $n-1$. By the pigeonhole principle, there must exist at least two vertices with the same degree.

P3. If each player has played at most two games so far, then the total number of games played is $\leq \frac{2\cdot(\text{number of players})}{2} = \text{number of players} = 10$.

P4. There is 1 with no edges and 1 with 6 edges. There is 1 with 1 edge and 1 with 5 edges. There are 2 with 2 edges and 2 with 4 edges. Finally, on 3 edges, there is 1 with a degree 3 vertex, 1 with a triangle, and 1 with exactly 2 degree 2 vertices. The total number is $1+1+1+1+2+2+3 = 11$.

P5. By induction we can suppose that the result holds for dimension less than n, and try to prove the case of n. One can start from the origin, find a hamiltonian cycle on the $(n-1)$-dim. cube given by fixing the last coordinate to 0, and then finish the walk prior to returning to the origin at $(1, 0, 0, \ldots, 0, 0)$. From there, one can walk to $(1, 0, 0, \ldots, 0, 1)$ and use induction and symmetry of the cube to find a hamiltonian cycle on the $(n-1)$-dim. cube given by fixing the last coordinate to 1, and whose final vertex prior to completing the cycle is precisely is $(0, 0, 0, \ldots, 0, 1)$. From there, one goes back to the origin, completeing the proof.

P6. It is not connected, one of the connected components must have at most 3 vertices. Hence, any vertex in that connected component cannot have degree at least 3.

A1. All vertices have degree $\binom{5}{3} = 10$, which is even, so a classic theorem from the book shows that the graph does have an Eulerian trail.

A2. No, the two degree 8 vertices connect to all other vertices, so there cannot be a vertex of degree 1.

A3. No, for example a path on 4 vertices $\circ - \circ - \circ - \circ$.