HW 6, Solutions

18.314

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P1. The line notation of a n-permutation is a concatenation of idecomposable permutations, each on a block of a partition of $[n]$. Hence, $G(x) = \frac{1}{1-F(x)}$, so $F(x) = 1 - \frac{1}{G(x)}$.

P2. Let $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$. Then, we can check that $A(x) = 1 + x + x(A(x)-1) + x^2 A(x)$, so $A(x) = \frac{1}{1-x-x^2}$. Let $F(x) = \sum_{n\geq 0} F_n x^n$ be the ordinary generating function found in Exercise 4. We see that $F(x) = A(x)$, but the solution gives the recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = F_1 = 1$, which is precisely the recurrence for Fibonacci numbers, i.e. F_n is equal to the n-th Fibonacci number. As a consequence, $a_n = n!F_n$.

P3. We can check that $\sum_{n\geq 1} \frac{x^n}{n} = -\log(1-x)$. Then, by the product formula for exponential generating functions we see that $G(x) = \frac{(-1)^k \log(1-x)^k}{k!}$, once we note that the order of the cycles does not matter and accordingly correct dividing by $k!$.

P4. The 2 possibilities for 1×1 rectangles are represented by $2x$, and the 3 possibilities for 1×2 rectangles are represented by $3x^2$. We see that we are actually looking for the compositional formula to create the tiling of the $1 \times n$ rectangle, so $H(x) = \frac{1}{1-2x-3x^2}$.

P5. Any concatenation of AB's and B's gives rise to one such word, and also any concatenation of them plus a final A in the last position works. Also, any such word can be uniquely decomposed as concatenations of the kind described. Hence, $H(x) = \frac{1+x}{1-x-x^2}$.

P6. Mainly, we can check that $\prod_{n\geq 1} \frac{1}{1+x^n} = \prod_{n\geq 0}(1 - x^{2n+1})$ as formal power series, using the identity $(1-x^n)(1+x^n) = (1+x^{2n})$. Now, the LHS is precisely $\sum_{n\geq 0}(p_{\text{even}}(n) - p_{\text{odd}}(n))x^n$, and the RHS is precisely $\sum_{n\geq 0}(-1)^n p_{\text{distict odd}}(n) x^n$.

A1. (Long solution) The problem is equivalent to proving the identity of formal power series

$$\prod_{i\geq 1}\left(\frac{1}{1-q^i} - q^i\right) = \prod_{\substack{i\geq 1 \\ i\not\equiv 1,5 \pmod 6}} \frac{1}{1-q^i}.$$

Now, $\frac{1}{1-q^i} - q^i = \frac{1-q^i+q^{2i}}{1-q^i} = \frac{1+q^{3i}}{(1-q^i)(1+q^i)}$. Multiplying both sides of the identity by $\prod_{i\geq 1}(1-q^i)$ we obtain

$$\prod_{i\geq 1}\frac{1+q^{3i}}{1+q^i} = \prod_{\substack{i\geq 1 \\ i\equiv 1,5 \pmod 6}} (1-q^i).$$

This is equivalent to having

$$\prod_{\substack{i\geq 1\\ i\equiv 1,5 \pmod 6}} (1-q^i) \prod_{\substack{k\geq 1\\ k\equiv 1,2 \pmod 3}} (1+q^k) = 1.$$

Every $k \equiv r \pmod 3$ with $r \in \{1, -1\}$ can be written uniquely as $k = 2^a b$ with b odd, a and b nonnegative integers. We have $b \equiv (-1)^a r \pmod 3$ and hence, $b \equiv (-1)^a r \pmod 6$. We can then write

$$\prod_{\substack{k\geq 1\\ k\equiv 1,2 \pmod 3}} (1+q^k) = \prod_{\substack{k\geq 1\\ j\geq 1\\ k\equiv 1,5 \pmod 6}} (1+q^{2^j k}).$$

However,

$$\prod_{\substack{i\geq 1\\ i\equiv 1,5 \pmod 6}} (1-q^i) \prod_{\substack{k\geq 1\\ j\geq 1\\ k\equiv 1,5 \pmod 6}} (1+q^{2^j k})$$

$$= \prod_{\substack{i\geq 1\\ i\equiv 1,5 \pmod 6}} (1-q^i) \prod_{j\geq 1}(1+q^{2^j i}) = \prod_{\substack{i\geq 1\\ i\equiv 1,5 \pmod 6}} 1 = 1.$$

A2. (Short solution, due to M. Zimet) Let f_n be the number of partitions of n in which no part appears more than twice, and set $F(x) = 1 + \sum_{n\geq 1} f_n x^n$. Let g_n be the number of partitions of n for which no part is divisible by 3, and set $G(x) = 1 + \sum_{n\geq 1} g_n x^n$. Then, we can check that,

$$F(x) = \prod_{n\geq 1}(1+x^n+x^{2n}) = \prod_{n\geq 1} \frac{1-x^{3n}}{1-x^n} = \prod_{\substack{n\geq 1\\ \text{3 doesn't divide } n}} \frac{1}{1-x^n}.$$

On the other hand,

$$G(x) = \prod_{\substack{n\geq 1\\ \text{3 doesn't divide } n}} (1+x^n+x^{2n}+\dots) = \prod_{\substack{n\geq 1\\ \text{3 doesn't divide } n}} \frac{1}{1-x^n},$$

as required.