P1. The line notation of a \( n \)-permutation is a concatenation of indecomposable permutations, each on a block of a partition of \([n]\). Hence, \( G(x) = \frac{1}{1 - F(x)} \), so \( F(x) = 1 - \frac{1}{G(x)} \).

P2. Let \( A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!} \). Then, we can check that \( A(x) = 1 + x + x(A(x) - 1) + x^2 A(x) \), so \( A(x) = \frac{1}{1 - 2x - x^2} \). Let \( F(x) = \sum_{n \geq 0} F_n x^n \) be the ordinary generating function found in Exercise 4. We see that \( F(x) = A(x) \), but the solution gives the recurrence \( F_n = F_{n-1} + F_{n-2} \) for \( n \geq 2 \) with \( F_0 = F_1 = 1 \), which is precisely the recurrence for Fibonacci numbers, i.e. \( F_n \) is equal to the \( n \)-th Fibonacci number. As a consequence, \( a_n = n! F_n \).

P3. We can check that \( \sum_{n \geq 1} n x^n = -\log(1 - x) \). Then, by the product formula for exponential generating functions we see that \( G(x) = (\frac{1}{1 - \frac{1}{\log(1 - x)^k}}) \), once we note that the order of the cycles does not matter and accordingly correct dividing by \( k! \).

P4. The 2 possibilities for \( 1 \times 1 \) rectangles are represented by \( 2 \times 1 \), and the 3 possibilities for \( 1 \times 2 \) rectangles are represented by \( 3 \times 2 \). We see that we are actually looking for the compositional formula to create the tiling of the \( 1 \times n \) rectangle, so \( H(x) = \frac{1}{1 - 2x - 3x^2} \).

P5. Any concatenation of \( AB \)'s and \( B \)'s gives rise to one such word, and also any concatenation of them plus a final \( A \) in the last position works. Also, any such word can be uniquely decomposed as concatenations of the kind described. Hence, \( H(x) = \frac{1}{1 - 2x - 3x^2} \).

P6. Mainly, we can check that \( \prod_{i \geq 1} \left( 1 - q^i - q^i \right) = \prod_{i \geq 1} \left( 1 - x^{2(i+1)} \right) \) as formal power series, using the identity \((1 - x^n)(1 + x^n) = (1 + x^{2n})\). Now, the LHS is precisely \( \sum_{n \geq 0} (p_{\text{even}}(n) - p_{\text{odd}}(n)) x^n \), and the RHS is precisely \( \sum_{n \geq 0} (-1)^n p_{\text{distinct odd}}(n) x^n \).

A1. (Long solution) The problem is equivalent to proving the identity of formal power series

\[
\prod_{i \geq 1} \left( \frac{1}{1 - q^i} - q^i \right) = \prod_{i \geq 1} \frac{1}{1 - q^i}, \quad i \not\equiv 1, 5 \pmod{6}
\]

Now, \( \frac{1}{1 - q^i} - q^i = \frac{1 - q^{i+q^i}}{1 - q^i} = \frac{1 + q^{n}}{(1-q^i)(1+q^{i})} \). Multiplying both sides of the identity by \( \prod_{i \geq 1} (1 - q^i) \) we obtain

\[
\prod_{i \geq 1} \frac{1 + q^{2i}}{1 + q^i} = \prod_{i \geq 1} (1 - q^i) \quad \text{for } i \equiv 1, 5 \pmod{6}
\]
This is equivalent to having
\[
\prod_{i \equiv 1.5 \pmod{6}} (1 - q^i) \prod_{k \equiv 1.2 \pmod{3}} (1 + q^k) = 1.
\]
Every \( k \equiv r \pmod{3} \) with \( r \in \{1, -1\} \) can be written uniquely as \( k = 2^a b \) with \( b \) odd, \( a \) and \( b \) nonnegative integers. We have \( b \equiv (-1)^a r \pmod{3} \) and hence, \( b \equiv (-1)^a r \pmod{6} \). We can then write
\[
\prod_{k \equiv 1.2 \pmod{3}} (1 + q^k) = \prod_{j \equiv 1.5 \pmod{6}} (1 + q^{2j}).
\]
However,
\[
\prod_{i \equiv 1.5 \pmod{6}} (1 - q^i) \prod_{k \equiv 1.5 \pmod{6}} (1 + q^{2i})
= \prod_{i \equiv 1.5 \pmod{6}} (1 - q^i) \prod_{j \equiv 1.5 \pmod{6}} (1 + q^{2j}) = \prod_{i \equiv 1.5 \pmod{6}} 1 = 1.
\]

A2. (Short solution, due to M. Zimet) Let \( f_n \) be the number of partitions of \( n \) in which no part appears more than twice, and set \( F(x) = 1 + \sum_{n \geq 1} f_n x^n \). Let \( g_n \) be the number of partitions of \( n \) for which no part is divisible by 3, and set \( G(x) = 1 + \sum_{n \geq 1} g_n x^n \). Then, we can check that,
\[
F(x) = \prod_{n \geq 1} (1 + x^n + x^{2n}) = \prod_{n \geq 1} \frac{1 - x^{3n}}{1 - x^n} = \prod_{n \geq 1} \frac{1}{1 - x^{3n}}.
\]
On the other hand,
\[
G(x) = \prod_{n \geq 1} (1 + x^n + x^{2n} + \ldots) = \prod_{n \geq 1} \frac{1}{1 - x^n},
\]
as required.