PSET 5, Solutions

18.314

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P1. Let A, B and C be the sets of n-permutations containing (1), (2) and (3) as cycles, respectively. By the inclusion-exclusion principle, We have $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = 3 \cdot (n-1)! - 3 \cdot (n-2)! + (n-3)!$.

P2. Again, by inclusion-exclusion, the number of positive integers < 1000 that are either squares or cubes is $\lfloor (1000)^{1/3} \rfloor + \lfloor (1000)^{1/2} \rfloor - \lfloor (1000)^{1/6} \rfloor = 10 + 31 - 3 = 38$. Hence, the answer is $1000 - 38 = 962$.

P3. Just note that for $i \in \{0, 1, \ldots, n\}$, the number of n-permutations with exactly $n - i$ fixed points is $\binom{n}{i} D(i)$. Hence, the sum on the RHS must be precisely $n!$.

P4. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$.Then $A(x) = 3xA(x) + x\sum_{n=0}^{\infty} 2^n x^n + 1$, so $A(x) = \frac{1-x}{(1-3x)(1-2x)} = \frac{2}{1-3x} - \frac{1}{1-2x}$. Hence, we directly see that $a_n = [x^n]A(x) = 2 \cdot 3^n - 2^n$.

P5. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $A(x) = 8xA(x) - 16x^2 A(x) + 1 - 4x$. Hence, $A(x) = \frac{1-4x}{(1-4x)^2} = -\frac{1}{1-4x}$, so $a_n = [x^n]A(x) = 4^n$.

P6. The problem translates to $a_0 = 50$, and $a_n = 2a_{n-1} + 1000$ for $n > 0$. Using the same approach as the last two, define $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Then, $A(x) = 2xA(x) + x\sum_{n=0}^{\infty} 1000x^n + 50 = 2xA(x) + \frac{1000x}{1-x} + 50$. Hence, $A(x) = 50\left(\frac{1}{1-2x} + \frac{20}{(1-x)(1-2x)}\right) = 50\left(\frac{21}{(1-2x)} - \frac{20}{1-x}\right)$, so $a_n = [x^n]A(x) = 50(21 \cdot 2^n - 20)$.