HW 4 Solutions

18.314

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P1. This is equal to the number of permutations of six whose cycle decomposition is given by the product of cycles of length at most 2. Hence, we have the case of three transpositions, two transpositions, one transposision and zero transpositions. Respectively, this gives $\frac{1}{3!}\binom{6}{2,2,2} + \frac{1}{2!}\binom{6}{2,2,2} + \binom{6}{2} + 1 = 15 + 45 + 15 + 1 = 76$.

P2. Since there can only be one cycle of length 11, then we can simply pick the underlying set of the cycle, then pick the order for the cycle, and then notice that the remaining part of the permutation is a permutation of nine, so the answer is $\binom{20}{11} \cdot \frac{11!}{11} \cdot 9! = \frac{20!}{11}$.

P3. For this one note that we can have two cycles of length n. Using the same reasoning as above, but subtracting the overcounting caused by this new possibility, we obtain $\binom{2n}{n} \cdot \frac{n!}{n} \cdot n! - \frac{1}{2!}\binom{2n}{n}\frac{n!n!}{n \cdot n} = \frac{(2n)!}{n} - \frac{1}{2}\frac{(2n)!}{n^2} = \frac{(2n)!(2n-1)}{2n^2}$.

P4. These are permutations that be decomposed as products of cycles of length 1 and 3. We have cases $3+3$, $3+1+1+1$ and $1+1+1+1+1+1$. Hence and respectively, the answer is $\frac{1}{2!}\binom{6}{3}\frac{3!3!}{3 \cdot 3} + \binom{6}{3}\frac{3!}{3} + 1 = 40 + 40 + 1 = 81$.

P5. Suppose $n > 2$. Then, for any permutation of $[n]$, the number n belongs to a cycle. If the cycle has length 1, then these are counted by $a(n-1, k-1)$ after removing n. If the cycle has length > 1, then these are counted by $(n-1)a(n-1,k)$ (the $n-1$ coming from the number right before n in the cycle notation of our permutation) after removing n. Hence, $a(n,k) = a(n-1,k-1) + (n-1)a(n-1,k)$.

Now, $a(2,2) = 0$ and $a(2,1) = 1$. Hence, for $n = 2$ the polynomial becomes x, agreeing with the proposed result. But for $n > 2$, we also have that $x^2(x+2)\cdots(x+n-2) + (n-1)\cdot x(x+2)\cdots(x+n-2) = x(x+2)\cdots(x+n-2)(x+n-1)$, so the same recurrence is satisfied by our polynomials on both sides of the equality, and by induction on n the equality will hold for all $n \geq 2$.

P6. Call these permutations good and consider a good permutation of $[n]$. In its cycle decomposition, the number n can be in a cycle of length 1, 2 or 4. These are all disjoint cases. For the first case, the remaining part is a good permutation of $[n-1]$. For the second case, it is a good permutation of $[n-1]\backslash\{i\}$ where i is the other element in the cycle with n. For the third case, it is a good permutation of $[n-1]\backslash\{i,j,k\}$ where i,j,k occur in the same cycle as n. Hence, a recurrence may be given by $u(n) = u(n-1) + \binom{n-1}{1}\cdot u(n-2) + \binom{n-1}{3}\cdot 3! \cdot u(n-4)$.