HW 3

18.314

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P1. The same as the number of compositions of 5. Hence, $2^{5-1} = 16$.

P2. This is equal to the number of compositions of 30 into 5 even parts, and hence equal to the number of compositions of 15 into 5 parts, so $\binom{15-1}{5-1}$.

P3. We can only have $n-4$, $n-5$ or $n-6$ singleton parts. Notably, we cannot have $n-k < n-6$ singleton parts since then there would exist $k-3$ or more parts of size at least 2 such that the size of their union has size k, which is impossible. The possible cases give a total of $\binom{n}{4} + \binom{n}{5}\binom{5}{3} + \frac{1}{3!}\binom{n}{6}\binom{6}{2,2,2}$, respectively.

P4. Take a partition of $[n]$, fix the non-singleton blocks, then take the union of the singleton blocks and $\{n+1\}$. The partition so obtained is a partition of $[n+1]$ iff there is at least one singleton block in the original partition, and a partition of $[n]$ otherwise, in which case it is exactly the same original partition. This establishes a bijection between the set of partitions of $[n]$ and the union of the sets of partitions of $[n]$ and $[n+1]$ with no singleton blocks.

P5. It is easier to start with a fixed partition of $n-1$. From there, consider all the intervals of two or more consecutive integers contained in any block. From each such interval $\{k_1 < k_2 < \cdots < k_m\}$, select the terms indexed by all $m-i$ with i odd. Then, remove the selected terms from the original blocks of the partition and form a new block as the union of all these selected terms and $\{n\}$. This procedure establishes the bijection.

On the proposed direction, we check that the new block does not contain consecutive terms since, otherwise, we can consider the minimal among those terms and obtain a contradiction with the construction.

The inverse procedure takes all the elements in the same block as n, one element at a time, and patches them to the block which contains the next consecutive integer, i.e. if i and n belong to the same block, then remove i from this block and add i to the block which contains $i+1$. Finally, drop the block $\{n\}$.

P6. We have $a_n = a_{n-1} + a_{n-2}$. Take one such composition of n. If the first term is 2, delete it and keep the remaining terms to obtain one such composition of $n-2$. If, on the other hand, the first term is $k > 2$, then make this first term equal to $k-1 \geq 2$ and keep the remaining terms to obtain one such composition for a_{n+1}. These procedures are bijective on their disjoint domains, and hence, when combined, establish the equality.