PSET 10 Solutions

18.314

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P1. The sum of the degrees of the vertices on one side of the bipartition is equal to the sum of the degrees of the vertices on the other side. However, all numbers are multiples of three except for five, and this cannot happen.

P2. This is Hall's theorem, exactly, so the conditions are those that guarantee the existence of a saturated matching: let n be the number of student clubs, then for every set of $k \leq n$ student clubs there must exist at least k students who are members of either one of these k clubs.

P3. $\chi_{K_{3,3}}(x) = x(x-1)^3 + 3x(x-1)(x-2)^3 + x(x-1)(x-2)(x-3)^3$. For example, one can count directly for a positive integer k, what is the number of different ways in which one can color $K_{3,3}$ properly using k colors.

P4. The chromatic polynomial of the n-vertices cycle is $(x-1)^n + (-1)^n(x-1)$. For some k, if we want to properly color the wheel on $n+1$ vertices with k colors, we have to properly color the cycle with $k-1$ of the colors, for which there are $k\left((k-1)^n + (-1)^n(k-1)\right)$ different ways to do it, and then color the center with the remaining color, so the answer is

$$x\left((x-1)^n + (-1)^n(x-1)\right).$$

P5. a) Vertices of X have degree $n-k$, and vertices of Y have degree $k+1$, but $k+1 \leq n-k$. The number of edges leaving a set of m vertices on X is $m(n-k)$, and these edges connect to M vertices on Y each of whose degree is $k+1$, so $m(n-k) \leq M(k+1) \leq M(n-k)$, so $m \leq M$.

b) Call $(1,2),(2,3),\ldots,(n-1,n),(n,1)$ good pairs, and consider the set B of all $(k+1)$-subsets of $[n]$ that contain at least the two elements of one good pair. Let I be one such set, and consider the minimal $i \in I$ such that (i,j) is good and $j \in I$, but (j,k) is good and $k \notin I$, and map I to $I \backslash j$. **We will call this first map** b. Now, let A be the set of all k-subsets of $[n]$, and for $I \in A$, let $i \in I$ be the minimal element such that (i,j) is good but $j \notin I$, and map I to $I \cup j$. **We will call this second map** a. Restricting b to $a(A)$, we see that b is the inverse of a.

P6. Suppose there exists two maximal matchings, one in which u is adjacent to an edge of the matching but no edge of the matching is adjacent to v, and viceversa. Now, consider the graph on same vertex set whose edges are the matching edges of either one of these two matchings. This graph is a disjoint union of even cycles and paths. In such a graph, by assumption, u and v belong to different connected components, each of these connected components is an even path (because paths must have even number of edges as cycles are even and both initial matchings are equicardinal), and moreover, u and v are endpoints on these paths. If we consider the path made up of these two connected components and edge uv, we can form an augmenting path, which contradicts the maximility of the matchings.