PSET 1, SOLUTIONS

18.314

September 15, 2013

P1. SOLUTION ACCEPTABLE ONLY FOR PSET 1: $44^n - 1 \equiv 2^n - 1 (\mod 7)$, so $n = 3$ works.

SOLUTION EXPECTTED FROM NOW ON (using material covered in the class): Suppose not, then the set of possible remainders upon dividing by 7 are $\{1, 2, 3, 4, 5, 6\}$. So consider $n = 1, 2, 3, 4, 5, 6, 7$. By the Pigeonhole Principle, there exists $1 \leq i < j \leq 7$ such that $44^i - 1 = 7n_i + r$ and $44^j - 1 = 7n_j + r$ for some $1 \leq r \leq 6$. Subtracting gives $44^j - 44^i = 7(n_j - n_i)$, so in particular $44^j - 44^i$ is divisible by 7. But $44^j - 44^i = 44^i(44^{j-i} - 1)$ and 7 does not divide $44^{j-i} - 1$ by hypothesis, so it must divide 44^i, a contradiction.

P2. It is easy to check how to cover the entire triangle with three of these half-circles. Then, the result follows from the pigenhole since their union covers all 10 points, so at least one of the circles must cover 4.

P3. Let G be a graph with nodes representing the people (let's assume they're a bunch of men) and where an edge is drawn whenever two people met in the library. For each individual, either he met all the people that left the library before him or he met all the people that got to the library after him, or both. Without loss of generality, let's assume no person met everyone who went to the library. Then we can partition the set of people into two disjoint sets, those who met everyone who left before them and those who met everyone who got to the library after them. Ordering the departure time of the first block shows that it forms a clique of G. Ordering the arrival time of the second block shows that it also forms a clique of G. These two cliques show what happened.

In graph theory terms, we can say that the complement of an interval graph with independence number less than 3 is bipartite.

P4. First subdivide the cube into 8 cubes ($n = 1$). Then, subdivide any of the smaller cubes into 8 cubes ($n = 2$). Then again, subdivide any of the smaller cubes into 8 cubes ($n = 3$) and so on. By induction it is true for all n that th number of cubes is $7n + 1$.

P5. For $n = 0$ and $n = 1$ this is true. Suppose that this holds for all $n < k + 2$. Then $a_k = a_{k+1} + 5a_k \leq 3^{k+1} + 5 \cdot 3^k = 3^k(3 + 5) < 3^{k+2}$.

P6. The base case $n = 1$ is clear. Moreover, a_1 is both divisible by 5 and odd. Assume that all of these hold for $n = k - 1$, so the last $k - 1$ digits of a_{k-1} are the same as the last $k - 1$ digits of a_k. Hence, $10^{k-1} | (a_k - a_{k-1})$, or $10^{k-1} | a_{k-1}(a_{k-1} - 1)$. We have $a_{k+1} - a_k = a_k^2 - a_k = a_{k-1}^4 - a_{k-1}^2 = a_{k-1}(a_{k-1} - 1) \cdot a_{k-1}(a_{k-1} + 1)$. By induction, a_{k-1} is divisible by 5 and odd, so $a_{k-1} + 1$ is even and $10 | a_{k-1}(a_{k-1} + 1)$. We conclude that $10^k = 10^{k-1} \cdot 10$ divides $a_{k-1}(a_{k-1} - 1) \cdot a_{k-1}(a_{k-1} + 1) = a_{k+1} - a_k$, so the last k digits of a_{k+1} and a_k are the same.