ON THE ℓ-ADIC VALUATION OF CERTAIN JACOBI SUMS

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ABSTRACT. Jacobi sums are ubiquitous in number theory, and congruences often provide a helpful way to study them. A p-adic congruence for Jacobi sums comes from Stickelberger’s congruence, and various ℓ-adic congruences have been studied in [Eva98], [Mik87], [Iwa75], [Iha86], and [Ueh87]. We establish a new ℓ-adic congruence for certain Jacobi sums.

1. Introduction

The Jacobi sum is usually defined as follows.

Definition 1.1. Fix a finite field \( \mathbb{F}_q \) and two characters \( \chi, \psi : \mathbb{F}_q^\times \to \mathbb{C}^\times \). Then the Jacobi sum \( J(\chi, \psi) \) is defined to be

\[
J(\chi, \psi) = \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \chi(x)\psi(1-x).
\]

Jacobi sums have various applications in number theory; see [BWE98] for many examples. They appear as Frobenius eigenvalues for the Fermat curve in [Kat81] and quotients of the Fermat curve in [Jęd14], [Jęd16], and [Aru19]. The main result of this paper is used in [Aru19] to study torsion points on the curve \( y^n = x^d + 1 \), which is a quotient of the Fermat curve \( F_{nd} : X^{nd} + Y^{nd} + Z^{nd} = 0 \).

For many applications it is helpful to know congruences for Jacobi sums. In [Con95], an application of Stickelberger’s congruence is used to show the following result.

Theorem 1.2. Fix a prime power \( q = p^f \), where \( p \) is a prime. Suppose \( \mathfrak{p} \) is a prime of \( \mathbb{Q}(\zeta_{q-1}) \) lying over \( p \). Suppose \( \omega_{\mathfrak{p}} \) is a Teichmüller character on \( \mathbb{F}_q \); i.e., for every \( \alpha \in \mathbb{Z}[\zeta_{q-1}] \) we have \( \omega_{\mathfrak{p}}(\alpha) \equiv \alpha \mod \mathfrak{p} \). Then for any integers \( k_1, k_2 \) in the range \( 0 \leq k_1, k_2 < q - 1 \) such that \( k_1 \) and \( k_2 \) are not both zero, we have

\[
J(\omega_{\mathfrak{p}}^{-k_1}, \omega_{\mathfrak{p}}^{-k_2}) \equiv \frac{(k_1 + k_2)!}{k_1!k_2!} \mod \mathfrak{p}.
\]

Theorem 1.2 expands a Jacobi sum \( p \)-adically where \( p \) is the characteristic of \( \mathbb{F}_q \). A variant of this question would be to ask to expand the Jacobi sum \( \ell \)-adically, where \( \ell \) is a prime such that \( q \equiv 1 \mod \ell \). In [Eva98], Evans shows the following.

Theorem 1.3. Let \( \mathbb{F}_q \) be a finite field, \( k > 2 \) be an integer such that \( q = fk + 1 \) for some integer \( f \), \( \zeta \) be a primitive \( k \)th root of unity, and \( \chi \) a multiplicative character of order \( k \). Given integers \( a, b, \) and \( c \) with \( c = -a - b, k \nmid a, k \nmid b, k \nmid c, \gcd(a,b,k) = 1, \) we have the

\[
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following equation modulo $(1 - \zeta^a)(1 - \zeta^b)(1 - \zeta)$:

$$J(\chi^a, \chi^b) \equiv \begin{cases} 
2 - q & \text{if } k = 3, \\
1 + i(q - 1)/2 & \text{if } k = 4, \\
\chi(-1) & \text{if } k > 4.
\end{cases}$$

In [Mik87], Miki obtains an $\ell$-adic congruence for generalized Jacobi sums of the form $J(\chi^{a_1}, \chi^{a_2}, \cdots, \chi^{a_r})$ where $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ is a character of order $m = \ell^a$ (in particular, $q \equiv 1 \mod \ell^a$). Miki’s congruences generalize Iwasawa’s ([Iwa75], Theorem 1) and Ihara’s congruences ([Iha86], Corollary to Theorem 7) for Jacobi sums.

In [Ueh87], Uehara establishes an $\ell$-adic congruence for Jacobi sums of the form $J(\chi, \chi^{cf})$ where $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ is a character of order $f\ell$ and $q \equiv 1 \mod f\ell$. Certain cyclotomic units of $\mathbb{Q}(\zeta_{f\ell})$ appear in Uehara’s expansion.

Our setup will be very similar to that of Uehara’s, and we will also find a connection with cyclotomic units.

Fix a prime $\ell$ and a positive integer $f > 1$ coprime to $\ell$. Fix a finite field $\mathbb{F}_q$ satisfying $q \equiv 1 \mod f\ell$. Let $L = \mathbb{Q}(\zeta_{f\ell})$. Fix a primitive $f\ell$-th root of unity $\zeta_{f\ell}$ in $L$ and set $\zeta_f = \zeta_{f\ell}^{\ell}$, $\zeta_{f} = \zeta_{f\ell}^{\ell}$. Let $\chi : \mathbb{F}_q^\times \to \mathbb{Q}(\zeta_{f\ell})$ be a character of order $f\ell$. Then we define the Jacobi sum as follows.

**Definition 1.4.** For integers $a, b$, we define the Jacobi sum $J(a, b)$ to be the element of $\mathcal{O}_L$ given by the sum

$$J(a, b) := J(\chi^a, \chi^b) = \sum_{x \in \mathbb{F}_q \setminus \{0, 1\}} \chi^a(x)\chi^b(1 - x).$$

We state our main result here, and it will be proved in Section 5 as Theorem 5.2.

**Theorem 1.5.** Fix an integer $k$ in the range $2 \leq k \leq \ell - 1$. Then

1. $J(\ell, f) + 1$ is always divisible by $\pi^\ell$.
2. $J(\ell, f) + 1$ is divisible by $\pi^\ell$ if and only if for each $i$ in the range $0 \leq i \leq k - 2$ and $j$ in the range $1 \leq j \leq f - 1$, we have

$$\prod_{r=0}^{\ell-1} \left(1 - \zeta_j^{r\ell} \zeta_i^{r}\right)^{(1)} \in (\mathbb{F}_q^\times)^\ell.$$

Our methods allow us to even reach the case $k = \ell$. We prove this result in Section 6 as Corollary 6.4.

**Theorem 1.6.** We have that $J(\ell, f) + 1$ is divisible by $\pi^\ell$ if and only if the following conditions hold.

1. $q - 1$ is divisible by $\ell^2$.
2. For all $i, j$ in the range $0 \leq i \leq \ell - 1$ and $1 \leq j \leq f - 1$, we have $1 - \zeta_j^{i\ell} \zeta_i^{j} \in (\mathbb{F}_q^\times)^\ell$.

### 2. Notation

To work more explicitly with $J(a, b)$, define $g$ to be a generator of the multiplicative group $\mathbb{F}_q^\times$ and the index of an element as follows.
Definition 2.1. For $x \in \mathbb{F}_q \setminus \{0\}$, set $\text{ind}(x) \in \{0, 1, \cdots, q - 2\}$ to be such that

$$x = g^{\text{ind} x}.$$  

Then an alternative definition of the Jacobi sum is

Definition 2.2. For integers $a, b$, define

$$J(a, b) = \sum_{x \in \mathbb{F}_q \setminus \{0, 1\}} \zeta_f^{a \text{ind}(x) + b \text{ind}(1-x)} f^\ell.$$  

We will be concerned with the $\ell$-adic valuation of the Jacobi sum $J(\ell, f)$. We set our uniformizer to be $\pi_\ell := \zeta_\ell - 1$.

Lemma 2.3. We have the following equality in $\mathcal{O}_L = \mathbb{Z}[\zeta_\ell]$:

$$J(\ell, f) + 1 = \sum_{i=1}^{q-1} \sum_{x \in \mathbb{F}_q \setminus \{0, 1\}} \binom{\text{ind}(1-x)}{i} \zeta_f^{\text{ind}(x)} \pi_\ell^i.$$  

Proof. Using the definition of $J(\ell, f)$ gives

$$J(\ell, f) = \sum_{x \in \mathbb{F}_q \setminus \{0, 1\}} \zeta_f^{\text{ind}(x) + f \text{ind}(1-x)} \zeta_\ell^{\text{ind}(1-x)} \zeta_f^{\text{ind}(1-x)} (1 + \pi_\ell)^{\text{ind}(1-x)} = \sum_{x \in \mathbb{F}_q \setminus \{0, 1\}} \zeta_f^{\text{ind}(1-x)} (1 + \pi_\ell)^{\text{ind}(1-x)} \zeta_f^{\text{ind}(x)} \pi_\ell^i.$$  

Since $\text{ind}(1-x) < q - 1$, we can allow the inner sum to go up to $i = q - 1$ without affecting its value (since the higher binomial coefficients will be zero anyway). This allows us to swap sums, and gives

$$J(\ell, f) = \sum_{i=0}^{q-1} \sum_{x \in \mathbb{F}_q \setminus \{0, 1\}} \binom{\text{ind}(1-x)}{i} \zeta_f^{\text{ind}(x)} \pi_\ell^i = -1 + \sum_{i=1}^{q-1} \sum_{x \in \mathbb{F}_q \setminus \{0, 1\}} \binom{\text{ind}(1-x)}{i} \zeta_f^{\text{ind}(x)} \pi_\ell^i.$$  

Adding 1 to both sides now finishes the proof.  

We will only be concerned with $i \leq \ell - 1$, so we rewrite the lemma as follows.

Lemma 2.4. We have that $J(\ell, f) + 1 \in \pi_\ell \mathcal{O}_\ell$ and moreover,

$$\pi_\ell^{-1}(J(\ell, f) + 1) \in \left( \sum_{i=1}^{\ell-1} \sum_{x \in \mathbb{F}_q \setminus \{0, 1\}} \binom{\text{ind}(1-x)}{i} \zeta_f^{\text{ind}(x)} \pi_\ell^{i-1} \right) + \ell \mathcal{O}_L.$$  


Proof. From the previous lemma we see that \( J(\ell, f) + 1 \) is always divisible by \( \pi_\ell^k \). Using the fact that \( \pi_\ell^k \in \ell O_L \) holds for \( i \geq \ell - 1 \) now completes the proof. \( \square \)

Now let \( M \) be the field \( \mathbb{Q}(\zeta_f) \) and \( O_M \) be its ring of integers. From the previous lemma we conclude that

**Corollary 2.5.** Fix some integer \( k \) in the range \([1, \ell]\). Then \( J(\ell, f) + 1 \) is divisible by \( \pi_\ell^k \) if and only if for \( 1 \leq i \leq k - 1 \), we have

\[
\sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \binom{\text{ind}(1-x)}{i}_{\ell^k} \in \ell O_M.
\]

To be precise, the expression in Corollary 2.5 lives in \( O_M \cong \mathbb{Z}[T]/(T^j - 1 + \cdots + 1) \). It will be helpful to work in a slightly larger ring.

**Definition 2.6.** Define \( R = \mathbb{Z}[T]/(T^j-1) \). Then \( R \) has a natural quotient map to \( O_M \). For integers \( u, v \) define \( S(u, v) \) to be the following element of \( R \).

\[
S(u, v) := \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \binom{\text{ind}(1-x)}{u} \binom{\text{ind}(1-x)}{v} T^{\text{ind} x} \in R.
\]

It is also convenient to have notation for the image of \( S(u, v) \) under the natural quotient map \( R \to \mathbb{Z} \) given by evaluation at \( T = 1 \). Thus, define \( T(u, v) \) to be the following integer:

\[
T(u, v) = \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \binom{\text{ind}(x)}{u} \binom{\text{ind}(1-x)}{v} \in \mathbb{Z}.
\]

Therefore the expression in Corollary 2.5 is the image of \( S(0,i) \) under the quotient map to \( O_M \). We first observe that \( T(0,i) \) is always divisible by \( \ell \) whenever \( 1 \leq i \leq \ell - 2 \).

**Lemma 2.7.** Fix an integer \( i \) in the range \( 1 \leq i \leq \ell - 2 \). Then \( T(0,i) \in \ell \mathbb{Z} \).

**Proof.** Computing \( T(0,i) \), we have

\[
T(0,i) = \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \binom{\text{ind}(1-x)}{i}.
\]

As \( x \) ranges over \( \mathbb{F}_q \setminus \{0,1\} \), the value of \( \text{ind}(1-x) \) will go through every element of \( \{1,2,\cdots,q-2\} \) exactly once. Therefore,

\[
T(0,i) = \binom{1}{i} + \binom{2}{i} + \cdots + \binom{q-2}{i} = \binom{q-1}{i+1}.
\]

Since \( i + 1 \leq \ell - 1 \), the denominator of \( \binom{q-1}{i+1} \) is coprime to \( \ell \). Since \( \ell \) divides \( q-1 \), the numerator of \( \binom{q-1}{i+1} \) is divisible by \( \ell \). Hence \( T(0,i) = \binom{q-1}{i+1} \) is divisible by \( \ell \). \( \square \)

We need to define a subring of \( R \) to proceed.

**Definition 2.8.** Let \( R' \) be the subgroup of \( R \) generated by \( 1, \ell T, \ell T^2, \cdots, \ell T^{j-1} \). Then \( R' \) is a subring of \( R \).

Here is the chain of relationships between the various rings we have encountered:

\[
\ell \mathbb{Z} \subseteq R' \subseteq R \Rightarrow O_M
\]
Lemma 2.9. Fix some integer \( i \) in the range \( 1 \leq i \leq \ell - 2 \). The following conditions are equivalent.

\[
\begin{align*}
(1) \quad & \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \left( \text{ind}(1-x) \right)_i \zeta_f^{\text{ind}(x)} \in \mathcal{O}_M. \\
(2) \quad & S(0,i) \in \ell R \\
(3) \quad & S(0,i) \in R'.
\end{align*}
\]

**Proof.** Since \( S(0,i) \) maps to \( \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \left( \text{ind}(1-x) \right)_i \zeta_f^{\text{ind}(x)} \) under the quotient map \( R \to \mathcal{O}_M \), it follows that (2) implies (1). Also, (2) implies (3) since \( \ell R \) is a subring of \( R' \).

To check that (3) implies (2), we must show that the constant term of \( S(0,i) \) is divisible by \( \ell \). By Lemma 2.7, we know that the the sum of the coefficients of \( S(0,i) \) is divisible by \( \ell \). Since the nonconstant coefficients of \( S(0,i) \) are divisible by \( \ell \) (by assumption that \( S(0,i) \in R' \)), it follows that the constant coefficient must also be divisible by \( \ell \). Therefore \( S(0,i) \in \ell R \).

To check that (1) implies (2), use the Chinese Remainder Theorem to write \( R \to \mathcal{O}_M \oplus \mathbb{Z} \), where this map sends

\[
S(0,i) \mapsto \left( \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \left( \text{ind}(1-x) \right)_i \zeta_f^{\text{ind}(x)}, T(0,i) \right)
\]

This map is an isomorphism once one inverts \( f \) (when one inverts \( f \), the ideals \( (T^{j-1} + T^{f-2} + \cdots + 1) \) and \( (T - 1) \) become coprime). Since \( f \) and \( \ell \) are coprime, it follows that we have an isomorphism

\[
R/\ell R \cong (\mathcal{O}_M/\ell \mathcal{O}_M) \oplus (\mathbb{Z}/\ell \mathbb{Z}).
\]

By assumption, the first coordinate \( \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \left( \text{ind}(1-x) \right)_i \zeta_f^{\text{ind}(x)} \) lives in \( \ell \mathcal{O}_M \). By Lemma 2.7, the second coordinate \( T(0,i) \) lives in \( \ell \mathbb{Z} \). From our isomorphism, it follows that \( S(0,i) \in \ell R \).

We now state a version of Corollary 2.5 but with \( R' \) instead of \( \ell \mathcal{O}_M \).

**Corollary 2.10.** Fix some integer \( k \) in the range \( [1, \ell - 1] \). Then \( J(\ell,f) + 1 \) is divisible by \( \pi_\ell^k \) if and only if for \( 1 \leq i \leq k - 1 \), we have

\[
S(0,i) \in R'.
\]

**Proof.** This follows directly from Corollary 2.5 and Lemma 2.9. \( \square \)

3. The Connection Between \( S(i,1) \) and Cyclotomic Units

In this section, we study \( S(i,1) \) more closely. Upon doing so, we will find a criterion for \( S(i,1) \) to lie in \( \ell R \) in terms of cyclotomic units. To start, we consider the \( T^i \)-coefficient of \( S(i,1) \).

Recall that \( g \) is a generator for \( \mathbb{F}_q^\times \). We will abuse notation and let

\[
\zeta_{ft} := g^{\frac{q-1}{T^i}} \quad \zeta_f := \zeta_{f\ell}^i \quad \zeta_\ell := \zeta_{f\ell}^j
\]

be elements of \( \mathbb{F}_q \). That is, when we refer to \( \zeta_{ft}, \zeta_f, \zeta_\ell \) as elements of \( \mathbb{F}_q \), we mean these powers of \( g \). The expression \( \zeta_\ell^{1/f} \) will refer to an \( \ell \)-th root of unity whose \( f \)-th power is \( \zeta_\ell \); it is not a primitive \( f\ell \)-th root of unity. Similarly, \( \zeta_f^{1/\ell} \) will refer to a primitive \( f \)-th root of unity.
Lemma 3.1. Fix an integer $i$ in the range $0 \leq i \leq \ell - 1$. For $0 \leq j \leq f - 1$, the $T^j$-coefficient of $S(i, 1)$ is

$$[T^j]S(i, 1) \equiv \begin{cases} 
- \text{ind}(-f) \pmod{\ell} & \text{if } (i, j) = (0, 0) \\
\text{ind}(1 - \zeta_f^j) \pmod{\ell} & \text{if } i = 0 \text{ and } 1 \leq j \leq f - 1. \\
\sum_{r=0}^{\ell-1} \binom{\ell}{r} \frac{\text{ind}(1 - \zeta_f^{j/\ell} \zeta_{\ell}^r)}{\ell} \pmod{\ell} & \text{if } i \geq 1.
\end{cases}$$

Proof. From the definition, the $T^j$-coefficient is

$$[T^j]S(i, 1) = \sum_{\substack{r \in [1, q-2] \\text{mod } f \equiv j}} \binom{r}{i} \text{ind}(1 - g^r).$$

Let us handle $i = 0$ first. When $i = 0$, this becomes

$$[T^j]S(0, 1) = \text{ind} \left( \prod_{r \in [1, q-2] \\text{mod } f \equiv j} (1 - g^r) \right).$$

There are usually $\frac{q-1}{f}$ terms in this product; the only exception is when $j = 0$. When $j = 0$, the product becomes

$$(1 - g)^{(1 - g^2)} \cdots (1 - g^{q-1-f}).$$

Note that $g^0, g^f, \ldots, g^{q-1-f}$ form the complete set of roots of $X^{(q-1)/f} - 1$, so

$$(X - 1)(X - g^f) \cdots (X - g^{q-1-f}) = X^{(q-1)/f} - 1,$$

which implies

$$(X - g^f) \cdots (X - g^{q-1-f}) = 1 + X + \cdots + X^{(q-1)/f - 1}.$$  

Plugging in $X = 1$ now gives that

$$[T^0]S(0, 1) \equiv \text{ind} \left( \frac{q-1}{f} \right) \equiv - \text{ind}(-f) \pmod{\ell}.$$

Now when $j \geq 1$, we can use the identity

$$(1 - X)(1 - g^f X) \cdots (1 - g^{q-1-f} X) = 1 - X^{(q-1)/f},$$

and plug in $X = g^j$ to get

$$\prod_{\substack{r \in [1, q-2] \\text{mod } f \equiv j}} (1 - g^r) = 1 - g^{j(q-1)/f} = 1 - \zeta_f^j.$$  

Therefore, when $i = 0$ we have

$$[T^j]S(0, 1) \equiv \begin{cases} 
- \text{ind}(-j) \pmod{\ell} & \text{if } j = 0 \\
\text{ind}(1 - \zeta_f^j) \pmod{\ell} & \text{if } 1 \leq j \leq f - 1.
\end{cases}$$
Now suppose that \( i \geq 1 \). We observe that modulo \( \ell \), the binomial coefficient \( \binom{r}{i} \) only depends on \( r \) modulo \( \ell \) (since \( i \leq \ell - 1 \)). So the \( T^j \)-coefficient modulo \( \ell \) is

\[
[T^j]S(i, 1) \equiv \sum_{r \in \{j, j+t, \ldots, j+(\ell-1)f\}} \binom{r}{i} \sum_{s \in [1, q-2]} \text{ind}(1 - g^s) \pmod{\ell}
\]

\[
\equiv \sum_{t=0}^{\ell-1} \binom{j+tf}{i} \sum_{s \equiv j+tf \pmod{\ell}} \text{ind}(1 - g^s) \pmod{\ell}
\]

\[
\equiv \sum_{t=0}^{\ell-1} \binom{j+tf}{i} \text{ind} \left( \prod_{s \equiv j+tf \pmod{\ell}} (1 - g^s) \right) \pmod{\ell}
\]

Note that there are usually \( \frac{q-1}{f\ell} \) terms in the product; the only exception is when \((j, t) = (0, 0)\). In that case, the binomial coefficient \( \binom{j+tf}{i} \) is zero anyway (since \( i \geq 1 \)), so we can ignore such terms. As before, we observe that

\[
(1 - X)(1 - Xg^{f\ell}) \cdots (1 - Xg^{q-1-f\ell}) = 1 - X^{\frac{q-1}{f\ell}}
\]

Plugging in \( X = g^{j+tf} \) (and assuming that \((j, t) \neq (0, 0)\)) gives

\[
\prod_{s \equiv j+tf \pmod{\ell}} (1 - g^s) = 1 - (g^{j+tf})^{\frac{q-1}{f\ell}} = 1 - \zeta_{f\ell}^{j+tf}.
\]

Since \( f \) and \( \ell \) are coprime, we can write

\[
\zeta_{f\ell} = \zeta_f^{\frac{\ell}{f\ell}} \zeta_{\ell}^{\frac{f}{f\ell}} = \zeta_f^{\frac{1}{f}} \zeta_{\ell}^{\frac{1}{\ell}} = \zeta_{f\ell}^{\frac{1}{f}} \zeta_{f\ell}^{\frac{1}{\ell}}.
\]

To be clear, the expression \( \zeta_{f\ell}^{1/f} \) refers to an \( \ell \)-th root of unity whose \( f \)th power is \( \zeta_{\ell} \); it is not a primitive \( f\ell \)-th root of unity. Similarly, \( \zeta_{f\ell}^{1/\ell} \) is an \( f \)-th root of unity.

Substituting this into the previous expression gives

\[
\prod_{s \equiv j+tf \pmod{\ell}} (1 - g^s) = 1 - (g^{j+tf})^{\frac{q-1}{f\ell}} = 1 - \zeta_{f\ell}^{j/\ell} (\zeta_{f\ell}^{1/f})^{j+tf}
\]

Rewriting \( r = j + tf \) and reducing \( r \) modulo \( \ell \) (since \( \binom{r}{i} \) modulo \( \ell \) only depends on \( r \) modulo \( \ell \)) gives

\[
[T^j]S(i, 1) \equiv \sum_{r=0}^{\ell-1} \binom{r}{i} \text{ind}(1 - \zeta_{f\ell}^{j/\ell} \zeta_{f\ell}^{r/f}) \pmod{\ell}.
\]

This completes the proof of the lemma. \( \square \)
From the lemma, we are motivated to define the following elements of $O_L$.

**Definition 3.2.** For integers $i, j$ satisfying $0 \leq i \leq \ell - 1$ and $0 \leq j \leq f - 1$, define

$$
\eta_{i,j} = \prod_{r=0}^{\ell-1} \left(1 - \zeta_j^{r/\ell} \zeta^{r/f}\right)^{(i)} \in O_L = \mathbb{Z}[\zeta_{f\ell}].
$$

Define analogous elements in $\mathbf{F}^\times_q$, where by $\zeta_f$ and $\zeta_{\ell}$ we mean appropriate powers of $g$.

When $i, j > 0$ note that $\eta_{i,j}$ is a cyclotomic unit; in particular, it is an element of $O_L^\times$.

We restate our results in terms of this notation.

**Lemma 3.3.** The variables $i, j$ will always be in the range $0 \leq i \leq \ell - 1$, $0 \leq j \leq f - 1$. The following are true.

1. The condition that $S(0, 1) \in R'$ is equivalent to $1 - \zeta_f, 1 - \zeta_{f^2}, \ldots, 1 - \zeta_{f^{f-1}} \in (\mathbf{F}^\times_q)^f$.
2. For $i \geq 1$, the condition that $S(i, 1) \in R'$ is equivalent to $\eta_{i,1}, \eta_{i,2}, \ldots, \eta_{i,f-1} \in (\mathbf{F}^\times_q)^f$.

**Proof.** The condition that $S(i, 1) \in R'$ is equivalent to $[T^j]S(i, 1) \equiv 0 \pmod{\ell}$ for $j \in \{1, 2, \ldots, f - 1\}$. Hence we are done by Lemma 3.1. □

4. A Recursion for $S(u, v)$

In this section, we will investigate the product of expressions of the form $S(u, v)$. Before doing so, we need a lemma about binomial coefficients.

**Lemma 4.1.** For integers $m, i \geq 0$, we have

$$
m\binom{m}{i} = (i + 1)\binom{m}{i+1} + i\binom{m}{i}.
$$

**Proof.** If $m < i$ then all the terms are zero. If $m = i$ then the right hand side becomes $m$ and the left hand side also becomes $m$. So now suppose $m \geq i + 1$.

Then this is equivalent to checking that $(i + 1)\binom{m}{i+1} = (m - i)\binom{m}{i}$. Expanding out with factorials gives

$$(i + 1)\binom{m}{i+1} = (i + 1)\frac{m!}{(i+1)!(m-i-1)!} = \frac{m!}{i!(m-i-1)!},$$

and

$$(m - i)\binom{m}{i} = (m - i)\frac{m!}{i!(m-i)!} = \frac{m!}{i!(m-i-1)!},$$

so we are done. □

**Lemma 4.2.** Suppose $i, s$ are integers such that $1 \leq i \leq \ell - 1$ and $1 \leq s \leq i$. Then

$$(i - s + 1)S(i - s + 1, s) - (s + 1)S(i - s, s + 1)$$

$$\equiv \sum_{r=0}^{i-s} S(i - s - r, s)S(r, 1) - \sum_{t=1}^{s} (T(1, s-t) + \delta_{t=s} \cdot (i - 2s))S(i - s, t) \pmod{R'}.$$

8
Proof. Expanding out the product, we have

\[
\sum_{r=0}^{i-s} S(i-s-r,s)S(r,1) = \sum_{y,z \in \mathbb{F}_q \setminus \{0,1\}} \sum_{r=0}^{i-s} \binom{\text{ind}(y)}{i-s-r} \binom{\text{ind}(z)}{r} \binom{\text{ind}(1-y)}{s} \text{ind}(1-z)^{\text{ind} y T^{\text{ind} z}}.
\]

Since \( T^{q-1} = 1 \), we can express \( T^{\text{ind} y T^{\text{ind} z}} \) as \( T^{\text{ind} y z} \). Also, we have the identity

\[
\sum_{r=0}^{i-s} \binom{\text{ind}(y)}{i-s-r} \binom{\text{ind}(z)}{r} = \binom{\text{ind}(y) + \text{ind}(z)}{i-s}.
\]

Since \( i-s \leq \ell-2 \), we know that \( \binom{\text{ind}(y) + \text{ind}(z)}{i-s} \equiv \binom{\text{ind}(y z)}{i-s} \mod \ell \). Putting this all together gives

\[
\sum_{r=0}^{i-s} S(i-s-r,s)S(r,1) \equiv \sum_{y,z \in \mathbb{F}_q \setminus \{0,1\}} \binom{\text{ind}(1-y)}{s} \text{ind}(1-z)^{\text{ind} y z} T^{\text{ind} y z} \pmod{\ell R}.
\]

Now, perform the change of variables \( x = y z \). Then we get

\[
\sum_{r=0}^{i-s} S(i-s-r,s)S(r,1) \equiv \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \binom{\text{ind}(1-y)}{s} \text{ind} \left( 1 - \frac{x}{y} \right) \text{ind} \left( \frac{x}{i-s} \right) T^{\text{ind} x} \pmod{\ell R}.
\]

We can throw away the term when \( x = 1 \), at the cost of our equation only being true modulo \( R' \). So we get

\[
\sum_{r=0}^{i-s} S(i-s-r,s)S(r,1) \equiv \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \binom{\text{ind}(1-y)}{s} \text{ind} \left( 1 - \frac{x}{y} \right) \text{ind} \left( \frac{x}{i-s} \right) T^{\text{ind} x} \pmod{R'}.
\]

Now we write \( \text{ind} \left( 1 - \frac{x}{y} \right) = \text{ind}(y-x) - \text{ind} y \) to convert our expression to

(1) \[
\sum_{r=0}^{i-s} S(i-s-r,s)S(r,1) \equiv \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \binom{\text{ind}(1-y)}{s} \text{ind} (y-x) \text{ind} \left( \frac{x}{i-s} \right) T^{\text{ind} x}
\]

(2) \[
- \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \binom{\text{ind}(1-y)}{s} \text{ind}(y) \text{ind} \left( \frac{x}{i-s} \right) T^{\text{ind} x}
\]
We analyze each piece separately. The second one is a bit easier. We have that (2) equals
\[
\sum_{x \in F_q \setminus \{0,1\}} \left( \text{ind} \left( \frac{1-y}{s} \right) \text{ind} \left( \frac{x}{i-s} \right) T^{\text{ind} x} \right) = \sum_{x \in F_q \setminus \{0,1\}} \left( \text{ind} \left( \frac{1-y}{s} \right) \text{ind} \left( \frac{x}{i-s} \right) T^{\text{ind} x} \right)
\]
\[
- \sum_{x \in F_q \setminus \{0,1\}} \left( \text{ind} \left( \frac{1-y}{s} \right) \text{ind} \left( \frac{x}{i-s} \right) T^{\text{ind} x} \right)
\]
\[
= T(1, s) S(i - s, 0)
\]
\[
- \sum_{x \in F_q \setminus \{0,1\}} \left( \text{ind} \left( \frac{1-x}{s} \right) \text{ind} \left( \frac{x}{i-s} \right) T^{\text{ind} x} \right).
\]

Using Lemma 4.1 we can write
\[
\text{ind} \left( \frac{x}{i-s} \right) \left( \text{ind} \left( \frac{x}{i-s} \right) \right) = \left( i - s + 1 \right) \left( \text{ind} \left( \frac{x}{i-s} \right) \right) + \left( i - s \right) \left( \text{ind} \left( \frac{x}{i-s} \right) \right).
\]

Plugging this into our work for (2) gives that
\[
(2) = T(1, s) S(i - s, 0) - (i - s + 1) S(i - s, 1) - (i - s) S(i - s, s).
\]

Now consider the term (1). Note that since \( s \geq 1 \), the \( y = 0 \) term just gives zero. So we can add it back in to rewrite
\[
\text{ind} \left( \frac{x}{i-s} \right) \left( \text{ind} \left( \frac{x}{i-s} \right) \right) = \left( \sum_{x,y \in F_q \setminus \{0,1\}} \left( \text{ind} \left( \frac{1-y}{s} \right) \text{ind} \left( \frac{x-y}{i-s} \right) T^{\text{ind} x} \right) \right)
\]
\[
\text{ind} \left( \frac{x}{i-s} \right) \left( \text{ind} \left( \frac{x}{i-s} \right) \right) = \left( \sum_{x,y \in F_q \setminus \{0,1\}} \left( \text{ind} \left( \frac{1-x(1-y)}{s} \right) \text{ind} \left( \frac{1-y}{i-s} \right) T^{\text{ind} x} \right) \right)
\]
\[
\text{Since } s \leq \ell - 1 \text{ and we are working modulo } \ell, \text{ we can write } \text{ind}((1-x)(1-y)) = \text{ind}(1-x) + \text{ind}(1-y) \text{ and }
\]
\[
\left( \text{ind} \left( \frac{(1-x)(1-y)}{s} \right) \right) \equiv \left( \text{ind} \left( \frac{1-x}{s} \right) + \text{ind} \left( \frac{1-y}{s} \right) \right).
\]

Similarly,
\[
\text{ind}((1-x)y) \equiv \text{ind}(1-x) + \text{ind}(y).
\]

Now we use the Vandermonde identity to write
\[
\left( \text{ind} \left( \frac{1-x + 1-y}{s} \right) \right) = \sum_{t=0}^{s} \left( \text{ind} \left( \frac{1-x}{t} \right) \right) \left( \text{ind} \left( \frac{1-y}{s-t} \right) \right).
\]
Substituting everything back into (1) and expanding gives
\[
\sum_{x,y \in \mathbb{F}_q \setminus \{0,1\}} \left( \text{ind}((1-x)(1-y)) \right) \text{ind}((1-x)y) \frac{\text{ind} x}{s} T^{\text{ind} x}
\]
\[
= \sum_{t=0}^{s} \sum_{x,y \in \mathbb{F}_q \setminus \{0,1\}} \left( \text{ind} x \right) \text{ind}(1-x) \left( \text{ind}(1-x) \right) \left( \text{ind}(1-y) \right) T^{\text{ind} x}
\]
\[+ \sum_{t=0}^{s} \sum_{x,y \in \mathbb{F}_q \setminus \{0,1\}} \left( \text{ind} x \right) \left( \text{ind}(1-x) \right) \text{ind}(y) \left( \text{ind}(1-y) \right) T^{\text{ind} x}
\]
Using Lemma 4.1 to write \(\text{ind}(1-x)\left(\text{ind}(1-x)\right) = (t+1)\left(\text{ind}(1-x)\right) + (\text{ind}(1-x))\), we can simplify the previous expression to
\[
\sum_{t=0}^{s} T(0, s-t) ((t+1)S(i-s, t+1) + tS(i-s, t)) + T(1, s-t)S(i-s, t)
\]
From Lemma 2.7 we know that \(T(0, i) \equiv 0 \mod \ell\) if \(i \geq 1\). Furthermore, one can check that \(T(0, 0) = q - 2 \equiv -1 \mod \ell\), so we can simplify the previous expression to
\[
\sum_{t=0}^{s} T(1, s-t)S(i-s, t)
\]
Combining our expressions for (1) and (2) now gives
\[
\sum_{t=0}^{s} T(1, s-t)S(i-s, t)
\]
\[- T(1, s)S(i-s, 0) + (i-s+1)S(i-s+1, s) + (i-2s)S(i-s, s).
\]
Note that the \(t = 0\) term in the sum cancels with the term which follows it. We can write this expression more compactly then as
\[
(i-s+1)S(i-s+1, s) - (s+1)S(i-s, s+1) + \sum_{t=1}^{s} (T(1, s-t) + \delta_{t=s} \cdot (i-2s))S(i-s, t).
\]
This completes the proof.

We will use Lemma 4.2 in the following way.

**Corollary 4.3.** Suppose that \(i\) is an integer such that \(1 \leq i \leq \ell-2\). Assume that \(S(u, v) \in R'\) holds whenever \(u + v \leq i\) and \(v \geq 1\). Then if any of the expressions
\[S(i, 1), S(i - 1, 2), \ldots, S(0, i + 1)\]
happens to land in \(R'\), then they must all land in \(R'\).

**Proof.** Under the assumption that \(S(u, v) \in R'\) holds for \(u + v \leq i\) and \(v \geq 1\), we can take the expression in Lemma 4.2 modulo \(R'\) to see that the entire right hand side is zero modulo \(R'\) (that is, lands in \(R'\)). Therefore, we know that
\[(i-s+1)S(i-s+1, s) \equiv (s+1)S(i-s, s+1) \mod R'\]
holds for \(1 \leq s \leq i\). Since \(i-s+1, s+1 \leq \ell-1\), we know that these coefficients are invertible modulo \(\ell\). Hence from this expression we know that \(S(i-s+1, s) \in R'\) if and
only if \( S(i-s, s+1) \in R' \). As this holds for all \( s \) in the range \( 1 \leq s \leq i \), this completes the proof.

5. Main Theorem

Now we combine all of our results from the previous sections in the following lemma.

**Lemma 5.1.** Fix an integer \( k \) in the range \( 1 \leq k \leq \ell - 1 \). Then the following are equivalent.

1. \( S(0, 1), S(1, 1), \ldots, S(k-2, 1) \) lie in \( R' \).
2. \( S(u, v) \) lies in \( R' \) for \( u + v \leq k - 1 \) and \( v \geq 1 \).
3. \( J(\ell, f) + 1 \) is divisible by \( \pi_\ell^k \).

**Proof.** The implication (1) implies (2) comes from Corollary 4.3. Certainly (2) implies (1).

By Corollary 2.10, we know that (3) is equivalent to \( S(0, 1), S(0, 2), \ldots, S(0, k-1) \) lying in \( R' \). Therefore again by Corollary 4.3 we know that (3) implies (2). Certainly (2) implies (3).

**Theorem 5.2.** Fix an integer \( k \) in the range \( 3 \leq k \leq \ell - 1 \). Then

1. \( J(\ell, f) + 1 \) is always divisible by \( \pi_\ell \).
2. \( J(\ell, f) + 1 \) is divisible by \( \pi_\ell^2 \) if and only if
   \[
   1 - \zeta_f, 1 - \zeta_f^2, \ldots, 1 - \zeta_f^{f-1} \in (F_q^\times)^\ell.
   \]
3. \( J(\ell, f) + 1 \) is divisible by \( \pi_\ell^k \) if and only if
   \[
   1 - \zeta_f, 1 - \zeta_f^2, \ldots, 1 - \zeta_f^{f-1} \in (F_q^\times)^\ell
   \]
   \[
   \eta_{1,1}, \eta_{1,2}, \ldots, \eta_{1,f-1} \in (F_q^\times)^\ell
   \]
   \[
   \vdots
   \]
   \[
   \eta_{k-2,1}, \eta_{k-2,2}, \ldots, \eta_{k-2,f-1} \in (F_q^\times)^\ell
   \]

**Proof.** Combine Lemma 3.1 and Lemma 5.1 (using (1) \( \iff \) (3) in the latter).

6. The case \( k = \ell \)

Although Theorem 5.2 was stated only up to \( k = \max\{2, \ell - 1\} \), our work allows us to understand the case \( k = \ell \) as well. First we need a version of Corollary 2.10. We may as well assume that \( \ell \geq 3 \) from now on, since we understand the \( k = 2 \) case from Theorem 5.2.

**Lemma 6.1.** Suppose that \( \ell \geq 3 \). We have that \( J(\ell, f) + 1 \) is divisible by \( \pi_\ell^k \) if and only if the following lie in \( R' \):

\[
S(0, 1), S(0, 2), \ldots, S(0, \ell - 2), S(0, \ell - 1) - \frac{q-1}{\ell f}(1 + T + T^2 + \cdots + T^{f-1}).
\]

**Proof.** From Corollary 2.5 we know that \( J(\ell, f) + 1 \) is divisible by \( \pi_\ell^k \) if and only if the following holds:

\[
\sum_{x \in F_q \setminus \{0, 1\}} \left( \text{ind}(1-x) \right)_i \zeta_f^{\text{ind}(x)} \in \ell \mathcal{O}_M \text{ for } 1 \leq i \leq \ell - 1.
\]
From Lemma 2.9 this is equivalent to
\[ S(0, i) \in R' \quad \text{for } 1 \leq i \leq \ell - 2 \]
\[ \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \left( \frac{\text{ind}(1-x)}{\ell - 1} \right) \zeta_f^\text{ind(x)} \in \ell \mathcal{O}_M \quad \text{for } i = \ell - 1. \]

Therefore, we need to check that the following two conditions are equivalent.

1. \[ \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \left( \frac{\text{ind}(1-x)}{\ell - 1} \right) \zeta_f^\text{ind(x)} \text{ lies in } \ell \mathcal{O}_M. \]

2. \[ S(0, \ell - 1) - \frac{q - 1}{\ell f}(1 + T + T^2 + \cdots + T^{f-1}) \text{ lies in } R'. \]

To do so, write
\[ S(0, \ell - 1) \equiv a_0 + a_1 T + \cdots + a_{f-1} T^{f-1} \pmod{\ell R} \]
for integers \( a_0, \ldots, a_{f-1} \) in the range \([0, \ell - 1]\). Then

- Condition (1) is equivalent to the statement that all the \( a_i \) are equal.
- Condition (2) is equivalent to the statement that \( a_1, a_2, \ldots, a_{f-1} \equiv \frac{q - 1}{\ell f} \pmod{\ell} \)

Finally, we know that
\[ a_0 + a_1 + \cdots + a_{f-1} = \sum_{x \in \mathbb{F}_q \setminus \{0,1\}} \left( \frac{\text{ind}(1-x)}{\ell - 1} \right) = \sum_{r=1}^{q-2} \binom{r}{\ell - 1} = \frac{q - 1}{\ell} \left( \frac{q - 2}{\ell - 1} \right). \]

Since \( q - 2 \equiv \ell - 1 \mod{\ell} \) and the denominator of \( \frac{q - 2}{\ell - 1} \) is coprime to \( \ell \), it follows that \( \frac{q - 2}{\ell - 1} \equiv \frac{\ell - 1}{\ell - 1} = 1 \mod{\ell} \). Hence,
\[ a_0 + a_1 + \cdots + a_{f-1} \equiv \frac{q - 1}{\ell} \mod{\ell}. \]

Therefore, all the \( a_i \) are equal if all the \( a_1, a_2, \ldots, a_{f-1} \) are congruent to \( \frac{q - 1}{\ell f} \mod{\ell} \), as desired.

Now we apply our recursive formula to generate the \( S(u, v) \) to get an equivalent condition for \( J(\ell, f) + 1 \) to be divisible by \( \pi_\ell^f \).

**Lemma 6.2.** Suppose that \( \ell \geq 3 \). We have that \( J(\ell, f) + 1 \) is divisible by \( \pi_\ell^f \) if and only if the following lie in \( R' \):
\[ S(0, 1), S(1, 1), \ldots, S(\ell - 3, 1), S(\ell - 2, 1) + \frac{q - 1}{\ell f}(1 + T + T^2 + \cdots + T^{f-1}). \]

**Proof.** By Corollary 4.3, the assumption that \( S(0, 1), S(1, 1), \ldots, S(\ell - 3, 1) \) lie in \( R' \) is the same as \( S(u, v) \in R' \) whenever \( u + v \leq \ell - 2 \) and \( v \geq 1 \).

From Lemma 6.1 we need to check that under the assumption that \( S(u, v) \in R' \) for \( u + v \leq \ell - 2 \) and \( v \geq 1 \), that \( S(0, \ell - 1) + S(\ell - 2, 1) \) lies in \( R' \). Now taking \( i = \ell - 2 \) in Lemma 4.2 and noting that the right hand side is in \( R' \) by assumption, we see that
\[ (\ell - s - 1)S(\ell - s - 1, s) \equiv (s + 1)S(\ell - s - 2, s + 1) \mod{R'} \]
holds for \( 1 \leq s \leq \ell - 2 \). Since \( \ell R \subseteq R' \), we can rewrite this as
\[ -(s + 1)S(\ell - s - 1, s) \equiv (s + 1)S(\ell - s - 2, s + 1) \mod{R'} \]
Note that $s + 1$ is invertible modulo $\ell$, so we can cancel the $s + 1$ from both sides to get

$$-S(\ell - s - 1, s) \equiv S(\ell - s - 2, s + 1) \mod R'$$

Therefore,

$$S(0, \ell - 1) \equiv -S(1, \ell - 2) \equiv S(2, \ell - 3) \equiv \cdots \equiv (-1)^{\ell - 2}S(\ell - 2, 1) = -S(\ell - 2, 1) \pmod{R'}.$$  

This shows that $S(0, \ell - 1) + S(\ell - 2, 1)$ lies in $R'$, as desired. \(\square\)

Now we use Lemma 3.1 to restate the condition in Lemma 6.2 in terms of cyclotomic units. This gives the following.

**Lemma 6.3.** Suppose that $\ell \geq 3$. We have that $J(\ell, f) + 1$ is divisible by $\pi^i_\ell$ if and only if the following are divisible by $\ell$.

\[
\begin{array}{cccc}
\text{ind}(1 - \zeta_\ell^i) & \text{ind}(1 - \zeta_\ell^j) & \cdots & \text{ind}(1 - \zeta_\ell^{\ell - 1}) \\
\text{ind}(\eta_{1,1}) & \text{ind}(\eta_{1,2}) & \cdots & \text{ind}(\eta_{1,3}) \\
\vdots & \vdots & \ddots & \vdots \\
\text{ind}(\eta_{\ell - 3,1}) & \text{ind}(\eta_{\ell - 3,2}) & \cdots & \text{ind}(\eta_{\ell - 3,\ell - 1}) \\
\text{ind}(\eta_{\ell - 2,1}) + \frac{q - 1}{\ell} & \text{ind}(\eta_{\ell - 2,2}) + \frac{q - 1}{\ell} & \cdots & \text{ind}(\eta_{\ell - 2,\ell - 1}) + (-1)^{\ell - 2}\frac{q - 1}{\ell}
\end{array}
\]

We can slightly improve this criterion.

**Corollary 6.4.** Suppose that $\ell \geq 3$. We have that $J(\ell, f) + 1$ is divisible by $\pi^i_\ell$ if and only if the following are divisible by $\ell$.

1. $\frac{q - 1}{\ell}$
2. $\text{ind}(1 - \zeta_\ell^i \zeta_\ell^j)$, for all $i, j$ in the range $1 \leq i \leq f - 1$ and $0 \leq j \leq \ell - 1$.

**Proof.** Choose an integer $s$ in the range $0 \leq s \leq \ell - 2$ and consider the following product:

$$\prod_{i=0}^{s} \eta_{\ell - 2 - i, j} = \prod_{r=0}^{s} \prod_{i=0}^{r} \left(1 - \zeta_\ell^{s + r}\zeta_\ell^{r}\right)^{\binom{s}{i}}$$

Again by an application of the Vandermonde identity, one can check that when $r$ is fixed, that

$$\sum_{i=0}^{s} \binom{s}{i} \binom{r}{\ell - 2 - i} = \binom{s + r}{\ell - 2}.$$  

When $r$ is in the range $0 \leq r \leq \ell - 3 - s$ or in the range $\ell - s \leq r \leq \ell - 1$, this expression is divisible by $\ell$ since the numerator will contain a factor of $\ell$, but the denominator will not. When $r = \ell - 2 - s$ then this expression is 1 and when $r = \ell - 1 - s$ this expression is $\ell - 1$. Therefore, we see that

$$\sum_{i=0}^{s} \binom{s}{i} \text{ind}(\eta_{\ell - 2 - i, j}) \equiv \text{ind}(1 - \zeta_\ell^{s + r}\zeta_\ell^{s + r - 2}) - \text{ind}(1 - \zeta_\ell^{s + r}\zeta_\ell^{s + r - 2}) \pmod{\ell}.$$  

Taking $s = \ell - 2$ first, note that $\eta_{0,j} = 1 - \zeta_\ell^j$. Therefore if we assume the conclusion of Lemma 6.3, then the left hand side equals $-\frac{q - 1}{\ell}$ modulo $\ell$. On the right hand side, the first term will be divisible by $\ell$, so we get

$$\text{ind}(1 - \zeta_\ell^{s + r}\zeta_\ell^{s + r - 2}) \equiv \frac{q - 1}{\ell} \pmod{\ell}.$$
Now taking \( s = \ell - 3 \) we know that
\[
\frac{q - 1}{\ell f} \equiv \text{ind}(1 - \zeta_f^{j/\ell} \zeta_\ell^{2/\ell}) - \text{ind}(1 - \zeta_f^{j/\ell} \zeta_\ell^{2/\ell}) \pmod{\ell}
\]
so we conclude that
\[
\text{ind}(1 - \zeta_f^{j/\ell} \zeta_\ell^{2/\ell}) \equiv 2 \left( \frac{q - 1}{\ell f} \right) \pmod{\ell}
\]
Inducting on \( r \), we get
\[
\text{ind}(1 - \zeta_f^{j/\ell} \zeta_\ell^{r/\ell}) \equiv r \left( \frac{q - 1}{\ell f} \right) \pmod{\ell}
\]
for \( r \) in the range \( 0 \leq r \leq \ell - 1 \). Now we use the identity
\[
(1 - \zeta_f^{j/\ell} \zeta_\ell^{r/\ell}) = -\zeta_f^{j/\ell} \zeta_\ell^{r/\ell} (1 - \zeta_f^{-j/\ell} \zeta_\ell^{-r/\ell})
\]
and take ind of both sides modulo \( \ell \) to get that
\[
\left( \frac{q - 1}{\ell f} \right) \equiv \text{ind}(-\zeta_f^{j/\ell} \zeta_\ell^{r/\ell}) - \text{ind}(1 - \zeta_f^{-j/\ell} \zeta_\ell^{-r/\ell}) \pmod{\ell}.
\]
By definition of \( \zeta_{f\ell}, \zeta_f, \) and \( \zeta_\ell \), we know that \( \text{ind} \zeta_{f\ell} = \frac{q - 1}{f \ell}, \zeta_{f\ell} = \zeta_f^{1/\ell} \zeta_\ell^{1/\ell} \), so we can write
\[
-\zeta_f^{j/\ell} \zeta_\ell^{r/\ell} = -\zeta_f^{(j-r)/\ell} \zeta_{f\ell}.
\]
Note that \( \text{ind} \zeta_f^{1/\ell} \equiv 0 \pmod{\ell} \) and \( \text{ind}(-1) = \frac{q - 1}{2} \equiv 0 \pmod{\ell} \), so
\[
\text{ind}(-\zeta_f^{j/\ell} \zeta_\ell^{r/\ell}) = \text{ind}(-\zeta_f^{(j-r)/\ell} \zeta_{f\ell}) \equiv r \text{ind} \zeta_{f\ell} = r \left( \frac{q - 1}{f \ell} \right) \pmod{\ell}.
\]
Hence the right hand side of equation (4) is just 0 \( \pmod{\ell} \). Taking \( r = 1 \) for example, we now see that \( \frac{q - 1}{f \ell} \) is divisible by \( \ell \). Now from equation (3) we conclude that each \( \text{ind}(1 - \zeta_f^{j/\ell} \zeta_\ell^{r/\ell}) \) is divisible by \( \ell \). This proves the forward direction of the corollary. The backwards direction is easy since it immediately implies that all the quantities in Lemma 6.3 are divisible by \( \ell \).

We summarize the result of this section in the following theorem.

**Theorem 6.5.** Suppose \( \ell \geq 3 \). Then \( J(\ell, f) + 1 \) is divisible by \( \pi_\ell \) if and only if the following conditions hold.

1. \( q \equiv 1 \pmod{\ell^2 f} \)
2. For every \( i, j \) in the range \( 1 \leq i \leq f - 1 \) and \( 0 \leq j \leq \ell - 1 \), we have that \( 1 - \zeta_f^i \zeta_\ell^j \in (F_q^\times)^\ell \).

**References**


