1 Lecture review

1.1 Stokes’ theorem

1. Stokes’ theorem is a 3D generalization of the tangential form of the 2D Green’s theorem. For a surface $S$ that is bounded, piecewise smooth, and simple with boundary curve $C$ such that $C$ and $S$ are oriented by the right hand rule, and a vector field $\mathbf{F}$ that is continuously differentiable on $S$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS$$

2. This table organizes the relationships between the various theorems in the course.

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3. In Stokes’ theorem, when $S$ is a region in the $xy$-plane, one gets precisely the normal form of Green’s theorem, because in that case, $\mathbf{n} = \mathbf{k}$ and so

$$\text{integrand in Stokes’} = (\nabla \times (M \mathbf{i} + N \mathbf{j})) \cdot \mathbf{k} = N_x - M_y = \text{integrand in Green’s}.$$  

4. Stokes’ theorem lets us understand the meaning of $\text{curl} \mathbf{F}$; it is the vorticity of $\mathbf{F}$. Qualitatively, it gives the magnitude of swirl/angular velocity. Quantitatively, $\mathbf{u} \cdot \text{curl} \mathbf{F}$ is twice the angular velocity of $\mathbf{F}$ in the $\mathbf{u}$-direction.

5. Stokes’ theorem gives that the integral of $(\text{curl} \mathbf{F}) \cdot \mathbf{n}$ is the same for any two surfaces with the same boundary. In practice, this allows one to compute $\iint_S (\text{curl} \mathbf{F}) \cdot \mathbf{n} \, dS$ in one of three ways: (i) directly, using the formula for surface integrals, (ii) replacing $S$ with a (simpler) surface $T$ with the same boundary curve, (iii) calculating the work done by $\mathbf{F}$ along the boundary of $S$ (oriented via the right hand rule).
2 Problems

1. Verify Stokes’ theorem for the following vector fields $F$ and curves $C$/surfaces $S$.

   (a) $F = (y+z)i + (x-z)j + (-x+y)k$, $C$ is the curve given as the intersection of the paraboloid $z = 2 - x^2 - y^2$ and the plane $2x + 2y - z = 0$.

   (b) $F = yi - xj$, $S$ is the upper hemisphere centered at the origin of radius 2, oriented upward.

   (c) $F = (x+2z+z^2)i + (2x+y+x^2)j + (2y+z+y^2)k$, $C$ is the triangle with vertices at $(1,0,0), (0,1,0), (0,0,1)$, oriented clockwise when viewed from above.

   (d) $F = yi + xzj + yk$, $C$ is the boundary of the half-circular cylinder $S, x^2 + y^2 = 1$, $y \geq 0, 0 \leq z \leq 1$ with corners at $(1,0,0), (-1,0,0), (-1,0,1), (1,0,1)$, oriented in that order.

2. Let $F = 3yzi + (3x-2z)j + (xy-x)k$, $R$ be the portion of the ellipsoid $x^2 + y^2 / 4 + z^2 / 9 \leq 1$ in the first octant, $S_{xy}, S_{yz}, S_{xz}, S_{top}$ be the boundary pieces of $R$ where the first three lie in the $xy$-, $yz$-, $xz$-planes and the fourth is the curved top surface.

   (a) Compute the outward flux of $\nabla \times F$ across $S_{top}$ directly.

   (b) Using Stokes’ theorem, relate the above quantity to the sum of the fluxes across $S_{xy}, S_{yz}, S_{xz}$. Compute this sum and verify that it matches your answer in the previous part.

   (c) Using Stokes’ theorem, relate the above quantities to the integral of the work done by $F$ over the boundary. Compute this and verify that it matches your answer in the previous part.

3. Let $F = yzi - xzj + k$. Let $S$ be the portion of the surface of the paraboloid $z = 4 - x^2 - y^2$ lying above the first octant $x, y, z \geq 0$; and let $C$ be the closed curve $C = C_1 + C_2 + C_3$, where the curves $C_1, C_2, C_3$ are the three curves formed by intersecting $S$ with the $xy$-, $yz$-, and $xz$-planes respectively (so that $C$ is the boundary of $S$). Orient $C$ so that it is traversed counterclockwise when seen from above in the first octant.

   (a) Use Stokes’ theorem to compute the work integral $\int_C F \cdot dr$ by using the surface integral over the capping surface $S$.

   (b) Let $S_1, S_2, S_3$ be the three surfaces formed by intersecting the paraboloid with the $yz$-, $xz$-, and $xy$-planes respectively. Use Stokes’ theorem to compute the work integral $\int_C F \cdot dr$ by using the surface integrals over $S_1, S_2, S_3$.

   (c) Set up and evaluate the work integral $\int_C F \cdot dr$ by parametrizing each piece of the curve $C$ and then adding up the three line integrals.

4. Use Stokes’ theorem to calculate the flux of $\nabla \times F$ across $S$ by reducing it to the same flux but over a simpler surface.
(a) \( \mathbf{F} = x^3 \mathbf{i} + y^4 \mathbf{j} + z^3 \sin(xy) \mathbf{k} \), \( S \) is the upper half of the ellipsoid \( x^2 + y^2 + z^2/9 = 1 \) with downward orientation.

(b) \( \mathbf{F} = (y + xz) \mathbf{i} + (5 - x) \mathbf{j} + (2e^x) \mathbf{k} \), \( S \) is the lower hemisphere given by \( x^2 + y^2 + z^2 = 1 \) and \( z \leq 0 \) with downward orientation.

(c) Let \( \mathbf{F} = M \mathbf{i} + N \mathbf{j} + P \mathbf{k} \). Show that \( \nabla \cdot (\nabla \times \mathbf{F}) = 0 \). Use this to explain why the divergence theorem can also be used to explain why the flux in the previous two parts is unchanged upon replacing \( S \) by a simpler surface with the same boundary.

5. Let \( \mathbf{F} = 2z \mathbf{i} + 3x \mathbf{j} + 4y \mathbf{k} \) be a vector field and \( C \) a simple positively oriented curve lying in some plane \( P \). Assume that \( C \) encloses a region \( R \) of area 5. Suppose that \( P \) is chosen to maximize the work done by \( \mathbf{F} \) along \( C \). Compute the equation for the plane \( P \).

3 Answers

1. (a) \(-32\pi\), (b) \(-8\pi\), (c) \(-4\), (d) \(-\pi/2\)
2. (a) \(21\pi/4\), (b) \(S_{xy}: 3\pi/2, S_{yz}: 3\pi, S_{xz}: 3\pi/4\), (c) \(C_{xy}: 3\pi/2, C_{yz}: 3\pi, C_{xz}: 3\pi/4\)
3. (a) 0, (b) all 0, (c) \(C_1: 4, C_2: -4, C_3: 0\)
4. (a) 0, (b) \(2\pi\)
5. \(4x + 2y + 3z = d\)