1 Lecture review

1.1 Stokes’ theorem

1. Stokes’ theorem is a 3D generalization of the tangential form of the 2D Green’s theorem. For a surface $S$ that is bounded, piecewise smooth, and simple with boundary curve $C$ such that $C$ and $S$ are oriented by the right hand rule, and a vector field $F$ that is continuously differentiable on $S$, we have

$$\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot n \, dS$$

2. This table organizes the relationships between the various theorems in the course.

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3. In Stokes’ theorem, when $S$ is a region in the $xy$-plane, one gets precisely the normal form of Green’s theorem, because in that case, $n = k$ and so

$$\text{integrand in Stokes’} = (\nabla \times (M i + N j)) \cdot k = N_x - M_y = \text{integrand in Green’s}.$$ 

4. Stokes’ theorem lets us understand the meaning of $\text{curl} F$; it is the vorticity of $F$. Qualitatively, it gives the magnitude of swirl/angular velocity. Quantitatively, $u \cdot \text{curl} F$ is twice the angular velocity of $F$ in the $u$-direction.

5. Stokes’ theorem gives that the integral of $(\text{curl} F) \cdot n$ is the same for any two surfaces with the same boundary. In practice, this allows one to compute $\iint_S (\text{curl} F) \cdot n \, dS$ in one of three ways: (i) directly, using the formula for surface integrals, (ii) replacing $S$ with a (simpler) surface $T$ with the same boundary curve, (iii) calculating the work done by $F$ along the boundary of $S$ (oriented via the right hand rule).
2 Problems

1. Verify Stokes’ theorem for the following vector fields $\mathbf{F}$ and curves $C$/surfaces $S$.

   (a) $\mathbf{F} = (y + z)\mathbf{i} + (x - z)\mathbf{j} + (-x + y)\mathbf{k}$, $C$ is the curve given as the intersection of the paraboloid $z = 2 - x^2 - y^2$ and the plane $2x + 2y - z = 0$.

   (b) $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$, $S$ is the upper hemisphere centered at the origin of radius 2, oriented upward.

   (c) $\mathbf{F} = (x + 2z + z^2)\mathbf{i} + (2x + y + x^2)\mathbf{j} + (2y + z + y^2)\mathbf{k}$, $C$ is the triangle with vertices at $(1, 0, 0), (0, 1, 0),$ and $(0, 0, 1)$, oriented clockwise when viewed from above.

   (d) $\mathbf{F} = y\mathbf{i} + xz\mathbf{j} + y\mathbf{k}$, $C$ is the boundary of the half-circular cylinder $S$, $x^2 + y^2 = 1$, $y \geq 0$, $0 \leq z \leq 1$ with corners at $(1, 0, 0), (-1, 0, 0), (-1, 0, 1), (1, 0, 1)$, oriented in that order.

Solution.

(a) Computation yields that $\nabla \times \mathbf{F} = 2\mathbf{i} + 2\mathbf{j}$, so $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = (2\mathbf{i} + 2\mathbf{j}) \cdot (-2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) / 3 = -8/3$ and hence

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = -\frac{8}{3} \iiint_S 1 \, dS.$$

The region $S$ is an ellipse given by the graph of $z = 2x + 2y$ over the shadow region $R = \{(x + 1)^2 + (y + 1)^2 = 4\}$, so $dS = \sqrt{2^2 + 2^2 + 1^2} \, dA = 3 \, dA$ and hence

$$\iiint_S 1 \, dS = 3 \cdot (\pi \cdot 4) = 12\pi,$$

so

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \left[-32\pi\right].$$

Alternatively, parametrize $C$ as

$$\mathbf{r}(t) = (-1 + 2 \cos(t))\mathbf{i} + (-1 + 2 \sin(t))\mathbf{j} + (4 \cos(t) + 4 \sin(t) - 4)\mathbf{k}$$

and then one may compute that

$$\mathbf{F} \cdot d\mathbf{r} = -20 \sin^2(t) + 10 \sin(t) - 12 \cos^2(t) + 6 \cos(t)$$

and the integral from 0 to $2\pi$ of this is clearly $-20\pi + 0 - 12\pi + 0 = \left[-32\pi\right]$.

(b) We have

$$\nabla \times \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & 0 \end{pmatrix} = -2\mathbf{k}$$

$$\mathbf{n} = (\sin \varphi \cos \theta)\mathbf{i} + (\sin \varphi \sin \theta)\mathbf{j} + (\cos \theta)\mathbf{k}$$

$$dS = 2^2 \sin \varphi \, d\varphi \, d\theta$$
and hence
\[\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \int_0^{2\pi} \int_0^{\pi/2} (-2 \cos \varphi) (2^2 \sin \varphi \, d\varphi \, d\theta) = 2\pi [2 \cos 2\varphi]_0^{\pi/2} = -8\pi.\]

Alternatively, parametrizing \(C\) as
\[\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j}, \quad 0 \leq t \leq 2\pi\]
gives
\[
\mathbf{F}(t) = 2 \sin t \mathbf{i} - 2 \cos t \mathbf{j} \\
\mathbf{dr}(t) = (-2 \sin t \mathbf{i} + 2 \cos t \mathbf{j}) \, dt \\
\mathbf{F} \cdot \mathbf{dr} = (-4 \sin^2 t - 4 \cos^2 t) \, dt = -4 \, dt.
\]
and hence
\[\int_C \mathbf{F} \cdot \mathbf{dr} = \int_0^{2\pi} -4 \, dt = -8\pi.\]

(c) Notice that \(C\) has three pieces. We find the flux through the piece from \((1,0,0)\) to \((0,0,1)\). Parametrize this as
\[\mathbf{r}(t) = (1-t) \mathbf{i} + t \mathbf{k}, \quad 0 \leq t \leq 1\]
and then
\[
\mathbf{F} \cdot \mathbf{dr} = \left(\left((-1-t)+2(t)+(t)^2\right) \mathbf{i} + (2(1-t)+0+(1-t)^2) \mathbf{j} + (2(0)+t+(0)^2) \mathbf{k}\right) \cdot (-\mathbf{i} + \mathbf{k})
= (-1 + t + t^2) \, dt
= (-1 - t^2) \, dt
\]
Therefore the work done on this piece is
\[\int_0^1 (-1 - t^2) \, dt = -\frac{4}{3}.
\]
It turns out that the work done along the other two pieces also happens to be \(-4/3\), so the total work is \(-4\).

Now we find the flux of \(\nabla \times \mathbf{F} = 2(y+1) \mathbf{i} + 2(z+1) \mathbf{j} + 2(x+1) \mathbf{k}\) across the plane region \(S\) with boundary \(C\). There, \(\mathbf{n} = -(\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}\) and \(dS = \sqrt{3} \, dA\) so the flux becomes
\[\int_0^1 \int_0^{1-x} -2(x+y+z+3) \, dy \, dx = \int_0^1 \int_0^{1-x} -2(1+3) \, dy \, dx = -4.\]
(d) In cylindrical coordinates we have \( n \, dS = (x \mathbf{i} + y \mathbf{j}) \, d\theta \, dz \), so

\[
(\nabla \times \mathbf{F}) \cdot n \, dS = ((1 - x)\mathbf{i} + (z - 1)\mathbf{k}) \cdot (x \mathbf{i} + y \mathbf{j}) \, d\theta \, dz = x(1 - x) \, d\theta \, dz.
\]

The surface integral in Stokes’ theorem is then

\[
\iint_S (\nabla \times \mathbf{F}) \cdot n \, dS = \int_0^1 \int_0^\pi (\cos \theta)(1 - \cos \theta) \, d\theta \, dz = \frac{-\pi}{2}.
\]

Alternatively, if we parametrized \( C \) in four pieces we would have

i. work from \((1, 0, 0)\) to \((-1, 0, 0)\): here, \( \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} \) for \(0 \leq t \leq \pi\), and hence \( \mathbf{F} \cdot d\mathbf{r} = ((\sin t)(-\sin t) + 0(\cos t) + (\sin t)(0)) \, dt = -\sin^2 t \, dt \) and hence the work is \(\int_0^{\pi} (\sin^2 t) \, dt = -\pi/2\).

ii. work from \((-1, 0, 0)\) to \((-1, 0, 1)\): here, \( \mathbf{r}(t) = t \mathbf{k} \) for \(0 \leq t \leq 1\) so \( \mathbf{F} \cdot d\mathbf{r} = ((0)(0) + (-t)(0) + (0)(1)) \, dt = 0 \) and hence the work is 0.

iii. work from \((-1, 0, 1)\) to \((1, 0, 1)\): here, \( \mathbf{r}(t) = -\cos t \mathbf{i} + \sin t \mathbf{j} + \mathbf{k} \) for \(0 \leq t \leq \pi\), and hence \( \mathbf{F} \cdot d\mathbf{r} = ((\sin t)(\sin t) + (-\cos t)(\cos t) + (\sin t)(0)) \, dt = (\sin^2 t - \cos^2 t) \, dt \) and hence the work is \(\int_0^{\pi} (\sin^2 t - \cos^2 t) \, dt = 0\).

iv. work from \((1, 0, 1)\) to \((1, 0, 0)\): here, \( \mathbf{r}(t) = (1 - t) \mathbf{k} \) for \(0 \leq t \leq 1\) so \( \mathbf{F} \cdot d\mathbf{r} = 0 \) as before and hence the work is 0.

Adding up all these again gives \(\frac{-\pi}{2}\).
2. Let \( \mathbf{F} = 3yz\mathbf{i} + (3x^2z)\mathbf{j} + (xy-\mathbf{x})\mathbf{k} \), \( R \) be the portion of the ellipsoid \( x^2 + y^2/4 + z^2/9 \leq 1 \) in the first octant, \( S_{xy}, S_{yz}, S_{xz}, S_{\text{top}} \) be the boundary pieces of \( R \) where the first three lie in the \( xy-, yz-, xz- \) planes and the fourth is the curved top surface.

(a) Compute the outward flux of \( \nabla \times \mathbf{F} \) across \( S_{\text{top}} \) directly.

(b) Using Stokes’ theorem, relate the above quantity to the sum of the fluxes across \( S_{xy}, S_{yz}, S_{xz} \). Compute this sum and verify that it matches your answer in the previous part.

(c) Using Stokes’ theorem, relate the above quantities to the integral of the work done by \( \mathbf{F} \) over the boundary. Compute this and verify that it matches your answer in the previous part.

Solution.

We have 
\[ \nabla \times \mathbf{F} = (x + 2)\mathbf{i} + (2y + 1)\mathbf{j} + (3 - 3z)\mathbf{k} \]

(a) We have 
\[ z = \frac{3}{2} \sqrt{4 - 4x^2 - y^2} \]

so
\[ -f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k} = \frac{6x}{\sqrt{4 - 4x^2 - y^2}}\mathbf{i} + \frac{3y}{\sqrt{4 - 4x^2 - y^2}}\mathbf{j} + \mathbf{k} \]

and hence
\[ (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \left( \frac{6x^2 + 12x + 3y^2 + 3y/2}{\sqrt{4 - 4x^2 - y^2}} + 3 - \frac{9}{2} \sqrt{4 - 4x^2 - y^2} \right) dA \]

Upon performing the change of variables \( x = \cos(t) \) and \( y = 2r \sin(t) \) we see that 
\( dA = 2r \, dr \, d\theta \) and the integrand becomes
\[ (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \frac{6r^3 + 12r^2 \cos(t) + 6r^3 \sin^2(t) + 3r^2 \sin(t)}{\sqrt{1 - r^2}} + 6r - 18r^2 \sqrt{1 - r^2} \, dr \, d\theta \]

and the flux is then
\[ \int_0^1 \int_0^{\pi/2} \left( \frac{6r^3 + 12r^2 \cos(t) + 6r^3 \sin^2(t) + 3r^2 \sin(t)}{\sqrt{1 - r^2}} + 6r - 18r^2 \sqrt{1 - r^2} \right) \, d\theta \, dr \]
\[ = \int_0^1 (9/2) \pi r^3 + 15r^2 \sqrt{1 - r^2} \, dr + 3\pi r - 9\pi r \sqrt{1 - r^2} \, dr \]
\[ = \frac{9\pi}{2} \left( \frac{2}{3} \pi + \frac{\pi}{4} + \frac{3\pi}{2} - 9\pi \left( \frac{1}{3} \right) \right) = \frac{21\pi}{4} \]

(b) For \( S_{xy} \), we have \( z = 0 \) so \( \mathbf{F} \cdot \mathbf{n} = \mathbf{F} \cdot \mathbf{k} = 3 \) so the flux is thrice the area of this region, which is \( 3\pi(1)(2)/4 = 3\pi/2 \). Similarly for \( S_{yz} \) we get \( 2\pi(2)(3)/4 = 3\pi \) and for \( S_{xz} \) we get \( 1\pi(1)(3)/4 = 3\pi/4 \) so the total is \( 21\pi/4 \).
(c) Let the boundary pieces be $C_{xy}$, $C_{yz}$, $C_{xz}$, where they are labeled according to which plane they belong.

Parametrize $C_{xy}$ by $\mathbf{r}(t) = \cos(t)\mathbf{i} + 2\sin(t)\mathbf{j}$ from $t = 0$ to $t = \pi/2$. Then $\mathbf{F}(t) = 3\cos(t)\mathbf{j} + (2\sin(t)\cos(t) - \cos(t))\mathbf{k}$, so that $\mathbf{F}(t) \cdot d\mathbf{r} = 6\cos^2(t)\,dt$. The integral from 0 to $\pi/2$ of this is $3\pi/2$.

Parametrize $C_{yz}$ by $\mathbf{r}(t) = 2\cos(t)\mathbf{j} + 3\sin(t)\mathbf{k}$ from $t = 0$ to $t = \pi/2$. Then $\mathbf{F}(t) = 18\sin(t)\cos(t)\mathbf{i} + (-6\sin(t))\mathbf{j}$, so that $\mathbf{F}(t) \cdot d\mathbf{r} = 12\sin^2(t)\,dt$. The integral from 0 to $\pi/2$ of this is $3\pi$.

Parametrize $C_{xz}$ by $\mathbf{r}(t) = \sin(t)\mathbf{i} + 3\cos(t)\mathbf{k}$ from $t = 0$ to $t = \pi/2$. Then $\mathbf{F}(t) = (3\sin(t) - 6\cos(t))\mathbf{j} - \sin(t)\mathbf{k}$, so that $\mathbf{F} \cdot d\mathbf{r} = 3\sin^2(t)\,dt$. The integral of this from 0 to $\pi/2$ is $3\pi/4$.

The sum of all these works is then $3\pi/2 + 3\pi + 3\pi/4 = \boxed{21\pi/4}$.
3. Let \( \mathbf{F} = yz \mathbf{i} - xz \mathbf{j} + \mathbf{k} \). Let \( S \) be the portion of the surface of the paraboloid \( z = 4 - x^2 - y^2 \) lying above the first octant \( x, y, z \geq 0 \); and let \( C \) be the closed curve \( C = C_1 + C_2 + C_3 \), where the curves \( C_1, C_2, C_3 \) are the three curves formed by intersecting \( S \) with the \( xy \)-, \( yz \)-, and \( xz \)-planes respectively (so that \( C \) is the boundary of \( S \)). Orient \( C \) so that it is traversed counterclockwise when seen from above in the first octant.

(a) Use Stokes’ theorem to compute the work integral \( \oint_C \mathbf{F} \cdot d\mathbf{r} \) by using the surface integral over the capping surface \( S \).

(b) Let \( S_1, S_2, S_3 \) be the three surfaces formed by intersecting the paraboloid with the \( yz \)-, \( xz \)-, and \( xy \)-planes respectively. Use Stokes’ theorem to compute the work integral \( \oint_C \mathbf{F} \cdot d\mathbf{r} \) by using the surface integrals over \( S_1, S_2, \) and \( S_3 \).

(c) Set up and evaluate the work integral \( \oint_C \mathbf{F} \cdot d\mathbf{r} \) by parametrizing each piece of the curve \( C \) and then adding up the three line integrals.

**Solution.**

(a) We have

\[
\nabla \times \mathbf{F} = \det \begin{pmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\partial/\partial x & \partial/\partial y & \partial/\partial z \\
yz & -xz & 1
\end{pmatrix} = xi + yj - 2zk
\]

\[\mathbf{n} dS = \left(-\frac{\partial}{\partial x}(4 - x^2 - y^2)i - \frac{\partial}{\partial y}(4 - x^2 - y^2)j + k\right) dA = (2xi + 2yj + k) dA\]

and hence

\[(\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = (2x^2 + 2y^2 - 2z) dA = (2x^2 + 2y^2 - 2(4 - x^2 - y^2)) dA = 4(x^2 + y^2 - 2) dA\]

which means that the relevant flux is

\[
\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = 4 \int_R (x^2 + y^2 - 2) dA
\]

\[
= 4 \int_0^{\pi/2} \int_0^2 (r^2 - 2)(r dr d\theta)
\]

\[
= 2\pi \int_0^2 (r^3 - 2r) dr
\]

\[
= 2\pi(4 - 4)
\]

\[= 0.\]

(b) On \( S_1 \) we have \( \mathbf{n} = \mathbf{i} \) and hence \((\nabla \times \mathbf{F}) \cdot \mathbf{n} = x = 0\), so the flux is zero.

On \( S_2 \) we have \( \mathbf{n} = \mathbf{j} \) and hence \((\nabla \times \mathbf{F}) \cdot \mathbf{n} = y = 0\), so the flux is zero.

On \( S_3 \) we have \( \mathbf{n} = \mathbf{k} \) and hence \((\nabla \times \mathbf{F}) \cdot \mathbf{n} = -2z = 0\), so the flux is zero.

Adding up the three fluxes gives 0.
(c) Since $C_1$ is in the $yz$-plane, a parametrization for $C_1$ is $r(t) = tj + (4 - t^2)k$ for $t$ going from 2 to 0, and $F = yz i - 0z j + k = yz i + k$ it follows that $F \cdot dr = -2t \, dt$ and hence the work is $\int_2^0 (-2t) \, dt = 4$.

Since $C_2$ is in the $xz$-plane, a parametrization for $C_2$ is $r(t) = ti + (4 - t^2)k$ for $t$ going from 0 to 2, and $F = 0z i - xz j + k = -xz j + k$ it follows that $F \cdot dr = -2t \, dt$ and hence the work is $\int_0^2 (-2t) \, dt = -4$.

Since $C_3$ is in the $xy$-plane we have $z = 0$ and $dz = 0$ so $F \cdot dr = 0$. Therefore the work here is zero.

Adding up all these works gives $4 + (-4) + 0 = 0$. 

4. Use Stokes’ theorem to calculate the flux of $\nabla \times \mathbf{F}$ across $S$ by reducing it to the same flux but over a simpler surface.

(a) $\mathbf{F} = x^3 \mathbf{i} + y^4 \mathbf{j} + z^3 \sin(xy) \mathbf{k}$, $S$ is the upper half of the ellipsoid $x^2 + y^2 + z^2/9 = 1$ with downward orientation.

(b) $\mathbf{F} = (y + xz) \mathbf{i} + (5 - x) \mathbf{j} + (2e^x) \mathbf{k}$, $S$ is the lower hemisphere given by $x^2 + y^2 + z^2 = 1$ and $z \leq 0$ with downward orientation.

(c) Let $\mathbf{F} = M \mathbf{i} + N \mathbf{j} + P \mathbf{k}$. Show that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$. Use this to explain why the divergence theorem can also be used to explain why the flux in the previous two parts is unchanged upon replacing $S$ by a simpler surface with the same boundary.

Solution.

(a) If $T$ is the unit disk in the $xy$-plane, then Stokes’ theorem gives that

$$\text{downward flux}(S) = \text{downward flux}(T) = \iint_T (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS.$$ 

Since $\mathbf{n} = -k$ on $T$ and the $k$-component of $\nabla \times \mathbf{F}$ happens to be $\frac{\partial}{\partial y}(y^4) - \frac{\partial}{\partial x}(x^3) = 0 - 0 = 0$, it follows that the downward flux through $T$ is zero. Hence the same applies to $S$; the flux is $0$.

Alternatively: via Stokes’ theorem, we have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$ 

Here, $\partial S = \{x^2 + y^2 + 0^2/9 = 1\}$ is the unit circle with clockwise orientation, so this may be parametrized as

$$\mathbf{r}(t) = \cos t \mathbf{i} - \sin t \mathbf{j}, \quad 0 \leq t \leq 2\pi$$

and hence

$$\mathbf{F}(t) = \cos^3 \mathbf{i} + \sin^4 t \mathbf{j}$$

$$d\mathbf{r}(t) = (-\sin t \mathbf{i} - \cos t \mathbf{j}) \, dt$$

$$\mathbf{F} \cdot d\mathbf{r} = (-\cos^3 t \sin t - \sin^4 t \cos t) \, dt.$$ 

Therefore the flux is

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_0^{2\pi} (-\cos^3 t \sin t - \sin^4 t \cos t) \, dt$$

$$= \left[ \frac{1}{4} \cos^4 t - \frac{1}{5} \sin^5 t \right]_0^{2\pi}$$

$$= 0.$$
(b) If $T$ is the unit disk in the $xy$-plane centered at the origin, then Stokes’ theorem gives that

\[
\text{downward flux}(S) = \text{downward flux}(T) = \oint_T (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS.
\]

Since $\mathbf{n} = -\mathbf{k}$ on $T$ and $\nabla \times \mathbf{F} \cdot (-\mathbf{k}) = 2$, it follows that the downward flux across $T$ is $2 \cdot \text{Area}(T) = 2\pi$, so the downward flux for $S$ is the same; it is $2\pi$.

(c) We have

\[
\nabla \cdot (\nabla \times \mathbf{F}) = \nabla \cdot ((P_y - N_z)\mathbf{i} + (M_z - P_x)\mathbf{j} + (N_x - M_y)\mathbf{k})
\]

\[
= (P_y - N_z)_x + (M_z - P_x)_y + (N_x - M_y)_z
\]

\[
= (M_{yz} - M_{yz}) + (N_{xz} - N_{xz}) + (P_{yx} - P_{xy})
\]

\[
= 0 + 0 + 0
\]

\[
= 0.
\]

We know that the flux of a divergence-free vector field (in this case, $\nabla \times \mathbf{F}$ is the divergence-free vector field) through a closed surface is zero, so if $D$ is the region bound by $S$ and $T$, then the difference in the flux between $S$ and $T$ is the integral of the divergence (which is zero) through $D$; hence the flux through $S$ is the same as the flux through $T$. 
5. Let \( \mathbf{F} = 2z \mathbf{i} + 3x \mathbf{j} + 4y \mathbf{k} \) be a vector field and \( C \) a simple positively oriented curve lying in some plane \( P \). Assume that \( C \) encloses a region \( R \) of area 5. Suppose that \( P \) is chosen to maximize the work done by \( \mathbf{F} \) along \( C \). Compute the equation for the plane \( P \).

**Solution.**

Via Stokes’ theorem, we have

\[
\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \oint_C \mathbf{F} \cdot d\mathbf{r},
\]

where \( S \) is any surface whose boundary is \( C \), oriented using the right hand rule. Let \( S \) be the part of the plane \( P \) that encloses \( C \).

Since

\[
\nabla \times \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ 2z & 3x & 4y \end{pmatrix} = 4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k},
\]

and the upward unit normal vector \( \mathbf{n} \) to \( P \) is a constant, it follows that \((\nabla \times \mathbf{F}) \cdot \mathbf{n}\) is a constant. Therefore Stokes’ theorem gives

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS
\]

\[
= ( (\nabla \times \mathbf{F}) \cdot \mathbf{n} ) \iint_S 1 \, dS
\]

\[
= ( (\nabla \times \mathbf{F}) \cdot \mathbf{n} ) \cdot \text{Area}(S)
\]

\[
= 5 \cdot ( (\nabla \times \mathbf{F}) \cdot \mathbf{n} )
\]

So we must choose \( \mathbf{n} \) in a way that maximizes \((\nabla \times \mathbf{F}) \cdot \mathbf{n}\). By 3D vector geometry, this quantity is maximized exactly when \( \mathbf{n} \) is in the same direction as \( \nabla \times \mathbf{F} \).

This means that \( \nabla \times \mathbf{F} = 4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \) is normal to \( P \). So any plane whose equation is of the form \( 4x + 2y + 3z = d \) (for any value of \( d \)) will work.