1 Lecture review

1.1 Surface integrals, flux

1. Recall: if \( \mathbf{F} \) is a vector field, \( dS \) is a surface element, and \( d\mathbf{S} \) is a vector with magnitude \( dS \) pointing in the outward normal direction, then

\[
d\mathbf{S} = \mathbf{n} \, dS
\]

and the flux of \( \mathbf{F} \) through that surface element is

\[
\mathbf{F} \cdot d\mathbf{S} = \mathbf{F} \cdot \mathbf{n} \, dS
\]

so the total flux through all of \( S \) is

\[
\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \mathbf{n} \, dS
\]

2. Suppose \( S \) is the graph of \( z = f(x, y) \) over a region \( R \) in the \( xy \)-plane. Then

\[
\int_S \mathbf{F} \cdot d\mathbf{S} = \int_R \mathbf{F} \cdot (-f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k}) \, dA
\]

3. Define \( g(x, y, z) = z - f(x, y) = 0 \). Then

\[
\nabla g = -f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k} \text{ is normal to } S
\]

\[
\mathbf{n} = \frac{\nabla g}{|\nabla g|} \text{ is the unit normal.}
\]

\[
d\mathbf{S} = |\nabla g| \, dA = \sqrt{f_x^2 + f_y^2 + 1} \, dA
\]

\[
d\mathbf{S} = \mathbf{n} \, d\mathbf{S} = \nabla g \, dA = (-f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k}) \, dA
\]

4. Formulas for the four main types of surfaces:

(a) Flat surfaces

If \( S \) is part of the \( xy \)-plane, then \( \mathbf{n} = \mathbf{k} \) and \( d\mathbf{S} = dA \). If \( S \) is part of the \( yz \)-plane or the \( xz \)-plane, similar formulas hold.

(b) Curved surfaces

If \( S \) is part of the surface \( z = f(x, y) \), then

\[
d\mathbf{S} = \sqrt{f_x^2 + f_y^2 + 1} \, dA \quad \mathbf{n} = \frac{-f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k}}{\sqrt{f_x^2 + f_y^2 + 1}} \quad d\mathbf{S} = \mathbf{n} \, d\mathbf{S} = (-f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k}) \, dA.
\]

(c) Parts of a cylinder

If \( S \) is part of a cylinder of radius \( a \) with a central \( z \)-axis, then

\[
d\mathbf{S} = a \, d\theta \, dz \quad \mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j}}{a} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \quad d\mathbf{S} = \mathbf{n} \, d\mathbf{S} = a(\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) \, d\theta \, dz
\]

(d) Parts of a sphere

If \( S \) is part of a sphere of radius \( a \) centered at the origin, then

\[
d\mathbf{S} = a^2 \sin \varphi d\varphi \, d\theta
\]

\[
\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}
\]

\[
= \sin \varphi \cos \theta \mathbf{i} + \sin \varphi \sin \theta \mathbf{j} + \cos \varphi \mathbf{k}
\]

\[
d\mathbf{S} = \mathbf{n} \, d\mathbf{S} = a^2 \sin \varphi (\sin \varphi \cos \theta \mathbf{i} + \sin \varphi \sin \theta \mathbf{j} + \cos \varphi \mathbf{k}) \, d\varphi \, d\theta
\]
2 Problems

1. Compute the indicated flux of $F$ through the surface $S$.

(a) Outward flux of $F = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through the unit cube with opposite vertices $(0,0,0)$ and $(1,1,1)$ and faces parallel to the coordinate planes.

(b) Upward flux of $F = e^{xyz}\mathbf{i} - e^{xyz}\mathbf{j} + \mathbf{k}$ through the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

(c) Upward flux of $F = e^{x^2}\mathbf{i} + \ln(e^y + 1)\mathbf{j} + x\mathbf{k}$ through the square $0 \leq x, y \leq 1$.

(d) Downward flux of $F = (1 + x^2)^{1/2}(1 + y^2)^{1/2}(1 + z^2)^{1/2}\mathbf{j} + (1 + y^2)^{1/2}\mathbf{k}$ through the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y = z$.

Solution.

(a) On the $x = 0$ portion we have $\mathbf{n} = -\mathbf{i}$ and so $\mathbf{F} \cdot \mathbf{n} = -x = 0$. The flux here is zero. The same goes for the $y = 0$, $z = 0$ portions.

On the $x = 1$ portion we have $\mathbf{n} = \mathbf{i}$ and so $\mathbf{F} \cdot \mathbf{n} = x = 1$. The flux here is then $\iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS = \iint_S dS = \text{Area}(S) = 1$. The same goes for the $y = 0$, $z = 0$ portions.

Adding up gives that the total flux is $3$.

(b) Here we have that $\mathbf{n} = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$ and so $\mathbf{F} \cdot \mathbf{n} = 1/\sqrt{3}$. Therefore the total flux is $\iint_S (1/\sqrt{3}) \, dS = (1/\sqrt{3})\text{Area}(S)$. The area of this triangle is given by half the magnitude of the cross product $\langle -1,1,0 \rangle \times \langle -1,0,1 \rangle = \langle 1,1,1 \rangle$ which is $\sqrt{3}/2$. So the total flux is $1/2$.

(c) Here we have that $\mathbf{n} = \mathbf{k}$ so $\mathbf{F} \cdot \mathbf{n} = x$. So our flux is

$$\iint_S x \, dS = \int_0^1 \int_0^1 x \, dy \, dx = 1/2.$$

(d) Here we have that $\mathbf{n} = (\mathbf{j} - \mathbf{k})/\sqrt{2}$ and so $\mathbf{F} \cdot \mathbf{n} = -1/\sqrt{2}$. So the flux is $-\text{Area}(S)/\sqrt{2}$. The intersection happens to be an ellipse with semi-axes of length 1 and $\sqrt{2}$, so the flux is $-\pi(1)(\sqrt{2})/\sqrt{2} = -\pi$. 


2. (a) Let $S$ be the portion of the paraboloid $z = 4 - x^2 - y^2$ in the region $x, y, z \geq 0$. Compute the upward flux of $\mathbf{F} = zi$ through $S$.

(b) Let $S$ be the portion of the cone $z = \sqrt{x^2 + y^2}$ lying in the region $1 \leq z \leq 2$. Compute the upward flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$ through $S$.

Solution.

(a) The region in the $xy$-plane over which $S$ lies is

\[
4 - x^2 - y^2 \geq 0
\]

$x, y \geq 0$.

In polar coordinates, this becomes

\[
0 \leq r \leq 2
\]

\[
0 \leq \theta \leq \frac{\pi}{2}.
\]

Moreover we also have

\[
d\mathbf{S} = \left( -\frac{\partial}{\partial x} (4 - x^2 - y^2) \mathbf{i} - \frac{\partial}{\partial y} (4 - x^2 - y^2) \mathbf{j} + \mathbf{k} \right) \, dA
\]

\[
= (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}) \, (dA)
\]

Therefore,

\[
\mathbf{F} \cdot d\mathbf{S} = ((2x)(z) + (2y)(0) + (1)(0)) \, dA
\]

\[
= 2xz \, dA
\]

\[
= 2x(4 - x^2 - y^2) \, dA
\]

\[
= 2(r \cos \theta)(4 - r^2)(r \, dr \, d\theta)
\]

\[
= 2r^2(4 - r^2) \cos \theta \, dr \, d\theta
\]

and hence the flux is

\[
\int_{0}^{\pi/2} \int_{0}^{2} 2r^2(4 - r^2) \cos \theta \, dr \, d\theta = [\sin \theta]_{0}^{\pi/2} \cdot \left[ \frac{8}{3} r^3 - \frac{2}{5} r^5 \right]_{0}^{2} = \frac{128}{15}.
\]

(b) The region in the $xy$-plane over which $S$ lies is

\[
1 \leq \sqrt{x^2 + y^2} \leq 2.
\]

In polar coordinates, this becomes

\[
1 \leq r \leq 2
\]

\[
0 \leq \theta \leq 2\pi.
\]
Moreover we also have
\[ dS = \left( -\frac{\partial}{\partial x} \left( \sqrt{x^2 + y^2} \right) \mathbf{i} - \frac{\partial}{\partial y} \left( \sqrt{x^2 + y^2} \right) + \mathbf{k} \right) dA \]
\[ = \left( -\frac{x}{\sqrt{x^2 + y^2}} \mathbf{i} - \frac{y}{\sqrt{x^2 + y^2}} \mathbf{j} + \mathbf{k} \right) dA \]
\[ = \left( -\frac{x}{r} \mathbf{i} - \frac{y}{r} \mathbf{j} + \mathbf{k} \right) (r dr d\theta) \]
\[ = (-x \mathbf{i} - y \mathbf{j} + r \mathbf{k}) dr d\theta. \]

Therefore,
\[ \mathbf{F} \cdot dS = ((x)(-x) + (y)(-y) + (2z)(r)) dr d\theta = (-x^2 - y^2 + 2r^2) dr d\theta = r^2 dr d\theta \]
and hence the flux is
\[ \int_0^{2\pi} \int_1^2 r^2 dr d\theta = (2\pi) \left[ \frac{1}{3} r^3 \right]_1^2 = \frac{14\pi}{3}. \]
3. (a) Let \( S \) be the portion of the cylinder \( x^2 + y^2 = 1 \) in the octant \( x, y, z \geq 0 \) that lies below \( z = 1 \). Compute the outward flux of \( \mathbf{F} = (x^3z^2 + y^2z)i + (x^2yz^2 - xyz)j + (xz^4 - y^5)k \) through \( S \).

(b) Let \( S \) be the portion of the sphere of radius 2 centered at the origin between \( z = 0 \) and \( z = \sqrt{3} \). Compute the outward flux of \( \mathbf{F} = xzi/\sqrt{4 - z^2} \) through \( S \).

(c) Compute the outward flux of \( \mathbf{F} = -xi - yj + zk \) across the boundary of the portion of the solid unit sphere lying in the first octant (that is, \( x, y, z \geq 0 \)).

(d) Compute the outward flux of \( \mathbf{F} = x^2i - yj + zk \) through the boundary of the solid cylinder \( y^2 + z^2 \leq 9, 0 \leq x \leq 2 \).

**Solution.**

(a) In this instance the normal vector is \( \mathbf{n} = xi + yj \) so

\[
\mathbf{F} \cdot \mathbf{n} = (x^3z^2 + y^2z)x + (x^2yz^2 - xyz)y = x^2z^2(x^2 + y^2) = x^2z^2 = z^2\cos^2 \theta
\]

and hence the flux is

\[
\int_0^{\pi/2} \int_0^1 z^2\cos^2 \theta (dz \, d\theta) = \frac{\pi}{12}.
\]

(b) First we convert \( \mathbf{F} \) to spherical coordinates:

\[
\mathbf{F} = \left( \frac{xz}{\sqrt{4 - z^2}} \right) i = \left( \frac{(2\sin \varphi \cos \theta)(2\cos \varphi)}{\sqrt{4 - (2\cos \varphi)^2}} \right) i = \left( \frac{(2\sin \varphi \cos \theta)(2\cos \varphi)}{2 \sin \varphi} \right) i = (2\cos \varphi \cos \theta)i
\]

Therefore,

\[
\mathbf{F} \cdot \mathbf{n} = (2\cos \varphi \cos \theta)i \cdot \rho = (2\cos \varphi \cos \theta)(\sin \varphi \cos \theta) = 2\sin \varphi \cos \varphi \cos^2 \theta
\]

and hence

\[
\mathbf{F} \cdot \mathbf{n} \, dS = (2\sin \varphi \cos \varphi \cos^2 \theta)(2^2 \sin \varphi \, d\varphi \, d\theta) = 8\sin^2 \varphi \cos \varphi \cos^2 \theta \, d\varphi \, d\theta
\]

so the flux is

\[
\int_0^{2\pi} \int_{\pi/6}^{\pi/2} (8\sin^2 \varphi \cos \varphi \cos^2 \theta) \, d\varphi \, d\theta = \frac{7\pi}{3}.
\]

(c) The boundary has three pieces: the portions on the \( xy \), \( yz \), and \( xz \)-coordinate planes, and then the spherical portion.

For the \( xy \)-plane portion note that \( \mathbf{n} = k \), yet the \( k \)-component of \( \mathbf{F} \) is zero. Hence the flux across this piece is zero. The same holds for the other coordinate plane portions.

Now consider the spherical portion. Here we have

\[
\mathbf{F} \cdot \mathbf{n} = (-xi - yj + 3zk) \cdot \left( \frac{xi + yj + zk}{\rho} \right) = 4z^2 - 1 = 4\cos^2 \phi - 1
\]
and

\[ dS = \sin \phi \, d\phi \, d\rho \]

so the flux is

\[
\int_0^{\pi/2} \int_0^{\pi/2} (4 \cos^2 \phi - 1) \sin \phi \, d\phi \, d\theta = \int_0^{\pi/2} \left[ -\frac{4}{3} \cos^3 \phi + \cos \phi \right]_0^{\pi/2} \, d\theta = \int_0^{\pi/2} \frac{1}{3} \, d\theta = \frac{\pi}{6}.
\]

Adding up all the contributions gives \( \frac{\pi}{6} \).

(d) On the \( x = 0 \) part we have \( \mathbf{n} = -\mathbf{i} \) and so \( \mathbf{F} \cdot \mathbf{n} = -x^2 = 0 \). Hence the flux is zero here.

On the \( x = 2 \) part we have \( \mathbf{n} = \mathbf{i} \) and so \( \mathbf{F} \cdot \mathbf{n} = x^2 = 4 \). Hence the flux is \( 4 \cdot \text{Area} = 4 \cdot 9\pi = 36\pi \).

On the side part we have \( \mathbf{n} = (\mathbf{y} + \mathbf{z})/3 \) and so it follows that \( \mathbf{F} \cdot \mathbf{n} = (z^2 - y^2)/3 \). By symmetry it follows the flux here is zero.

Adding up the contributions gives a total outward flux of \( 36\pi \).