1 Lecture review

1.1 Triple integrals

Triple integration allows one to integrate over volumes. To each point in the volume \( D \) is assigned a function \( f(x, y, z) \). One can then integrate this to get a triple integral

\[
\iiint_D f(x, y, z) \, dV.
\]

1. When \( f = \delta(x, y, z) \) is the density, then the triple integral yields the mass of \( D \).

2. For a density function \( \delta \), the center of mass of \( D \) is the point \((x_{CM}, y_{CM}, z_{CM})\) given by

\[
\begin{align*}
x_{CM} &= \frac{\iiint_D x \delta \, dV}{\iiint_D \delta \, dV} \\
y_{CM} &= \frac{\iiint_D y \delta \, dV}{\iiint_D \delta \, dV} \\
z_{CM} &= \frac{\iiint_D z \delta \, dV}{\iiint_D \delta \, dV}
\end{align*}
\]

The denominator in each of the three formulas above is the mass of \( D \). Oftentimes one can see without integration that some of these expressions must be zero by symmetry.

1.2 Evaluation of triple integrals

1. Rectangular coordinates are good when the bounds are given nicely in \( x, y, z \) coordinates. The bounds usually look like

\[
x_1 \leq x \leq x_2 \quad y_1(x) \leq y \leq y_2(x) \quad g(x, y) \leq z \leq h(x, y) \quad dV = dz \, dy \, dx
\]

which would make the integral

\[
\iiint_D f(x, y, z) \, dV = \int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} \int_{g(x,y)}^{h(x,y)} f(x, y, z) \, dz \, dy \, dx.
\]

(In other cases, the roles of \( x, y, z \) may be interchanged with each other.)

2. Cylindrical coordinates are good for problems with a line of symmetry, for volumes that are pieces of a cylinder, or those bounded by paraboloids. Take

\[
x = r \cos \theta \quad y = r \sin \theta \quad z = z \quad dV = r \, dz \, dr \, d\theta
\]

The bounds almost always look like

\[
\theta_1 \leq \theta \leq \theta_2 \quad r_1(\theta) \leq r \leq r_2(\theta) \quad g(r, \theta) \leq z \leq h(r, \theta)
\]

which would make the integral

\[
\iiint_D f(x, y, z) \, dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{g(r,\theta)}^{h(r,\theta)} f(r \cos \theta, r \sin \theta, z) r \, dz \, dr \, d\theta.
\]
2 Problems

1. Let $V$ be the region over the unit square $0 \leq x, y \leq 1$ between $z = x^2y^2$ and $z = 2e^{xy}$ and with density $\delta = x$. Compute the $z$-coordinate of the center of mass of $V$.

Solution.

The mass is

$$\int_0^1 \int_0^1 \int_{x^2y^2}^{2e^{xy}} x \, dz \, dy \, dx = \int_0^1 \int_0^1 (2xe^{xy} - x^3y^2) \, dy \, dx$$

$$= \int_0^1 \left[ 2e^{xy} - \frac{1}{3}x^3y^3 \right]_0^1 \, dx$$

$$= \int_0^1 2(e^x - 1) - \frac{1}{3}x^3 \, dx$$

$$= 2(e - 2) - \frac{1}{12}$$

$$= 2e - \frac{49}{12}$$

and we have

$$\int_0^1 \int_0^1 \int_{x^2y^2}^{2e^{xy}} xz \, dz \, dy \, dx = \int_0^1 \int_0^1 x(2e^{2xy} - \frac{1}{2}x^4y^4) \, dy \, dx$$

$$= \int_0^1 \left[ e^{2xy} - \frac{1}{10}x^5y^5 \right]_0^1 \, dx$$

$$= \int_0^1 (e^{2x} - 1) - \frac{1}{10}x^5 \, dx$$

$$= \left( \frac{e^2}{2} - \frac{3}{2} \right) - \frac{1}{60}$$

$$= \frac{e^2}{2} - \frac{91}{60},$$

so the $z$-coordinate of the center of mass must be the ratio

$$\frac{\frac{e^2}{2} - \frac{91}{60}}{2e - \frac{49}{12}}.$$
2. Let $D$ be the region that lies between the downward facing paraboloid $z = 3 - x^2 - y^2$ and the cone $z = 2\sqrt{x^2 + y^2}$ with density $\delta = 1/r$.

(a) Compute the mass of $D$.
(b) Compute the centroid of $D$.

Solution.

(a)

$$
\iiint_D \delta \, dV = \int_0^{2\pi} \int_0^1 \int_0^{3-r^2} \frac{1}{r} \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (3 - r^2 - 2r) \, dr \, d\theta = \frac{10}{3} \pi.
$$

(b) The centroid only has a $z$-component by symmetry. So it is

$$
\left( \frac{\iiint_D z \, dV}{\iiint_D 1 \, dV} \right) \hat{k} = \left( \frac{\int_0^{2\pi} \int_0^1 \int_0^{3-r^2} zr \, dz \, dr \, d\theta}{\int_0^{2\pi} \int_0^1 \int_0^{3-r^2} r \, dz \, dr \, d\theta} \right) \hat{k} = \left( \frac{13\pi/6}{7\pi/6} \right) \hat{k} = \frac{13}{7} \hat{k}.
$$
3. Find the volume between the surfaces $x + 4y + z^2 = 9$ and $x - y^2 + 4z = 1$.

Solution.

Projecting onto the $yz$-plane, we seek to find the volume given by $1 + y^2 - 4z \leq x \leq 9 - 4y - z^2$. This means that the region in the $yz$-plane looks like $1 + y^2 - 4z \leq 9 - 4y - z^2$, which can be rearranged to give

$$R : \{(y + 2)^2 + (z - 2)^2 \leq 16\}.$$ 

Therefore the volume is

$$\iiint_R (8y - y^2 + 4z - z^2) \, dz \, dy.$$ 

Making the change of coordinates $y = -2 + r \cos \theta$ and $z = 2 + r \sin \theta$, the region $R$ becomes $0 \leq r \leq 4$, $0 \leq \theta \leq 2\pi$ and $dz \, dy = r \, dr \, d\theta$. Therefore the volume is

$$\int_0^{2\pi} \int_0^4 (16 - r^2)(r \, dr \, d\theta) = \left( \int_0^{2\pi} d\theta \right) \left( \int_0^4 (16 - r^2)(r \, dr) \right)$$

$$= (2\pi) (128 - 64)$$

$$= 128\pi.$$
4. Find the volume between the surfaces $x^2 + y + 4z^2 = 3$ and $2x + y = 0$.

Solution.

Projecting onto the $xz$-plane, we seek to find the volume given by $-2x \leq y \leq 3 - x^2 - 4z^2$. This means that the region in the $xz$-plane looks like $-2x \leq 3 - x^2 - 4z^2$, which can be rearranged to give

$$R : \{(x - 1)^2 + 4z^2 \leq 4\}.$$ 

Therefore the volume is

$$\iiint_R \int_{-2x}^{3 - x^2 - 4z^2} dy \, dz \, dx = \iiint_R (3 - x^2 - 4z^2 + 2x) \, dz \, dx.$$ 

Making the change of coordinates $x = 1 + 2r \cos \theta$ and $z = r \sin \theta$, the region $R$ becomes $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$ and $dz \, dx = 2r \, dr \, d\theta$. Therefore the volume is

$$\int_0^{2\pi} \int_0^1 (4 - 4r^2)(2r \, dr \, d\theta) = \left(\int_0^{2\pi} d\theta\right) \left(\int_0^1 (4 - 4r^2)(2r \, dr)\right)$$

$$= (2\pi) (2)$$

$$= 4\pi.$$
5. (5A) These problems involve triple integrals in cylindrical coordinates.

(a) Find limits in cylindrical coordinates for the region bounded below by the cone \( z^2 = x^2 + y^2 \), and above by the sphere of radius \( \sqrt{2} \) and center at the origin.

(b) A solid right circular cone of height \( h \) with 90° vertex angle has density at point \( P \) numerically equal to the distance from \( P \) to the central axis. Find its mass.

(c) Let \( D \) be the upper half \( (z > 0) \) of a solid hemisphere centered at the origin of radius \( a \). Compute the triple integral of the function \( f(x, y, z) = x^2 + y^2 \) over \( D \).

(d) The paraboloid \( z = x^2 + y^2 \) is shaped like a wine-glass, and the plane \( z = 2x \) slices off a finite piece \( D \) of the region above the paraboloid (i.e., inside the wine-glass). Find the triple integral of the function \( f(x, y, z) = x^2 + y^2 \) on \( D \).

Solution.

(a) The sphere has equation \( r^2 + z^2 = 2 \) in cylindrical coordinates. The cone has equation \( z^2 = r^2 \), or \( z = r \). The circle in which they intersect is given by solving \( z = r \) and \( z^2 + r^2 = 2 \) simultaneously, giving \( r = 1 \). Putting it all together, we get

\[
\int_0^{2\pi} \int_0^1 \int_r^\sqrt{2-r^2} r \, dz \, dr \, d\theta.
\]

(b) The cone in cylindrical coordinates is \( z = r \) and its density is also \( \delta = r \). Therefore its mass will be

\[
\iiint_D r \, dV = \int_0^{2\pi} \int_0^h \int_r^h r^2 \, dz \, dr \, d\theta = \frac{\pi h^4}{6}.
\]

(c) In cylindrical coordinates, the base is \( 0 \leq \theta \leq 2\pi \), \( 0 \leq r \leq a \). The sphere is \( z = \sqrt{a^2 - r^2} \), so the \( z \)-bounds are \( 0 \leq z \leq \sqrt{a^2 - r^2} \). We seek to integrate the function \( r^2 \). This gives

\[
\int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2-r^2}} r^2 (r \, dz \, dr \, d\theta) = 2\pi \int_0^a r^3 \sqrt{a^2 - r^2} \, dr.
\]

Using the \( u \)-substitution \( u = a^2 - r^2 \), this becomes

\[
2\pi \int_0^a r^3 \sqrt{a^2 - r^2} \, dr = \pi \int_0^{a^2} (a^2 - u)\sqrt{u} \, du = \frac{4\pi a^5}{15}.
\]

(d) To determine the region in the \( xy \)-plane over which \( D \) lies, we set the two surfaces equal to each other: \( z = 2x \) and \( z = x^2 + y^2 \) gives \( x^2 + y^2 = 2x \). The interior of this is \( x^2 + y^2 \leq 2x \), which in polar coordinates is \( r^2 \leq 2r \cos \theta \), or \( r \leq 2 \cos \theta \). From a picture we see that the bounds for \( \theta \) will be \( -\pi/2 \leq \theta \leq \pi/2 \). Finally the \( z \)-bounds are given by the surfaces themselves, so this is \( x^2 + y^2 \leq z \leq 2x \).
Converting this to cylindrical gives \( r^2 \leq z \leq 2r \cos \theta \). Integrating the function \( r^2 \) now gives

\[
\int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos \theta} \int_{r^2}^{2r \cos \theta} r^2 \, dz \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos \theta} r^4 (2 \cos \theta - r) \, dr \, d\theta
\]

\[
= \int_{-\pi/2}^{\pi/2} \frac{32}{15} \cos^6 \theta \, d\theta = \frac{2\pi}{3}
\]