1 Lecture review

1.1 Green’s theorem

Let $C$ be a closed (no endpoints, or alternatively, endpoints are equal to each other), simple (no self intersections), piecewise smooth, positively oriented (oriented counterclockwise) curve bounding a region $R$. Suppose $F$ is continuously differentiable (meaning its components have continuous partial derivatives).

1. (Work-curl “tangential” form) Then

$$\int_C F \cdot dr = \iint_R \text{curl} F \, dA$$

If $F = M(x,y)i + N(x,y)j$, then

$$F \cdot dr = M \, dx + N \, dy$$

$$\text{curl} F = N_x - M_y$$

The left hand side is the work done by $F$ along $C$.

2. (Flux-divergence “normal” form) Then

$$\int_C F \cdot n \, ds = \iint_R \text{div} F \, dA$$

If $F = M(x,y)i + N(x,y)j$, then

$$F \cdot n \, ds = -N \, dx + M \, dy$$

$$\text{div} F = M_x + N_y$$

so Green’s theorem gives

$$\int_C -N \, dx + M \, dy = \iint_R (M_x + N_y) \, dA$$

The left hand side is the flux of $F$ across $C$. 
2 Problems

1. Use Green’s theorem to find the work done by the vector field \( \mathbf{F} = (y^2 - 4x^2y)i + 2xyj \) along the unit circle centered at the origin, traversed counterclockwise.

2. Let \( \mathbf{F} = (4y - 4x^2y + \sin(x^2))i + (y^3 + xy^2)j \).

   (a) Find the positively oriented closed path along which the work done by \( \mathbf{F} \) is minimal.

   (b) Compute the work done by \( \mathbf{F} \) along the closed path computed in the previous part.

3. For these problems, calculate the flux by parametrizing the curve.

   (a) \( \mathbf{F} = xi + yj \), \( C \) is the portion of \( y = e^x \) from \( x = 0 \) to \( x = 1 \)

   (b) \( \mathbf{F} = y^3i - x^2j \), \( C \) is the portion of \( y = \arcsin x \) from \( x = 0 \) to \( x = 1 \)

4. For the following problems, compute the flux of \( \mathbf{F} \) across \( C \).

   (a) \( \mathbf{F} = x^3i + y^3j \); \( C \) is the counterclockwise-oriented boundary of the lemniscate \( r^2 = \cos \theta \) in the region where \( x \) is positive.

   (b) \( \mathbf{F} = y \ln(x + y)i + x \ln(x + y)j \); \( C \) is the clockwise-oriented boundary of the region given by \( 1 \leq r \leq 2, \pi/6 \leq \theta \leq \pi/3 \)

   (c) \( \mathbf{F} = \sin yi - \cos xj \); \( C \) is the clockwise-oriented part of the unit circle from \((0,1)\) to \((1,0)\).

5. Let \( \mathbf{F} = \frac{x^2y}{1-xy}(i - y^2j) \) and \( C \) be the piece of the curve \( y = 4\sqrt{1 - x^{2/7}} \) from \((0, 4)\) to \((1, 0)\). Use Green’s theorem to compute the flux of \( \mathbf{F} \) through \( C \).

   Warning: this curve is not closed, so be careful when applying Green’s theorem.

6. Let \( \mathbf{F} \) be the vector field

\[
\mathbf{F} = \frac{(x - 2y)i + (2x + y)j}{x^2 + y^2}.
\]

Compute curl \( \mathbf{F} \) and div \( \mathbf{F} \). For which closed curves can one immediately deduce that the work done by \( \mathbf{F} \) is zero? How about flux?

3 Answers

1. \( \pi \)

2. (a) the ellipse \( 4x^2 + y^2 = 4 \), (b) \(-4\pi \)

3. (a) \(-e + 2 \), (b) \( \frac{e^4}{10} + \frac{1}{3} \)

4. (a) \( \frac{3\pi}{4} \), (b) \(-\frac{\pi}{4} \), (c) \( \sin 1 + \cos 1 - 1 \)

5. \(-1 \)

6. curl and div are 0; work and flux are zero for any closed curve whose interior avoids origin