1 Lecture review

1.1 The divergence theorem

1. Recall the 2D version: for a closed, simple, piecewise smooth, positively oriented curve $C$ and a vector field $\mathbf{F}$ we have

$$\oint_{C} \mathbf{F} \cdot \mathbf{n} \, ds = \iint_{R} \nabla \cdot \mathbf{F} \, dA$$

(We also called this the normal form of Green’s theorem.)

2. The 3D version is as follows. Let $S$ be a closed piecewise smooth surface bounding a space region $D$ with outward unit normal $\mathbf{n}$. Then for a vector field $\mathbf{F}$ we have

$$\oiint_{S} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_{D} \nabla \cdot \mathbf{F} \, dV$$

3. Just as in the 2D version, the 3D divergence theorem can be helpful to find the flux through non-closed surfaces; just make sure to close it off first and subtract off the contribution from the surface that was added.

1.2 3D Line Integrals

<table>
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<th>line integrals</th>
<th>curl</th>
<th>conservativity</th>
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<tr>
<td>$\int_{C} (M , dx + N , dy)$</td>
<td>$\nabla \times (Mi + Nj)$</td>
<td>check curl is 0 in simply-connected domain</td>
</tr>
<tr>
<td>parametrize $C$ to evaluate</td>
<td>$(Nx - Ny)k$</td>
<td></td>
</tr>
<tr>
<td>$\int_{C} M , dx + N , dy + P , dz$</td>
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<td>parametrize $C$ to evaluate</td>
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</table>

The notion of conservativity is equivalent to any of the following:

$$\mathbf{F} = \nabla f \iff \oint_{C} \mathbf{F} \cdot d\mathbf{r} = 0$$

$$\iff \oint_{C} \mathbf{F} \cdot d\mathbf{r} \text{ depends only on endpoints}$$

$$\iff \nabla \times \mathbf{F} = 0 \text{ (in regions with no holes)}$$
2 Problems

1. Verify that the divergence theorem holds.
   (a) \( \mathbf{F} = -xi - yj + 3zk \), \( S \) is the portion of the unit sphere lying in the first octant (that is, \( x, y, z \geq 0 \)).
   (b) \( \mathbf{F} = x^2i - yj + zk \), \( D \) is the solid cylinder \( y^2 + z^2 \leq 9, 0 \leq x \leq 2 \).
   (c) \( \mathbf{F} = y^2z^3i + 2yzj + 4z^2k \), \( D \) is the solid between \( z = x^2 + y^2 \) and the plane \( z = 9 \).

2. Use the divergence theorem to compute the outward flux of \( \mathbf{F} \) through \( S \).
   (a) \( \mathbf{F} = (2x^3 + y^3)i + (y^3 + z^3)j + 3y^2zk \), \( S \) is the surface of the solid bounded by \( z = 1 - x^2 - y^2 \) and the \( xy \)-plane.
   (b) \( \mathbf{F} = \ln(1 + e^y)i + \ln(1 + e^x)j + \ln(1 + e^z)k \), \( S \) is the surface of an icosahedron centered at the origin of side length 2.
   (c) \( \mathbf{F} = (x^2 + e^{y^2} + \sin z)i + (y + e^{\arctan z})j + zk \), \( S \) is the surface of the unit sphere centered at \( (1, 1, 1) \).

3. Calculate the flux of \( \mathbf{F} \) through the non-closed surface \( S \) by closing it off first, and then applying the divergence theorem.
   (a) \( \mathbf{F} = z^2xi + (\frac{1}{3}y^3 + \tan z)j + (x^2z + y^2)k \), \( S \) is the top half of the unit sphere centered at the origin; upward flux.
   (b) \( \mathbf{F} = \ln(e^x + e^y)(i + j) + (x^2 + y^2 + z^2)k \), \( S \) is the “side” portion of a cylinder with central \( z \)-axis of radius \( R \) between \( z = 0 \) and \( z = H \); outward flux.

4. Let \( \mathbf{F} = (1 + x - x^3)i + (1 + y - y^3)j + (1 + z - z^3)k \).
   (a) Find the closed surface \( S \) that maximizes the outward flux of \( \mathbf{F} \) through \( S \). Find that maximal flux.
   (b) Let \( C \) the the unit circle in the \( xy \)-plane. Find the surface \( S \) in the region \( z \geq 0 \) with boundary \( C \) which maximizes the upward flux of \( \mathbf{F} \) through \( S \). Find that maximal flux.

5. Find the closed surface \( S \) in the region \( z \geq 0 \) through which the flux of \( \mathbf{F} = (y^3 - xy^2 - x^3)i + (6y + \sin z)j - (z^2 + \log(x^2 + 1))k \) is maximal. Find that flux.

6. Calculate the work done by \( \mathbf{F} \) along the curve.
   (a) \( \mathbf{F} = (x + 2z + z^2)i + (2x + y + x^2)j + (2y + z + y^2)k \), \( C \) is the triangle with vertices at \( (1, 0, 0) \), \( (0, 1, 0) \), and \( (0, 0, 1) \), oriented clockwise when viewed from above.
   (b) \( \mathbf{F} = xy^2i + 2z^2j + xk \), the curve is given by \( r(t) = \sin(t)i + tj + \cos(t)k \) from \( t = -\pi/2 \) to \( t = \pi/2 \).

7. Let \( \mathbf{F} \) be the force field \( \mathbf{F} = 2xyz^3i + x^2z^2j + 3x^2yz^2k \).
   (a) Show that \( \mathbf{F} \) is conservative, and find a potential function for it.
   (b) Find the maximum and minimum values of work done by all paths lying in the unit sphere centered at the origin.

8. Let \( \mathbf{F} \) be the vector field \( \mathbf{F} = (3x^2 + ayz)i + b(xz + z^2)j + (cxy + 2yz)k \)
   (a) Find the values of \( a, b, c \) which make \( \mathbf{F} \) conservative.
   (b) Using the values of \( a, b, c \) found above, find a potential for \( f \) using the (i) algebraic method and the (ii) integration method. Check that your answers match.
   (c) Using the values of \( a, b, c \) found above, calculate the work done by \( \mathbf{F} \) along any path from \( (0, 0, 0) \) to \( (1, 1, 1) \).

3 Answers

1. (a) \( \pi/6 \); (b) \( 36\pi \); (c) \( 2430\pi \)
2. (a) \( \pi \); (b) \( 0 \); (c) \( 8\pi/3 \)
3. (a) \( 13\pi/20 \); (b) \( \pi R^2 H \)
4. (a) unit sphere, centered at origin, \( 8\pi/5 \); (b) surface of unit hemisphere centered at origin, \( z \geq 0, 9\pi/5 \)
5. \( 6\pi\sqrt{3} \)
6. (a) \(-4 \); (b) \( \pi/2 \)
7. (a) \( x^2y^3 \); (b) max = \( \sqrt{3}/18 \), min = \( -\sqrt{3}/18 \)
8. (a) \( a = b = c = 1 \); (b) \( x^3 + xyz + yz^2 \); (c) \( 3 \)