1 Lecture review

1.1 Surface integrals, flux

1. Recall: if $F$ is a vector field, $dS$ is a surface element, and $dS$ is a vector with magnitude $dS$ pointing in the outward normal direction, then

$$dS = n \, dS$$

and the flux of $F$ through that surface element is

$$F \cdot dS = F \cdot n \, dS$$

so the total flux through all of $S$ is

$$\iint_S F \cdot dS = \iint_S F \cdot n \, dS$$

2. If a region $S$ is $z$-simple with shadow $R$ in the $xy$-plane given by $z = f(x, y)$, then

$$\iint_S F \cdot dS = \iint_R F \cdot (-f_x \hat{i} - f_y \hat{j} + \hat{k}) \, dA$$

3. Define $g(x, y, z) = z - f(x, y) = 0$. Then

$$\vec{\nabla} g = -f_x \hat{i} - f_y \hat{j} + \hat{k}$$

is normal to $S$

$$n = \frac{\vec{\nabla} g}{|\vec{\nabla} g|}$$

is the unit normal.

$$dS = |\vec{\nabla} g| \, dA = \sqrt{f_x^2 + f_y^2 + 1} \, dA$$

$$dS = n \, dS = \vec{\nabla} g \, dA = (-f_x \hat{i} - f_y \hat{j} + \hat{k}) \, dA$$

4. Formulas for the four main types of surfaces:

(a) Flat surfaces

If $S$ is part of the $xy$-plane, then $n = \hat{k}$ and $dS = dA$. If $S$ is part of the $yz$-plane or the $xz$-plane, similar formulas hold.

(b) Curved surfaces

If $S$ is part of the surface $z = f(x, y)$, then

$$dS = \sqrt{f_x^2 + f_y^2 + 1} \, dA$$

$$n = \frac{-f_x \hat{i} - f_y \hat{j} + \hat{k}}{\sqrt{f_x^2 + f_y^2 + 1}}$$

(c) Parts of a cylinder

If $S$ is part of a cylinder of radius $a$ with a central $z$-axis, then

$$dS = a \, d\theta \, dz$$

$$n = r = \frac{x \hat{i} + y \hat{j}}{a} = \cos \theta \hat{i} + \sin \theta \hat{j}$$

(d) Parts of a sphere

If $S$ is part of a sphere of radius $a$ centered at the origin, then

$$dS = a^2 \sin \varphi \, d\varphi \, d\theta$$

$$n = \rho = \frac{x \hat{i} + y \hat{j} + z \hat{k}}{a} = \sin \varphi \cos \theta \hat{i} + \sin \varphi \sin \theta \hat{j} + \cos \varphi \hat{k}$$
2 Problems

1. Let $S$ be the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$ lying above the shadow region $R = \{0 \leq x, y \leq 1\}$. Compute the $x$-coordinate of the centroid of $S$.

2. Let $S$ be the portion of the cylinder $x^2 + y^2 = 1$ in the octant $x, y, z \geq 0$ that lies below $z = 1$. Compute the outward flux of $\mathbf{F} = (x^3 z^2 + y^2 z) \mathbf{i} + (x^2 y z^2 - x y z) \mathbf{j} + (x z^4 - y^2) \mathbf{k}$ through $S$.

3. Let $S$ be the portion of the sphere of radius 2 centered at the origin between $z = 0$ and $z = \sqrt{3}$. Compute the outward flux of $\mathbf{F} = x z \mathbf{i}/\sqrt{4 - z^2}$ through $S$.

4. Calculate the surface area of the part of the unit sphere that lies above the lemniscate $r^2 = \cos 2\theta$ in the region $x \geq 0$.

5. What is the area of the patch of the paraboloid $z = 16 - x^2 - y^2$ lying above the unit disk centered at the origin?

6. Compute the outward flux of the vector field $\mathbf{F} = -x \mathbf{i} - y \mathbf{j} + 3z \mathbf{k}$ across the boundary of the portion of the solid unit sphere lying in the first octant (that is, $x, y, z \geq 0$).

7. Compute the outward flux of $\mathbf{F} = x^2 \mathbf{i} - y \mathbf{j} + z \mathbf{k}$ through the boundary of the solid cylinder $y^2 + z^2 \leq 9$, $0 \leq x \leq 2$.

8. Let $S$ be the surface of the cone $z = \sqrt{x^2 + y^2}$ from $z = 0$ to $z = h$ with density function $\delta(x, y, z) = z$.
   (a) Compute the moment of inertia of $S$ about its symmetry axis.
   (b) Find the center of mass of $S$.

9. Let $S$ be the plate $x^2 + y^2 \leq a^2$ in the $xy$-plane with density $\delta = x^2 + y^2$. Compute the moment of inertia of $S$ about
   (a) its symmetry axis
   (b) an axis parallel to its symmetry axis, but passing through its edge

3 Answers

1. $\left(\frac{2(9\sqrt{3} + 4\sqrt{2})}{21(9\sqrt{3} - 8\sqrt{2} + 1)}\right)$.

2. $\pi/12$

3. $7\pi/3$

4. $\frac{\pi}{2} - 2(\sqrt{2} - 1)$.

5. $\frac{\pi}{5}(5\sqrt{5} - 1)$.

6. $\pi/6$

7. $36\pi$

8. (a) $2\pi h^2 \sqrt{2}/5$; (b) $3/4$ \( \mathbf{k} \)

9. (a) $\pi a^6/3$; (b) $5\pi a^6/6$