1 Lecture review

1.3 Gravity

1. The gravitational attraction felt by two point masses is \( F = GM_1 M_2 / r^2 \).

2. The gravitational force felt by a point of mass \( M \) by a volume \( V \) of density \( \delta \) is

\[
F = \iiint_{V} \left( \frac{GM\delta(x, y, z)}{\rho^2} \right) \rho \, dV
\]

3. Use symmetry extensively to simplify calculation. If \( V \) has an axis of symmetry \( L \) and the point is on \( L \), then the gravitational attraction must be a vector in the direction of \( L \).

1.4 Surface integrals, flux

1. If \( F \) is a vector field, \( dS \) is a surface element, and \( dS \) is a vector with magnitude \( dS \) pointing in the outward normal direction, then

\[
dS = n \, dS
\]

and the flux of \( F \) through that surface element is

\[
F \cdot dS = F \cdot n \, dS
\]

so the total flux through all of \( S \) is

\[
\iint_{S} F \cdot dS = \iint_{S} F \cdot n \, dS
\]

2. If a region \( S \) is \( z \)-simple with shadow \( R \) in the \( xy \)-plane given by \( z = f(x, y) \), then

\[
\iint_{S} F \cdot dS = \iint_{R} F \cdot (-f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k}) \, dA
\]
2 Problems

1. (a) Let $H$ be a solid hemisphere of radius $a$ in the region $z \geq 0$ whose density function is $\delta(x, y, z) = bz$. Compute the moment of inertia of $H$ about the $z$-axis.

(b) Let $P$ be the portion of the sphere centered at the origin of radius 2 between $z = 1$ and $z = \sqrt{3}$. Compute the volume of $P$ using
   
   i. spherical coordinates
   ii. cylindrical coordinates

(c) Let $C$ be a solid “ice cream cone” above the cone $z = \sqrt{3(x^2 + y^2)}$ and inside the sphere centered at the origin of radius 2 whose density function is $\delta = \rho$. Compute the center of mass of $C$.

(d) Let $S$ be the solid sphere centered at $(0, 0, a)$ with radius $a$ and density $\delta(x, y, z) = z^2$. Use spherical coordinates to compute the mass of $S$.

2. (a) Let $V$ be a solid sphere with center $(1, 0, 0)$ and radius 1 of density $\delta = (x - 2)^2$. Compute the gravitational force exerted by $V$ at a unit point mass at $(2, 0, 0)$.

(b) Let $C$ be the portion of the cone $3z^2 = x^2 + y^2$ between $z = 1$ and $z = 2$ with density $\delta = z$. Compute the gravitational force exerted by $C$ at the origin on a unit point mass.

(c) Let $U$ be the volume below the surface $z^4 = x^2 + y^2$ and above the cone $z^2 = x^2 + y^2$. Suppose $U$ has uniform density $\delta = \cos^2 \varphi$. Find the gravitational force exerted by $U$ at the origin on a unit point mass.

(d) Let $D$ be the domain given in spherical coordinates by $0 \leq \rho \leq 1 + \cos \varphi$ with density $\delta = \cos \varphi$. Compute the gravitational attraction that $D$ exerts on a point at the origin with unit mass.

(e) Let $D$ be a solid circular disk in the $yz$-plane with center $(0, a, 0)$ and radius $a$. If we rotate $D$ around the tangent line $z = 0$, we obtain a donut shape $T$ called a torus whose central hole has radius 0. Calculate by using triple integration in spherical coordinates the gravitational attraction of $T$, where $T$ has density $\delta(x, y, z) = y$, on a unit point mass $P$ at the origin.

3. Compute the indicated flux of $\mathbf{F}$ through the surface $S$.

   (a) Outward flux of $\mathbf{F} = xi + yj + zk$ through the unit cube with opposite vertices $(0, 0, 0)$ and $(1, 1, 1)$ and faces parallel to the coordinate planes.

   (b) Upward flux of $\mathbf{F} = e^{xy}i - e^{xy}j + k$ through the triangle with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.

   (c) Upward flux of $\mathbf{F} = e^{x^2}i + \ln(e^y + 1)j + xk$ through the square $0 \leq x, y \leq 1$.

   (d) Downward flux of $\mathbf{F} = \frac{1+x^2j-(1+y^2)j+(1+z^2)k}{2+y^2+z^2}$ through the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $y = z$.

3 Answers

1. (a) $\pi a^4 b / 12$; (b) $(3\sqrt{3} - 11/3)\pi$; (c) $2(2 + \sqrt{3})k/5$; (d) $8\pi a^5 / 5$

2. (a) $-16\pi GJ/21$; (b) $3\pi Gk/2$; (c) $\sqrt{2}\pi Gk/6$; (d) $4\pi Gk/3$; (e) $32\pi a^2 Gj/15$

3. (a) 3; (b) 1/2; (c) 1/2; (d) $-\pi$