1 Lecture review

1.1 Generalizations of Green’s Theorem

1. Non-simple/non-positively oriented curves
   Split the curve into multiple curves, each of which is simple. For each of those pieces, apply Green’s theorem, taking care to put a minus sign if it is negatively oriented.

2. Non-closed curves
   Close the curve by adding another curve, apply Green’s theorem to the combined curve, and subtract the contribution of the line integral from the piece that was added.

3. Singular vector fields, non-simply-connected regions
   A region \( R \) is simply connected if it has no holes. When the vector field has a singularity, one may have to apply Green’s theorem to a region which is not simply connected.
   Deform the curve \( C \) to a simpler curve \( C_0 \) around the singularity. Apply Green’s theorem to the region \( R \) between \( C \) and \( C_0 \) to relate the line integral around \( C \) to that around \( C_0 \).

1.2 Triple integrals

1. Triple integration allows one to integrate over volumes. To each point in the volume \( V \) is assigned a function \( f(x,y,z) \). One can then integrate this to get a triple integral

\[
\iiint_V f(x,y,z) \, dV.
\]

(a) When \( f = \delta(x,y,z) \) is the density, then the triple integral yields the mass of \( V \).

(b) The centroid of \( V \) is the point \((\overline{x}, \overline{y}, \overline{z})\) given by

\[
\overline{x} = \iiint_V x \, dV / \iiint_V 1 \, dV, \quad \overline{y} = \iiint_V y \, dV / \iiint_V 1 \, dV, \quad \overline{z} = \iiint_V z \, dV / \iiint_V 1 \, dV
\]

The denominator in each of the three formulas above is the volume of \( V \). Oftentimes one can see without integration that some of these expressions must be zero by symmetry.

(c) For a density function \( \delta \), the center of mass of \( V \) is the point \((x_{CM}, y_{CM}, z_{CM})\) given by

\[
x_{CM} = \iiint_V x \delta \, dV / \iiint_V \delta \, dV, \quad y_{CM} = \iiint_V y \delta \, dV / \iiint_V \delta \, dV, \quad z_{CM} = \iiint_V z \delta \, dV / \iiint_V \delta \, dV
\]

The denominator in each of the three formulas above is the mass of \( V \). Oftentimes one can see without integration that some of these expressions must be zero by symmetry.

2. Evaluation method (in rectangular/cylindrical coordinates)

(a) Rectangular: Let \( R \) be the projection of \( V \) to the \( xy \)-plane (the “vertical shadow” of \( D \)). For each point \((x,y)\) in \( R \), suppose that the points in \( V \) are given by \( g(x,y) \leq z \leq h(x,y) \). Then

\[
\iiint_V f(x,y,z) \, dV = \int_R \left( \int_{g(x,y)}^{h(x,y)} f(x,y,z) \, dz \right) \, dA.
\]

becomes a double integral which can be evaluated using the standard techniques.

(b) Cylindrical: Let \( R \) be the projection of \( V \) to the \( xy \)-plane (the “vertical shadow” of \( D \)). For each point \((r,\theta)\) in \( R \), suppose that the points in \( V \) are given by \( g(r,\theta) \leq z \leq h(r,\theta) \). Then

\[
\iiint_V f(r,\theta,z) \, dV = \int_R \left( \int_{g(r,\theta)}^{h(r,\theta)} f(r,\theta,z) \, dz \right) \, (r \, dr \, d\theta).
\]

becomes a double integral which can be evaluated using the standard polar-coordinate techniques.

3. Spherical coordinates will be covered later. (They will not be on the midterm.)
2 Problems

1. Let \( \mathbf{F} \) be the vector field
\[
\mathbf{F} = \frac{(x-2y)i + (2x+y)j}{x^2+y^2}.
\]
Compute \( \text{curl} \mathbf{F} \) and \( \text{div} \mathbf{F} \). If a curve loops \( k \) times around the origin, what is the work done by \( \mathbf{F} \) along the curve? How about flux?

2. Let \( \mathbf{F} \) be the vector field
\[
\mathbf{F} = \frac{-y i + x j}{x^2+y^2}.
\]
Compute the work done by \( \mathbf{F} \) across the curve \( C \), drawn to the right with prescribed orientation.

3. Let \( V \) be the region over the unit square \( 0 \leq x, y \leq 1 \) between \( z = x^2y^2 \) and \( z = 2e^{xy} \) and with density \( \delta = x \). Compute the \( z \)-coordinate of the center of mass of \( V \).

4. Let \( D \) be the region that lies between the downward facing paraboloid \( z = 3-x^2-y^2 \) and the cone \( z = 2\sqrt{x^2+y^2} \) with density \( \delta = 1/r \).
   
   (a) Compute the mass of \( D \).
   
   (b) Compute the centroid of \( D \).

5. Let \( V \) be the volume between the two solid cylinders \( x^2 + z^2 \leq 1 \) and \( y^2 + z^2 \leq 1 \). Compute the volume of \( V \).

6. Let \( V \) be the volume between the surfaces \( x+4y+z^2 = 9 \) and \( x - y^2 + 4z = 1 \). Find the volume of \( V \).

7. Let \( V \) be the volume between the surfaces \( x^2 + y + 4z^2 = 3 \) and \( 2x + y = 0 \). Find the volume of \( V \).

3 Answers

1. work = \( 4\pi k \), flux = \( 2\pi k \)
2. \( 4\pi \)
3. \( \left( \frac{2}{7} - \frac{91}{8} \right) / (2e - \frac{49}{12}) \)
4. (a) \( \frac{16\pi}{3} \) (b) \( \frac{11}{7} \mathbf{k} \)
5. \( \frac{16}{3} \)
6. 128\pi
7. \( 4\pi \)